К О Р О Т К І П О В І Д О М Л Е Н Н Я

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C2 PROPERTY OF COLUMN FINITE MATRIX RINGS*
C2 ВЛАСТИВІСТЬ СТОВПЧИКОВИХ СКІНЧЕННИХ МАТРИЧНИХ КІЛЕЦЬ

A ring R is called a right C2 ring if any right ideal of R isomorphic to a direct summand of R_R is also a direct summand. The ring R is called a right C3 ring if any sum of two independent summands of R_R is also a direct summand. It is well known that a right C2 ring must be a right C3 ring but the converse assertion is not true. The ring R is called J-regular if $R/J(R)$ is von Neumann regular, where $J(R)$ is the Jacobson radical of R. Let $\mathbb{N}$ be the set of natural numbers and Λ be any infinite set. The following assertions are proved to be equivalent for a ring R : (1) $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C2 ring; (2) $\mathbb{CFM}_{\Lambda}(R)$ is a right C2 ring; (3) $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C3 ring; (4) $\mathbb{CFM}_{\Lambda}(R)$ is a right C3 ring; (5) $\mathbb{CFM}_{\mathbb{N}}(R)$ is a J-regular ring and $\mathbb{M}_n(R)$ is a right C2 (or right C3) ring for all integers $n \geq 1$.

Кільце R називається правим C2 кільцем, якщо будь-який правий ідеал R, що є ізоморфним до прямого доданка в R_R, також є прямим доданком. Кільце R називається правим C3 кільцем, якщо будь-яка сума двох незалежних доданків в R_R також є прямим доданком. Відомо, що праве C2 кільце має бути правим C3 кільцем, але протилежне твердження є невірним. Кільце R називається J-регулярним, якщо $R/J(R)$ є регулярним у сенсі фон Ноймана, де $J(R)$ — радикал Якобсона для R. Нехай $\mathbb{N}$ — множина натуральних чисел, а Λ — деяка нескінченна множина. Доведено, що наступні твердження є еквівалентними для кільця R: (1) $\mathbb{CFM}_{\mathbb{N}}(R)$ — праве C2 кільце; (2) $\mathbb{CFM}_{\Lambda}(R)$ — праве C2 кільце; (3) $\mathbb{CFM}_{\mathbb{N}}(R)$ — праве C3 кільце; (4) $\mathbb{CFM}_{\Lambda}(R)$ — праве C3 кільце; (5) $\mathbb{CFM}_{\mathbb{N}}(R)$ — J-регулярне кільце, а $\mathbb{M}_n(R)$ — праве C2 (або праве C3) кільце для всіх цілих $n \geq 1$.

1. Introduction. Throughout this paper, rings are associative with identity and modules are unitary modules. We denote by $\mathbb{N}$ the set of natural numbers. For a ring R, $\mathbb{M}_n(R)$ denotes the ring of all $(n \times n)$-matrices over R and $J(R)$ means the Jacobson radical of R. Let Λ be an infinite set. $\mathbb{CFM}_{\Lambda}(R)$ means the column finite $\mathrm{card}(\Lambda) \times \mathrm{card}(\Lambda)$ matrix ring over a ring R, where $\mathrm{card}(\Lambda)$ is the cardinality of Λ. For a module M, $M^{(A)}$ is the direct sum of copies of M indexed by a set A. We use $N \leq_{\oplus} M$ to show that N is a direct summand of M. And use End(M) to denote the ring of endomorphisms of M.

The following are three well-known generalizations of the injective condition of a module M.

C_1) Every submodule of M is essential in a direct summand of M.

C_2) Every submodule that is isomorphic to a direct summand of M is itself a direct summand of M.

C_3) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B \leq_{\oplus} M$. M is called a Ci module if it satisfies condition Ci, $i = 1, 2, 3$. C1 modules are also called CS (or extending) modules. A C2 module is always a C3 module and the converse is not true. A ring R is called a *right Ci* ring if the right R-module R_R is a Ci module, $i = 1, 2, 3$. Much more information about these conditions can be referred to [5].

