ON RINGS WITH WEAKLY PRIME CENTERS*

PRO КИЛЬЯЗ З СЛАБКИМИ ПРОСТИМИ ЦЕНТРАМИ

We introduce a class of rings obtained as a generalization of rings with prime centers. A ring $R$ is called weakly prime center (or, briefly, $WPC$) if $ab \in Z(R)$ implies that $aRb$ is an ideal of $R$, where $Z(R)$ stands for the center of $R$. The structure and properties of these rings are studied, the relationships between prime center rings, strongly regular rings, and $WPC$ rings are discussed, parallel with the relationship between $WPC$ to commutativity.

1. Introduction. Throughout this article, all rings considered are associative with identity, and all modules are unital, the symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$, $Z(R)$ and $\text{Max}_l(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the center and the set of all maximal left ideals of $R$. For any nonempty subset $X$ of a ring $R$, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of $X$ and the left annihilator of $X$, respectively.

A ring $R$ is called

1. reduced if $N(R) = 0$;
2. Abel if $E(R) \subseteq Z(R)$;
3. left quasiduo if every maximal left ideal of $R$ is an ideal;
4. MELT if every essential maximal left ideal of $R$ is an ideal.

Recall that a ring $R$ has prime (semiprime) center [8] if $ab \in Z(R)$ implies $a \in Z(R)$ or $b \in Z(R)$ ($a^n \in Z(R)$ implies $a \in Z(R)$). Clearly, commutative rings have prime center. In [8], some basic properties of prime center rings are studied.

A ring $R$ is called periodic [3] if for each $x \in R$, there exist distinct positive integers $m$ and $n$ for which $x^m = x^n$. In [8] (Theorem 1), it is shown that for a periodic ring $R$, $R$ is commutative if and only if $R$ has prime center.

In this paper, a new class of rings is introduced, which is a proper generalization of rings with prime centers. A ring $R$ is called weakly prime center (or, briefly, $WPC$) if $ab \in Z(R)$ implies $aRb$ is an ideal of $R$. Remark 2.1 points out that $WPC$ rings are proper generalization of rings with prime centers. Proposition 2.6 shows that strongly regular rings are a class of $WPC$ rings. Proposition 2.8 shows that a ring $R$ is a division ring if and only if $R$ is a $WPC$ primitive ring.

Let $R$ be a ring and $e \in E(R)$. $e$ is called left minimal idempotent if $Re$ is a minimal left ideal of $R$. We write $ME_l(R)$ for the set of all left minimal idempotents of $R$. A ring $R$ is called left min-Abel if $(1-e)Re = 0$ for each $e \in ME_l(R)$. In [13] (Theorem 1.2), it is shown that a ring $R$

* This work is supported by the Foundation of Natural Science of China (11471282, 11171291) and Natural Science Fund for Colleges and Universities in Jiangsu Province(11KJB110019).

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ISSN 1027-3190. Укр. мат. журн., 2014, т. 66, № 12

UDC 512.5

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ПРО КИЛЬЯЗ З СЛАБКИМИ ПРОСТИМИ ЦЕНТРАМИ

Введено клас кiлець, що № узагальненням кiлець з простими центрами. Кiльце $R$ all modules are unital, the symbols $J$ throughout this article, all rings considered are associative with identity, and $1. Introduction.$ Вивчено структуру i властивостi таких kiлець та проаналiзовано спiввiдношення мiж простими центральними

kiльцями, сильно регулярними кiльцями та кiльцями з слабко простим центром паралельно зi спiввiдношенням мiж слабко простим центром та комутативнiстю.

1. Introduction. Throughout this article, all rings considered are associative with identity, and all modules are unital, the symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$, $Z(R)$ and $\text{Max}_l(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the center and the set of all maximal left ideals of $R$. For any nonempty subset $X$ of a ring $R$, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of $X$ and the left annihilator of $X$, respectively.

A ring $R$ is called

1. reduced if $N(R) = 0$;
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Recall that a ring $R$ has prime (semiprime) center [8] if $ab \in Z(R)$ implies $a \in Z(R)$ or $b \in Z(R)$ ($a^n \in Z(R)$ implies $a \in Z(R)$). Clearly, commutative rings have prime center. In [8], some basic properties of prime center rings are studied.

A ring $R$ is called periodic [3] if for each $x \in R$, there exist distinct positive integers $m$ and $n$ for which $x^m = x^n$. In [8] (Theorem 1), it is shown that for a periodic ring $R$, $R$ is commutative if and only if $R$ has prime center.