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Let R be a ring and Λ be an infinite set whose cardinality is not $\aleph_0$. It can be proved that $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C1 ring may not inform that $\mathbb{CFM}_{\Lambda}(R)$ is a right C1 ring (see [4], Example). In this short article, we concentrate on the C2 property of column finite matrix rings. Some interesting results are obtained. It is proved in Theorem 2.3 that, for any infinite set Λ, $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C2 ring if and only if $\mathbb{CFM}_{\Lambda}(R)$ is a right C2 ring if and only if $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C3 ring if and only if $\mathbb{CFM}_{\Lambda}(R)$ is a right C3 ring.

2. Results. First we look at some basic results on column finite matrix rings. Let R be a ring and Λ be an infinite set. We consider the right R-module $R_R^{(\Lambda)}$ as the set of all $\mathrm{card}(\Lambda)\times 1$ column matrices with finite nonzero entries in R. We have the following results.

Proposition 2.1. *Let R be a ring and Λ be an infinite set. Then every right ideal I of $\mathbb{CFM}_{\Lambda}(R)$ has the form $I=\big\{[\alpha\ \beta\ \gamma\ \ldots]|\alpha,\beta,\gamma,\ldots\in T\big\}$, where T is a submodule of $R_R^{(\Lambda)}$. In particular, I is an essential right ideal of $\mathbb{CFM}_{\Lambda}(R)$ if and only if T is an essential submodule of $R_R^{(\Lambda)}$, and I is a direct summand of $\mathbb{CFM}_{\Lambda}(R)_{\mathbb{CFM}_{\Lambda}(R)}$ if and only if T is a direct summand of $R_R^{(\Lambda)}$.*

Proof. Set $A=\big\{[\alpha\ \beta\ \gamma\ \ldots]|\alpha,\beta,\gamma,\ldots\in T\big\}$, where T is a submodule of $R_R^{(\Lambda)}$. It is easy to verify that A is a right ideal of $\mathbb{CFM}_{\Lambda}(R)$. Now let T be the set of columns those appear in all the matrices of I. It is clear that T is a submodule of $R_R^{(\Lambda)}$ and $I=\big\{[\alpha\ \beta\ \gamma\ \ldots]|\alpha,\beta,\gamma,\ldots\in T\big\}$.

Proposition 2.2. *Let R be a ring and Λ be an infinite set. Assume $e^2=e\in R$. Set $M=eR$ and $S=eRe$. Then* End$(M_R^{(\Lambda)})\cong\mathbb{CFM}_{\Lambda}(S)$.

Proof. We prove the case $\Lambda=\mathbb{N}$. The others are similar. To be convenient, we consider $M_R^{(\mathbb{N})}$ as the set of all column $(\mathbb{N}\times 1)$-matrices with finite nonzero entries in M. Then for any $\alpha\in M_R^{(\mathbb{N})}$ and $A\in\mathbb{CFM}_{\mathbb{N}}(S)$, $A\alpha\in M_R^{(\mathbb{N})}$. Now define a map F from $\mathbb{CFM}_{\mathbb{N}}(S)$ to End$\Big(M_R^{(\mathbb{N})}\Big)$ such that for every $A\in\mathbb{CFM}_{\mathbb{N}}(S)$ and any $\alpha\in M_R^{(\mathbb{N})}$, $F(A)(\alpha)=A\alpha$. It is clear that F is a ring homomorphism from $\mathbb{CFM}_{\mathbb{N}}(S)$ to End$(M_R^{(\mathbb{N})})$. Next we show that F is an isomorphism. It is easy to see that F is a monomorphism. We only need to show that F is epic. Let ε_i be the element in $M_R^{(\mathbb{N})}$ with the ith entry equal to e and the others are zero, $\forall i\in\mathbb{N}$. Assume $\varphi\in$ End $\Big(M_R^{(\mathbb{N})}\Big)$. Let $B=\big[\varphi(\varepsilon_1),\varphi(\varepsilon_2),\ldots,\varphi(\varepsilon_n),\ldots\big]$ and $E=\big[\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,\ldots\big]$ be the matrices with the ith column equal to $\varphi(\varepsilon_i)$ and ε_i, respectively, $i\in\mathbb{N}$. It is clear that $E^2=E$ and $BE\in\mathbb{CFM}_{\mathbb{N}}(S)$. For each $X\in M_R^{(\mathbb{N})}$, there exists finite nonzero elements $r_i\in eR, i\in\mathbb{N}$, such that $X=\sum_{i=1}^{\infty}\varepsilon_i r_i$. Let C be the column $(\mathbb{N}\times 1)$-matrix with the ith entry equal to $r_i,\ i\in\mathbb{N}$. Then $X=EC$. Thus $\varphi(X)=\varphi\Big(\sum_{i=1}^{\infty}\varepsilon_i r_i\Big)=\sum_{i=1}^{\infty}\varphi(\varepsilon_i)r_i=\sum_{i=1}^{\infty}\varphi(\varepsilon_i)er_i=BEC=BEEC=BEX$. Set $A=BE$. It is clear that $\varphi=F(A)$. Therefore, F is an epimorphism.

Lemma 2.1 ([6], Theorem 7.14). *Let M_R be a module and write $E=\mathrm{End}(M_R)$. Then*

(1) *If E is a right* C2 *ring, then M_R is a* C2 *module.*

(2) *The converse in* (1) *holds if* Ker(α) *is generated by M whenever $\alpha\in E$ is such that $\mathbf{r}_E(\alpha)$ is a direct summand of E_E.*

Theorem 2.1. *Let R be a ring and Λ be an infinite set. Then*

(1) $\mathbb{CFM}_{\Lambda}(R)$ *is right* C1 *if and only if $R_R^{(\Lambda)}$ is a* C1 *module.*

(2) $\mathbb{CFM}_{\Lambda}(R)$ *is right* C2 *if and only if $R_R^{(\Lambda)}$ is a* C2 *module.*

(3) $\mathbb{CFM}_{\Lambda}(R)$ *is right* C3 *if and only if $R_R^{(\Lambda)}$ is a* C3 *module.*

Proof. (1) and (3) are directly obtained by Proposition 2.1.

(2) By Proposition 2.2, End$(R_R^{(\Lambda)}) \cong \mathbb{CFM}_\Lambda(R)$. Since $R_R^{(\Lambda)}$ is a generator of right R-modules, according to the above lemma, $\mathbb{CFM}_\Lambda(R)$ is a right C2 ring if and only if $R_R^{(\Lambda)}$ is a C2 module.

Applying a similar proof, we have the following theorem.

Theorem 2.2. *Let R be a ring and n be a positive integer. Consider R_R^n as direct sum of n copies of R_R. Then*

(1) $\mathbb{M}_n(R)$ *is right* C1 *if and only if R_R^n is a* C1 *module.*

(2) $\mathbb{M}_n(R)$ *is right* C2 *if and only if R_R^n is a* C2 *module.*

(3) $\mathbb{M}_n(R)$ *is right* C3 *if and only if R_R^n is a* C3 *module.*

Recall that a ring R is called *right* (*countably*) Σ-*CS* if every (countable) direct sum of copies of R_R is CS. And a right countably Σ-CS ring may not be right Σ-CS. In fact, a von Neumann regular right self-injective ring is right countably Σ-CS but not right Σ-CS unless it is semisimple (see [4], Example). Thus, by Theorem 2.1, $\mathbb{CFM}_\mathbb{N}(R)$ is a right C1 ring may not imply that $\mathbb{CFM}_\Lambda(R)$ is a right C1 ring for every infinite set Λ. But if C1 is replaced by C2 or C3, the results will be different and interesting. Before giving our main results, we need some lemmas.

The next result was firstly obtained by Yiqiang Zhou. To be self-contained, we write down the proof.

Lemma 2.2 (Zhou's lemma)**.** *Let R be a ring and M be a right R-module. If the direct sum $M \oplus M$ is a* C3 *module*, *then M is a* C2 *module.*

Proof. Assume K is a submodule of M that is isomorphic to a direct summand L of M. We want to show that K is also a direct summand of M. Let f be the isomorphism from K to L. Set $K' = \{(x, f(x)) \colon x \in K\}$, $L' = 0 \oplus L$ and $M' = M \oplus 0$. Then $K' + M' = M \oplus L$ is a direct summand of $M \oplus M$. Since $K' \cap M' = 0$, K' is also a direct summand of $M \oplus M$. It is clear that $K' \cap L' = 0$ and L' is a direct summand of $M \oplus M$. Because $M \oplus M$ is a C3 module, $K' + L' = K \oplus L$ is a direct summand of $M \oplus M$. As $K \oplus 0$ is a direct summand of $K \oplus L$, $K \oplus 0$ is also a direct summand of $M \oplus M$. This shows that $K \oplus 0$ is a direct summand of $M \oplus 0$. It is clear that K is a direct summand of M.