In this paper, a new class of rings is introduced, which is a proper generalization of rings with prime centers. A ring $R$ is called weakly prime center (or, briefly, $WPC$) if $ab \in Z(R)$ implies $aRb$ is an ideal of $R$. Remark 2.1 points out that $WPC$ rings are proper generalization of rings with prime centers. Proposition 2.6 shows that strongly regular rings are a class of $WPC$ rings. Proposition 2.8 shows that a ring $R$ is a division ring if and only if $R$ is a $WPC$ primitive ring.

Let $R$ be a ring and $e \in E(R)$. $e$ is called left minimal idempotent if $Re$ is a minimal left ideal of $R$. We write $ME_l(R)$ for the set of all left minimal idempotents of $R$. A ring $R$ is called left min-Abel if $(1-e)Re = 0$ for each $e \in ME_l(R)$. In [13] (Theorem 1.2), it is shown that a ring $R$

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ISSN 1027-3190. Укр. мат. журн., 2014, т. 66, № 12

UDC 512.5

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is a left quasiduo ring if and only if \( R \) is a left min-Abel \( MELT \) ring. The study of left min-Abel rings appears in [13, 15, 16]. Proposition 2.5 shows that \( WPC \) rings are left min-Abel.

Following [10], an element \( a \) of a ring \( R \) is called clean if \( a \) is a sum of a unit and an idempotent of \( R \), and \( a \) is said to be exchange if there exists \( e \in E(R) \) such that \( e \in aR \) and \( 1 - e \in (1 - a)R \). A ring \( R \) is called clean if every element of \( R \) is clean, and \( R \) is said to be exchange if every element of \( R \) is exchange. According to [10], clean rings are always exchange, but the converse is not true, in general. In [18], it is shown that left quasiduo exchange rings are clean; in [19], it is shown that Abel exchange rings are clean; in [15], it is shown that quasinormal exchange rings are clean; in [16], it is shown that weakly normal exchange rings are clean. Theorem 3.1 shows that \( WPC \) exchange rings are clean and have stable range 1.

Following [4], a ring \( R \) is said to be \textit{semiperiodic} if for each \( x \in R \setminus (J(R) \cup Z(R)) \), there exist \( m, n \in \mathbb{Z} \), of opposite parity, such that \( x^n - x^m \in N(R) \). Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings. Theorem 4.2 shows that for a semiperiodic ring \( R \) with \( J(R) \neq N(R) \), \( R \) is \( WPC \) if and only if \( R \) is commutative.

2. Some properties of \( WPC \) rings.

**Definition 2.1.** A ring \( R \) is called \textit{weakly prime center} (\( WPC \)) if for any \( a, b \in R, ab \in Z(R) \) implies \( aRb \) is an ideal of \( R \).

Clearly commutative rings are \( WPC \).

**Proposition 2.1.** Prime center rings are \( WPC \).

**Proof.** Let \( R \) be a prime center ring and \( a, b \in R \) with \( ab \in Z(R) \). Since \( R \) is prime center, \( a \in Z(R) \) or \( b \in Z(R) \), one has \( aRb = abR = Rab \). Hence \( aRb \) is an ideal of \( R \) and \( R \) is \( WPC \).

Recall that a ring \( R \) is directly finite if \( ab = 1 \) implies \( ba = 1 \) for any \( a, b \in R \). In [8], it is shown that prime center rings are directly finite.

**Lemma 2.1.** \( WPC \) rings are directly finite.

**Proof.** Let \( a, b \in R \) with \( ab = 1 \). Let \( e = ba \). Then \( a = ae \) and \( e \in E(R) \). Since \( a(1 - e) = 0 \in Z(R) \), \( aR(1 - e) \) is an ideal of \( R \), that is, \( R(1 - e) \) is an ideal of \( R \) because \( aR = R \). Hence \( (1 - e)a \in R(1 - e) \), which implies \( (1 - e)a = (1 - e)ae = 0 \). Then \( a = ea \) and \( 1 = ab = eab = e = ba \), this shows that \( R \) is a directly finite ring.