We define a ring R to be *J-regular* if $R/J(R)$ is a von Neumann regular ring.

Lemma 2.3. *A ring R is right perfect if and only if $\mathbb{CFM}_\mathbb{N}(R)$ is a J-regular ring.*

Proof. See [3], Theorem 1.

Lemma 2.4 ([1], Lemma 19.18)**.** *Let R be a ring and V be a flat right R-module and suppose that the sequence*

$$0 \to K \to V \to V^{'} \to 0$$

is exact. Then $V^{'}$ is flat if and only if for each (*finitely generated*) *left ideal $I \subseteq_R R$, $KI = K \cap VI$.*

Theorem 2.3. *The following are equivalent for a ring R.*

(1) $\mathbb{CFM}_\mathbb{N}(R)$ *is a right* C2 *ring.*

(2) $\mathbb{CFM}_\mathbb{N}(R)$ *is a right* C3 *ring.*

(3) *For any infinite set* Λ, $\mathbb{CFM}_\Lambda(R)$ *is a right* C2 *ring.*

(4) *For any infinite set* Λ, $\mathbb{CFM}_\Lambda(R)$ *is a right* C3 *ring.*

(5) $\mathbb{CFM}_\mathbb{N}(R)$ *is a J-regular ring and* $\mathbb{M}_n(R)$ *is right* C2 *for all integer* $n \geq 1$.

(6) $\mathbb{CFM}_\mathbb{N}(R)$ *is a J-regular ring and* $\mathbb{M}_n(R)$ *is right* C3 *for all integer* $n \geq 1$.

Proof. Let Λ be an infinite set. It is clear that $R_R^{(\Lambda)} \cong (R_R^{(\Lambda)} \oplus R_R^{(\Lambda)})$. Then by Theorem 2.1, Theorem 2.2 and Lemma 2.2, we have (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (6). Next we only need to prove (1) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5). If R satisfies (1), by Theorem 2.1, $R_R^{(\mathbb{N})}$ is a C2 module. For any integer $n \geq 1$, R_R^n can be looked on as a direct summand of $R_R^{(\mathbb{N})}$. Since a direct summand of a C2 module is always a C2 module, we have that R_R^n is a C2 module. Then by Theorem 2.2, $\mathbb{M}_n(R)$ is right C2 for all integer $n \geq 1$. Now we prove that $\mathbb{CFM}_{\mathbb{N}}(R)$ is a J-regular ring. According to Lemma 2.3, we need to show that R is a right perfect ring. By [1] (Theorem 28.4), we will prove that R satisfies DCC on principal left ideals of R. The following method is owing to Bass [2]. Let $Ra_1 \supseteq Ra_2a_1 \supseteq \ldots$ be any descending chain of principal left ideals of R. Set $F = R_R^{(\mathbb{N})}$ with free basis $x_1, x_2, \ldots$ and G be the submodule of F spanned by $y_i = x_i - x_{i+1}a_i, i \in \mathbb{N}$. By [1] (Lemma 28.1), G is free with basis $y_1, y_2, \ldots$. Thus $G \cong F$. F is a C2 module implies that G is a direct summand of F. Then by [1] (Lemma 28.2), the chain $Ra_1 \supseteq Ra_2a_1 \supseteq \ldots$ terminates.

(5) $\Rightarrow$ (3). By Theorem 2.1, We only need to show that $R_R^{(\Lambda)}$ is a C2 module. Assume K is a submodule of the free module $F = R_R^{(\Lambda)}$ and K is isomorphic to a direct summand of F. In order to show that K is also a direct summand of F, we only need to prove that F/K is a projective R-module. Since $\mathbb{CFM}_{\mathbb{N}}(R)$ is a J-regular ring, by Lemma 2.3, R is a right perfect ring. According to [1] (Theorem 28.4), every flat right R-module is projective. Thus, we just need to show that F/K is flat. As R is right perfect, R is semiperfect. Then R has a basic set of primitive idempotents $e_1, \ldots, e_m$. Since K is projective, by [1] (Theorem 27.11), there exist sets $A_1, \ldots, A_m$ such that $K \cong (e_1R)^{(A_1)} \oplus \ldots \oplus (e_mR)^{(A_m)}$. Set $\lambda = \mathrm{card}(\Lambda)$. Since K is isomorphic to a direct summand of F, K is λ-generated. So each $(e_iR)^{(A_i)}$ is also λ-generated, $i = 1, 2, \ldots, m$. As λ is an infinite cardinality, by [1] (Lemma 25.7), $\mathrm{card}(A_i) \leq \lambda$, $i = 1, 2, \ldots, m$. So $\mathrm{card}(A_1) + \ldots + \mathrm{card}(A_m) \leq$ $\leq m\lambda = \lambda$. Set $L = (e_1R)^{(A_1)} \oplus \ldots \oplus (e_mR)^{(A_m)}$. Then L can be considered as a direct summand of F. Let $\mathfrak{A} = \{L_\alpha \leq_\oplus L \colon L_\alpha \cong (e_1R)^{(A_{\alpha_1})} \oplus \ldots \oplus (e_mR)^{(A_{\alpha_m})}$ with $\mathrm{card}(A_{\alpha_1}) + \ldots + \mathrm{card}(A_{\alpha_m})$ is finite $\}$. It is clear that $L = \bigcup_{L_\alpha \in \mathfrak{A}} L_\alpha$ and, for any left ideal I of R, $LI = \bigcup_{L_\alpha \in \mathfrak{A}} L_\alpha I$. Now let f be the isomorphism from K to L. Set $\mathfrak{B} = \{K_\alpha = f^{-1}(L_\alpha) \colon L_\alpha \in \mathfrak{A}\}$. Since K is isomorphic to L, $K = \bigcup_{K_\alpha \in \mathfrak{B}} K_\alpha$ and, for any left ideal I of R, $KI = \bigcup_{K_\alpha \in \mathfrak{B}} K_\alpha I$. By Theorem 2.2, R_R^n is a C2 module for all integers $n \geq 1$. As L_α is a finitely generated direct summand of L for each $L_\alpha \in \mathfrak{A}$, it is easy to verify that K_α is a direct summand of F for each $K_\alpha \in \mathfrak{B}$. At last we apply Lemma 2.4 to show that F/K is a flat module. Let I be any left ideal of R, by Lemma 2.4, $K_\alpha \cap FI = K_\alpha I$, $K_\alpha \in \mathfrak{B}$. Then $K \cap FI = (\bigcup_{K_\alpha \in \mathfrak{B}} K_\alpha) \cap FI = \bigcup_{K_\alpha \in \mathfrak{B}} (K_\alpha \cap FI) = \bigcup_{K_\alpha \in \mathfrak{B}} K_\alpha I = KI$. Thus, by Lemma 2.4, F/K is flat.

(3) $\Rightarrow$ (1). If R satisfies (3), by Theorem 2.1, $R_R^{(\Lambda)}$ is a C2 module. Since Λ is an infinite set, $R_R^{(\mathbb{N})}$ can be looked on as a direct summand of $R_R^{(\Lambda)}$. As a direct summand of C2 module is always C2, we have $R_R^{(\mathbb{N})}$ is a C2 module. Applying Theorem 2.1 again, $\mathbb{CFM}_{\mathbb{N}}(R)$ is a right C2 ring.

Based on Theorem 2.1, Theorem 2.2, Lemma 2.3 and Theorem 2.3, we have the following corollary.

***Corollary* 2.1.** *The following are equivalent for a ring R.*

(1) $R_R^{(\mathbb{N})}$ *is a* C2 *module.*

(2) $R_R^{(\mathbb{N})}$ *is a* C3 *module.*

(3) *For any infinite set* Λ, $R_R^{(\Lambda)}$ *is a* C2 *module.*

(4) *For any infinite set* Λ, $R_R^{(\Lambda)}$ *is a* C3 *module.*

(5) *R is a right perfect ring and every finite direct sum of copies of R_R is a* C2 *module.*

(6) *R is a right perfect ring and every finite direct sum of copies of R_R is a* C3 *module.*

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