**Proposition 2.2.** Let \( R \) be a local ring. If \( J(R) \) is commutative, then \( R \) is \( WPC \).

**Proof.** Assume that \( ab \in Z(R) \). Then \( Rab \) is an ideal of \( R \). If \( Rab = R \), then \( ab \in U(R) \), by Lemma 2.1, \( a, b \in U(R) \), so \( aRb = R \) is an ideal of \( R \). If \( Rab \subseteq J(R) \), then \( ab \notin U(R) \). If \( a \notin U(R) \) and \( b \notin U(R) \), then \( aR, Rb \subseteq J(R) \). Since \( J(R) \) is commutative, \( RaRbR = R(a(Rb))R = RbaR = aRb \), this gives \( aRb \) is an ideal of \( R \). If \( a \in U(R) \) and \( b \notin U(R) \), then \( RaRbR = RbR = RabR = Rab = Rb = aRb \), so \( aRb \) is an ideal. Similarly, if \( a \notin U(R) \) and \( b \in U(R) \), we can show that \( aRb \) is an ideal. Hence \( R \) is \( WPC \).

**Proposition 2.3.** If \( R \) be a local prime center ring, then \( R \) is commutative.

**Proof.** It is an immediate result of [8] (Basic Lemma 2(b)).

**Remark 2.1.** By Proposition 2.2, one knows that division rings are \( WPC \). By Proposition 2.3, noncommutative division rings need not be prime center. Thus there exists a \( WPC \) ring (noncommutative division rings) which is not prime center. Hence \( WPC \) rings are proper generalization of prime center rings.

**Proposition 2.4.** If \( R \) is a semiprime \( WPC \) ring, then \( R \) is reduced.
Hence \( g \) which is a contradiction. Hence \( R \), which is a contradiction. Thus \( R \), is an ideal of \( ON \) rings with weakly prime centers 1617.

Lemma 2.1, we know that the converse of Corollary 2.1 is not true.

Hence, for any \( R \) if and only if \( R \) is a left min-Abel ring.

Recall that a ring \( R \) is left min-Abel [13] if for every \( e \in ME_1(R) = \{ e \in E(R) \mid Re \) is a minimal left ideal of \( R \} \), \( e \) is left semicentral in \( R \). Clearly, \( R \) is a left min-Abel ring if and only if \( (1 - e)Re = 0 \) for each \( e \in ME_1(R) \).

Proposition 2.5. If \( R \) is a WPC ring, then \( R \) is left min-Abel.

Recall that a ring \( R \) is left min-Abel [13] if for every \( e \in ME_1(R) = \{ e \in E(R) \mid Re \) is a minimal left ideal of \( R \} \), \( e \) is left semicentral in \( R \). Clearly, \( R \) is a left min-Abel ring if and only if \( (1 - e)Re = 0 \) and \( R \) is a left min-Abel ring.

Remark 2.3. Simple rings need not be WPC. For example, let \( D \) be a division ring and

\[
R = \begin{pmatrix} D & D \\ D & D \end{pmatrix}.
\]

Then \( R \) be a simple ring. Clearly, \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) \( Re = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) = 0 \( ∈ Z(R) \) and

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \).
\]

If \( R \) is WPC, then

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

which is a contradiction. Hence \( R \) is not a WPC ring.

Recall that a ring \( R \) is left min-Abel [13] if for every \( e \in ME_1(R) = \{ e \in E(R) \mid Re \) is a minimal left ideal of \( R \} \), \( e \) is left semicentral in \( R \). Clearly, \( R \) is a left min-Abel ring if and only if \( (1 - e)Re = 0 \) for each \( e \in ME_1(R) \).

Proposition 2.5. If \( R \) is a WPC ring, then \( R \) is left min-Abel.

Proof. Let \( e \in ME_1(R) \). Since \( R \) is a WPC ring and \( (1 - e)e = 0 \) \( ∈ Z(R) \), \( (1 - e)Re \) is an ideal of \( R \), this gives \( R(1 - e)Re \subseteq (1 - e)Re \). If \( (1 - e)Re \neq 0 \), then \( R(1 - e)Re = Re \), so \( e \in eRe = eR(1 - e)Re \subseteq e(1 - e)Re = 0 \), which is a contradiction. Therefore \( (1 - e)Re = 0 \) and \( R \) is a left min-Abel ring.

Remark 2.4. The converse of Proposition 2.5 is not true in general. For example, let

\[
R = \begin{pmatrix} \mathbb{Z}_5 & \mathbb{Z}_5 \\ 0 & \mathbb{Z}_5 \end{pmatrix}.
\]

Clearly, \( \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} R \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \in Z(R) \) and \( \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} R \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} \)

\( \notin Z(R) \), so \( R \) is not WPC. Since \( R \) is a left quasiduo ring by [18], \( R \) is a left min-Abel ring by [13] (Theorem 1.2).

Recall that a ring \( R \) is von Neumann regular if \( a \in aRa \) for any \( a \in R \), and \( R \) is said to be strongly regular if \( a \in a^2R \) for any \( a \in R \). It is well known that a ring \( R \) is a strongly regular ring if and only if \( R \) is a reduced von Neumann regular ring.

Proposition 2.6. The following conditions are equivalent for a ring \( R \):

1. \( R \) is a strongly regular ring;
2. \( R \) is a WPC von Neumann regular ring.

Proof. 1 \( \implies \) 2. Since \( R \) is a strongly regular ring, \( R \) is an Abel von Neumann regular ring. Hence, for any \( a, b \in R \), \( aR = bR = Re \) for some \( e \in E(R) \), this gives \( aRb = R(eb) = Rg = gR \) for some \( g \in E(R) \). Thus \( aRb \) is an ideal of \( R \) and \( R \) is a WPC ring.

2 \( \implies \) 1. Since von Neumann regular rings are semiprime, by Proposition 2.4, \( R \) is reduced. Hence \( R \) is a strongly regular ring.

Recall that a ring \( R \) is left SF if every simple left \( R \)-module is flat. It is well known that von Neumann regular rings are left SF. In [11] (Remark 3.13), it is shown that if \( R \) is a reduced left SF ring, then \( R \) is strongly regular. We can generalize this result as follows.
Proposition 2.7. If $R$ is a prime center left $SF$ ring, then $R$ is a commutative strongly regular ring.

Proof. Let $a \in R$ with $a^2 = 0$. By [8] (Basic Lemma 2(a)), $a \in Z(R)$. If $a \neq 0$, then $l(a) \neq R$ and there exists a maximal left ideal $M$ of $R$ such that $l(a) \subseteq M$. Since $R$ is a left $SF$ ring, $R/M$ is flat as left $R$-module. Since $a \in l(a) \subseteq M$, $a = am$ for some $m \in M$. Since $a \in Z(R)$, $a = ma$, one obtains $1 - m \subseteq l(a) \subseteq M$, 1 $\in M$, which is a contradiction. Hence $a = 0$, which implies $R$ is reduced, by [11] (Remark 3.13), $R$ is strongly regular. Now let $x \in R$. Then $x = xy$ for some $y \in R$. Write $e = xy$ and $g = yx$. Then $e, g \in E(R)$ and $x = ex = yg$. Since $R$ is Abelian, $e, g \in Z(R)$. Since $xy = g \in Z(R)$, $y \in Z(R)$ or $x \in Z(R)$. If $x \in Z(R)$, we are done. If $y \in Z(R)$, then for any $r \in R$, we have $x = xgr = xyg = xyx = eyx = exe = rer = xe = rx$, which implies $x \in Z(R)$. Hence $R$ is commutative.

Corollary 2.2. If $R$ is a prime center von Neumann regular ring, then $R$ is a commutative strongly regular ring.

Remark 2.5. Since strongly regular rings need not be commutative, by Corollary 2.2, strongly regular rings need not be prime center.

Corollary 2.3. $R$ is a field if and only if $R$ is a prime center division ring.

Proof. Fields are certainly prime center division rings. The converse is an immediate corollary of Corollary 2.2.

Proposition 2.8. $R$ is a division ring if and only if $R$ is a $WPC$ primitive ring.

Proof. Division rings are certainly $WPC$ primitive rings. Now let $R$ be a $WPC$ primitive ring. If $R$ is not a division ring, then there exists a subring $S$ of $R$ such that $S \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}$, where $D$ is a division ring. Clearly, $\begin{pmatrix} D & D \\ D & D \end{pmatrix}$ is not reduced, so $S$ is not reduced, this implies $R$ is not reduced. But by Proposition 2.4, $R$ is reduced, which is a contradiction. Hence $R$ is a division ring.

Corollary 2.3 and Proposition 2.8 give the following corollary.

Corollary 2.4. $R$ is a field if and only if $R$ is a prime center primitive ring.

A ring $R$ is called weakly regular if $a \in aRaR \cap RaRa$ for every $a \in R$. A left $R$-module $M$ is called $YJ$-injective ($Wnil$-injective (see [14])) if for each $0 \neq a \in R \neq 0 \neq a \in N(R)$, there exists a positive integer $n$ such that $a^n \neq 0$ and each left $R$-homomorphism $Ra^n \rightarrow M$ can be extended to $R \rightarrow M$. It is easy to see that $YJ$-injective modules are $Wnil$-injective.

Proposition 2.9. Let $R$ be a $WPC$ ring. If each singular simple left $R$-modules are $Wnil$-injective, then $R$ is reduced.

Proof. By Proposition 2.4, we only need to show that $R$ is semiprime. Assume that $a \in R$ with $aRa = 0$. If $a \neq 0$, then there exists a maximal left ideal $M$ of $R$ such that $r(aR) \subseteq M$. We claim that $M$ is an essential left ideal of $R$. If not, $M = l(e)$ for some $e \in ME_1(R)$. Since $R$ is a $WPC$ ring, $R$ is left min-Abel by Proposition 2.5. Hence $aRe = aeRe = 0$ because $a \in r(aR) \subseteq M = l(e)$, this leads to $e \in r(aR) \subseteq l(e)$, which is a contradiction. Thus $M$ is an essential left ideal of $R$ and $R/M$ is a singular simple left $R$-module, by hypothesis, $R/M$ is $Wnil$-injective. Then the left $R$-homomorphism $f: Ra \rightarrow R/M$ defined by $f(ra) = r + M$ can be extended into $R \rightarrow R/M$, so there exists $d \in R$ such that $1 - ad \in M$. Since $ad(ad) = 0$, $1 - ad$ is a unit of $R$, so $M = R$, a contradiction. Hence $a = 0$.

Corollary 2.5. Let $R$ be a $WPC$ ring whose singular simple left $R$-modules are $YJ$-injective, then $R$ is a reduced weakly regular ring.
Proof. By Proposition 2.9, $R$ is reduced. By [9] (Theorem 4), $R$ is a reduced weakly regular ring.

3. Exchange WPC rings. Following [10], an element $a$ of a ring $R$ is called clean if $a$ is a sum of a unit and an idempotent of $R$, and $a$ is said to be exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. A ring $R$ is called clean if every element of $R$ is clean, and $R$ is said to be exchange if every element of $R$ is exchange. According to [10], clean rings are always exchange, but the converse is not true unless $R$ satisfies one of the following conditions: (1) $R$ is a left quasi duo ring [18]; (2) $R$ is an Abel ring [19]; (3) $R$ is a quasinormal ring [15]; (4) $R$ is a weakly normal ring [16].

Theorem 3.1. Let $R$ be a WPC ring and $a \in R$. Then

(1) If $a$ is exchange, then $a$ is clean.

(2) If $R$ is an exchange ring, then $R$ is a clean ring.

(3) If $a^n$ is clean for some $n \geq 1$, then $a$ is clean.

(4) If $a^2$ is clean, then $a$ and $-a$ are clean.

Proof. (1) Let $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. Write $e = ab$ and $1 - e = (1 - a)c$ for some $b = be, c = c(1 - e) \in R$. Then $(a - (1 - e))(b - c) = ab - ac - (1 - e)b + (1 - e)c = ab + (1 - a)c - (1 - e)b - ec = 1 - (1 - e)b - ec$. Since $R$ is a WPC ring and $b(1 - e) = 0 \in Z(R)$, $bR(1 - e)$ is an ideal of $R$. Hence $bR(1 - e)R \subseteq bR(1 - e)$ and $bR(1 - e)Re = 0$, which implies $bR(1 - e)Rb = bR(1 - e)Rb = 0$. Therefore $(R(1 - e)Rb)^2 = 0$, this leads to $(1 - e)b \in (1 - e)Rb \subseteq J(R)$. Similarly, $ec \in J(R)$. Hence $1 - (1 - e)b - ec$ is a unit of $R$, by Lemma 2.1, one obtains $a - (1 - e)$ is an unit of $R$. Hence $a$ is a clean element.

(2) It is an immediate result of (1).

(3) Since $a^n$ is clean, there exist $u \in U(R)$ and $f \in E(R)$ such that $a^n = u + f$. Let $e = u(1 - f)$ and $e = a^n(1 - e) \in aR$, so $e = a^n + (a^n - a^{2n})u^{-1} \in aR$ and $1 - e \in (1 - a)R$, this implies $a$ is exchange, by (1), $a$ is clean.

(4) Since $a^2 = (-1)a^2$ is clean, by (3), $a$ and $-a$ are clean.

Corollary 3.1. Let $R$ be a WPC ring and idempotent can be lifted modulo $J(R)$. Let $a \in R$ be clean and $e \in E(R)$. Then

(1) $ae$ is clean.

(2) If $-a$ is also clean, then $a + e$ is clean.

Proof. (1) Since $a$ is clean, $\bar{a}$ is clean in $\bar{R} = R/J(R)$. Since $R$ is a WPC ring, $eR(1 - e)$ is an ideal of $R$, which implies $(1 - e)Re)^2 = (eR(1 - e))^2 = 0$. Hence $(1 - e)\bar{R}e = e\bar{R}(1 - e) = 0$, that is, $\bar{e}$ is a central idempotent in $\bar{R}$. Since $a$ is clean in $R$, there exist $u \in U(R)$ and $f \in E(R)$ such that $a = u + f$. Let $v \in R$ such that $uv = vu = 1$. Then, in $\bar{R}$, $\bar{a} = \bar{u}\bar{e} + \bar{e} + \bar{e} = (\bar{v}\bar{e} + \bar{e} - \bar{1})(\bar{u}\bar{e} + \bar{e} - \bar{1}) = \bar{1}$ and $\bar{f}\bar{e} + \bar{e} - \bar{1} = \bar{f}\bar{e} + \bar{e} - \bar{1}$, so $\bar{a}e = 0$. Since idempotent can be lifted modulo $J(R)$, there exists $g \in E(R)$ such that $\bar{g} = \bar{f}\bar{e} + \bar{e} - \bar{1}$. Let $w \in R$ such that $\bar{w} = \bar{u}\bar{e} + \bar{e} - \bar{1}$. Then $w \in U(R)$ and $ae - w - g \in J(R)$. Let $ae - w - g = x \in J(R)$. Then $ae = g + w(1 + w^{-1}x)$. Since $w(1 + w^{-1}x) \in U(R)$, $ae$ is clean in $R$.

(2) Since $-a$ is clean in $R$, $1 + a$ is clean in $R$. Hence $\bar{a}$ and $\bar{1} + \bar{a}$ are all clean in $\bar{R} = R/J(R)$. Let $\bar{a} = \bar{u} + \bar{f}$ and $\bar{1} + \bar{a} = \bar{v} + \bar{g}$ where $u, v \in U(R)$ and $f, g \in E(R)$. Clearly, $\bar{a} + \bar{e} = \bar{a}(\bar{1} - \bar{e}) + (\bar{1} + \bar{a})\bar{e}$, so $\bar{a} + \bar{e} = \bar{v}\bar{e} + \bar{u}(\bar{1} - \bar{e}) + \bar{g}\bar{e} + \bar{f}(\bar{1} - \bar{e})$. Clearly, $(\bar{v}\bar{e} + \bar{u}(\bar{1} - \bar{e}))(\bar{v}^{-1}\bar{e} + \bar{a}^{-1}(\bar{1} - \bar{e})) = \bar{1}$ and $\bar{g}\bar{e} + \bar{f}(\bar{1} - \bar{e}) \in E(\bar{R})$. Therefore, $\bar{a} + \bar{e}$ is clean in $\bar{R}$, similar to (1), we obtain $a + e$ is clean in $R$.

In [5], it is showed that if $R$ is a unit regular ring, then every element of $R$ is a sum of two units. A ring $R$ is called an $(S, 2)$-ring [5], if every element of $R$ is a sum of two units of $R$. In [2], it...
is proved that if \( R \) is an Abel \( \pi \)-regular ring, then \( R \) is an \((S, 2)\)-ring if and only if \( \mathbb{Z}/2\mathbb{Z} \) is not a homomorphic image of \( R \).

**Theorem 3.2.** Let \( R \) be a WPC \( \pi \)-regular ring. Then \( R \) is an \((S, 2)\)-ring if and only if \( \mathbb{Z}/2\mathbb{Z} \) is not a homomorphic image of \( R \).

**Proof.** Since \( R \) is a WPC \( \pi \)-regular ring, \( R/J(R) \) is \( \pi \)-regular ring. Since \( R \) is an exchange ring, idempotent can be lifted modulo \( J(R) \). By the proof of Corollary 3.1(1), \( R/J(R) \) is an Abelian ring. By [2], \( R/J(R) \) is an \((S, 2)\)-ring if and only if \( \mathbb{Z}/2\mathbb{Z} \) is not a homomorphic image of \( R/J(R) \). By [15] (Lemma 4.3), we are done.

In light of Theorem 3.2, we have the following corollaries:

**Corollary 3.2.** Let \( R \) be a WPC \( \pi \)-regular ring such that \( 2 = 1 + 1 \in U(R) \). Then \( R \) is an \((S, 2)\)-ring.

**Corollary 3.3.** Let \( R \) be a WPC \( \pi \)-regular ring. Then \( R \) is an \((S, 2)\)-ring if and only if for some \( d \in U(R), 1 + d \in U(R) \).

Recall that a ring \( R \) is said to have stable range 1 [12] if for any \( a, b \in R \) satisfying \( aR + bR = R \), there exists \( y \in R \) such that \( a + by \) is right invertible. Clearly, \( R \) has stable range 1 if and only if \( R/J(R) \) has stable range 1. In [19] (Theorem 6), it is showed that exchange rings with all idempotents central have stable range 1.

**Theorem 3.3.** WPC exchange rings have stable range 1.

**Proof.** Let \( R \) be a WPC exchange ring. Then \( R/J(R) \) is exchange with all idempotents central, so, by [19] (Theorem 6), \( R/J(R) \) has stable range 1. Therefore \( R \) has stable range 1.

In [17], A ring \( R \) is said to satisfy the unit 1-stable condition if for any \( a, b, c \in R \) with \( ab + c = 1 \), there exists \( u \in U(R) \) such that \( au + c \in U(R) \). It is easy to prove that \( R \) satisfies the unit 1-stable condition if and only if \( R/J(R) \) satisfies the unit 1-stable condition.

**Theorem 3.4.** Let \( R \) be a WPC exchange ring, then the following conditions are equivalent:

1. \( R \) is an \((S, 2)\)-ring.
2. \( R \) satisfies the unit 1-stable condition.
3. Every factor ring of \( R \) is an \((S, 2)\)-ring.
4. \( \mathbb{Z}_2 \) is not a homomorphic image of \( R \).

A ring \( R \) is called left topologically boolean, or a tb-ring [1] for short, if for every pair of distinct maximal left ideals of \( R \) there is an idempotent in exactly one of them.

**Theorem 3.5.** Let \( R \) be a WPC exchange ring. Then \( R \) is a left tb-ring.

**Proof.** Suppose that \( M \) and \( N \) are distinct maximal left ideals of \( R \). Let \( a \in M \setminus N \). Then \( Ra + N = R \) and \( 1 - xa \in N \) for some \( x \in R \). Clearly, \( xa \in M \setminus N \). Since \( R \) is a WPC exchange ring, \( R \) is clean by Theorem 3.1, there exist an idempotent \( e \in E(R) \) and a unit \( u \) in \( R \) such that \( xa = e + u \). If \( e \in M \), then \( u = xa - e \in M \) from which it follows that \( R = M \), a contradiction. Thus \( e \notin M \). If \( e \notin N \), then \( Re + N = R \). Since \( R \) is a WPC ring, by the proof of Corollary 3.1(1), \( (1 - e)ReR \subseteq J(R) \subseteq N \), \( 1 - e \in (1 - e)R = (1 - e)Re + (1 - e)N \subseteq N \). Hence \( u = (1 - e) + (xa - 1) \in N \). It follows that \( N = R \) which is also impossible. We thus have that \( e \) is an idempotent belonging to \( N \) only.

4. **WPC semiperiodic rings.** Following [4], a ring \( R \) is said to be semiperiodic if for each \( x \in R \), \((J(R) \cup \mathbb{Z}(R))\), there exist \( m, n \in \mathbb{Z} \), of opposite parity, such that \( x^n - x^m \in N(R) \). Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain nonnil periodic rings.

**Lemma 4.1.** If \( R \) is a WPC semiperiodic ring, then \( N(R) \subseteq J(R) \).
Proof. Let $a \in N(R)$ with $a^k = 0$, and let $x \in R$. If $ax \in J(R)$, then $ax$ is right quasiregular; and if $ax \in Z(R)$, then $ax$ is nilpotent and again $ax$ is right quasiregular. Suppose, then, that $ax \notin J \cup Z$, in which case [4] (Lemma 2.3 (iii)) gives $q \in \mathbb{Z}^+$ and an idempotent $e$ of form $ay$ such that $(ax)^q = (ax)^qe$. Since

$$
eq ay = eay = ea(1-e)y + eaey = ea(1-e)y + ea^2y^2 =$$

$$= ea(1-e)y + ea^2(1-e)y^2 + ea^2ey^2 = ea(1-e)y + ea^2(1-e)y^2 + ea^3y^3 = \ldots$$

$$\ldots = \sum_{i=1}^{k-1} ea^i(1-e)y^i + ea^k y^k = \sum_{i=1}^{k-1} ea^i(1-e)y^i.$$

Since $R$ is a WPC ring, $eR(1-e) \in J(R)$ by the proof of Corollary 3.1(1), which implies $e \in J(R)$, so $e = 0$ and $(ax)^q = 0$, which shows that $ax$ is right quasiregular. Thus $a \in J(R)$.

**Theorem 4.1.** If $R$ is a WPC semiperiodic ring, then $R/J(R)$ is commutative.

**Proof.** Let $\bar{R} = R/J(R)$. Clearly, $N(R) \subseteq J(R)$ by Lemma 4.1. Now let $\bar{a} \in \bar{R}$ with $\bar{a}^2 = 0$. Then $a^2 \in J(R) \subseteq N(R) \cup Z(R)$ by [4] (Lemma 2.6). If $a^2 \in N(R)$, then $a \in N(R)$. Hence $a \in J(R)$ by Lemma 4.1 and $\bar{a} = 0$. If $a^2 \in Z(R)$, then $\bar{a} = 0$. If $\bar{a} \in J(\bar{R})$, then $\bar{a} = 0$. Since $\bar{R}$ is semiprime, $\bar{a} = 0$. If $\bar{a} \notin Z(\bar{R})$, then $\bar{a} \notin J(\bar{R}) \cup Z(\bar{R})$. By [4] (Lemma 2.3(iii)), $a^q = a^q e$ for some $q \geq 1$ and $e \in E(R)$ with the form $ay$. Hence $e = eay = ea(1-e)y + eaey = ea(1-e)y + ea^2y^2 \in J(R)$. Thus $e = 0$ and $a^q = 0$. This implies $a \in N(R) \subseteq J(R)$ by Lemma 4.1, which is a contradiction. Hence $\bar{a} \in Z(\bar{R})$ and so $\bar{a} = 0$. Therefore $\bar{R}$ is a reduced ring. Since $\bar{R}$ is also semiperiodic, by [4] (Lemma 4.4), $\bar{R}$ is commutative.

**Theorem 4.2.** Let $R$ be a WPC semiperiodic ring. Then

1. $N(R)$ is an ideal of $R$.
2. If $J(R) \neq N(R)$, then $R$ is commutative.

**Proof.** (1) Let $a, b \in N(R)$ and $x \in R$. Then $a - b, ax \in J(R)$ by Lemma 4.1. By [4] (Lemma 2.6), $a - b, ax \in N(R) \cup Z(R)$. If $a - b, ax \in N(R)$, we are done. If $a - b, ax \in Z(R)$, then $(a-b)a = a(a-b)$ and $(ax)^n = a^nx^n$ for any $n \geq 1$, this gives $ab = ba$, thus $a - b, ax \in N(R)$. Similarly, $xa \in N(R)$. Therefore $N(R)$ is an ideal of $R$.

(2) By [4] (Lemma 2.6), it follows that

$$J(R) = (J(R) \cap N(R)) \cup (J(R) \cap Z(R)).$$  \hspace{1cm} (4.1)

By (1), viewing (4.1) as a relation holding on additive subgroup, we conclude that

$$J(R) = J(R) \cap N(R) \quad \text{or} \quad J(R) = J(R) \cap Z(R).$$

This implies that

$$J(R) \subseteq N(R) \quad \text{or} \quad J(R) \subseteq Z(R).$$

Since $J(R) \neq N(R)$, by Lemma 4.1, $J(R) \subseteq Z(R)$.

Now let $x \in R$. If $x \notin Z(R)$, then $x \notin J(R) \cup Z(R)$, so there exists positive integers $n, m$ ($n \geq m$) of opposite parity such that $x^n - x^m \in N(R)$. Let $k \geq 1$ such that $(x^n - x^m)^k = 0$. Then $((x^n - x^m)^k)^m = 0$, this gives $x - x^{n-m+1} \in N(R) \subseteq J(R) \subseteq Z(R)$. By Herstein’s theorem [6], $R$ is commutative.

Received 15.10.12

ISSN 1027-3190. Укр. мат. журн., 2014, т. 66, № 12