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GLOBAL EXISTENCE AND LONG-TIME BEHAVIOR OF A NONLINEAR EQUATION OF THE SCHRÖDINGER TYPE*

ГЛОБАЛЬНЕ ІСНУВАННЯ ТА ДОВГОЧАСОВА ПОВЕДІНКА НЕЛІНІЙНОГО РІВНЯННЯ ТИПУ ШРЬОДІНГЕРА

We study a global mixed problem for the nonlinear Schrödinger equation with nonlinear addition in which the coefficient is a generalized function. We prove a global solvability theorem for the considered problem with the use of the general solvability theorem from [*Soltanov K. N.* Perturbation of the mapping and solvability theorems in the Banach space // Nonlinear Anal.: Theory, Meth. and Appl. – 2010. – **72**, № 1]. Furthermore, we also investigate the behavior of the solution of the analyzed problem.

Вивчається глобальна мішана задача для нелінійного рівняння Шрьодінгера з нелінійним додаванням, в якій коефіцієнтом є узагальнена функція. Доведено теорему про глобальну розв'язність поставленої задачі на основі загальної теореми про розв'язність з [*Soltanov K. N.* Perturbation of the mapping and solvability theorems in the Banach space // Nonlinear Anal.: Theory, Meth. and Appl. – 2010. – **72**, № 1]. Крім того, досліджено поведінку розв'язку задачі, що вивчається.

We consider the following problem for the nonhomogeneous nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} - \Delta u + f(x,u) = h(t,x), \quad (t,x) \in R_+ \times \Omega \equiv Q, \tag{0.1}$$

$$u(0,x) = u_0(x), \quad x \in \Omega \subset R^n, \quad n \geq 1, \quad u\big|_{R_+\times\partial\Omega} = 0, \tag{0.2}$$

where $h(t,x)$ and $u_0(x)$ are complex functions, $f(x,\tau)$ is a distribution (generalized function) with respect to variable $x \in \Omega$, Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $i \equiv \sqrt{-1}$. We investigate this problem in the case, when the function $f(x,t)$ can be represented as $f(x,u) = q(x)\,|u(t,x)|^{p-2}\,u(t,x) + a(x)\,|u(t,x)|^{\widetilde{p}-2}\,u(t,x)$, i.e., the function f has the growth with respect to unknown function of the polynomial type, where $a\colon \Omega \longrightarrow R$ is some function and $q\colon \Omega \longrightarrow R$ is a generalized function, $p \geq 2$, $\widetilde{p} \geq 2$, $h \in L_2(Q)$ (i.e., $h(t,x) \equiv h_1(t,x) + ih_2(t,x)$ and $h_j \in L_2(Q)$, $j = 1,2$).

The nonlinear Schrödinger equation of the type (0.1), and also steady-state case of the equation (0.1) arises in several models of different physical phenomena corresponding to various function f. The equation of such type were studied in many articles under different conditions on the function f in the dynamic case (see, for example, [2, 4, 6–9, 14, 17, 18, 20, 22, 24, 32, 33, 35] and the references therein) and in the steady-state case (see, for example, [1, 3, 5, 10–16, 18, 19, 23–26, 28, 33, 34, 36] and references therein). It is known that in this case the equation (0.1) in the steady-state case (i.e., if u is independent of t) is an equation of the semiclassical nonlinear Schrödinger type (i.e., NLS) (see [1, 2, 3, 10, 13] and references therein). Considerable attention has been paid in recent years to the problem (0.1) for small $\varepsilon > 0$ as the coefficients of the linear part since the solutions are known as in the semiclassical states, which can be used to describe the transition from quantum to classical mechanics (see [3, 10–14, 23–25, 33–36] and references therein).

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In the above mentioned articles the equation (0.1), and also the steady-state case was considered with various functions $f(x,u)$ that are mainly Carathéodory functions[1] with some additional properties. Moreover, in some of these articles are presumed, that dates of considered problem possess more smoothness and study the behavior of a solution of the posed problem with use the Fourier mod. Although such cases when $f(x,u)$ possesses a singularity with respect to the variable x of certain type were also investigated (as equations Emden – Fowler, Yamabe, NLS etc.), but in all of these articles the coefficient $q(x)$ is a function in the usual sense (of a Lebesgue space or of a Sobolev space).

In this paper we study problem (0.1), (0.2) in the case when f have the above representation and the function q is a generalized function and the behavior of the solution. Moreover here for the proof of the existence theorem of the problem is used some different method, which allow us several other possibility. It should be noted that the steady-state case of the problem of such type were studied in [28]. Here we study problem (0.1), (02) globally in the dynamical case, which in [32] is studied for $t \in (0,T)$, $T < \infty$. Here an existence theorem (Section 1) for the problem (0.1), (0.2) is proved in the model case when $f(x,u)$ only has the above expression (Section 4).

In Section 2 we have defined how to understand the equation (0.1) with use of representation of certain generalized functions and properties of some special class of functions (see, for example, [27, 28]). In Section 3 we have conducted variants of the general results from [29, 31], on which the proof of the solvability theorem is based and in Section 5 is studied the behavior of the solutions of the considered problem under certain additions conditions.

1. Statement of the main solvability result. Let the operator $f\left(x,u\right)$ have the form

$$f\left(x,u\right) = q(x)|u|^{p-2}u + a\left(x\right)\left|u(t,x)\right|^{\widetilde{p}-2}u\left(t,x\right) \tag{1.1}$$

in the generalized sense, where $q \in W^{-1,p_0}(\Omega)$, $p_0 \geq 2$ (it should be noted that either $p_0 \equiv p_0\left(p\right)$ or $p \equiv p\left(p_0\right)$), $a\colon \Omega \longrightarrow R$ and $u\colon Q \longrightarrow \mathbb{C}$ is an element of the space of sufficiently smooth functions that will be determined below (see Section 2). Consequently the function $q(x)$ is a generalized function, which has singularity of the order 1.

We will set some necessary denotations. Everywhere later the expression of the type $u \in \in L^m\left(R_+; W_0^{1,2}(\Omega)\right) \cap L^2\left(R_+; W_0^{1,2}(\Omega)\right) \equiv L^{(2,m)}\left(R_+; W_0^{1,2}(\Omega)\right)$ for $u\colon Q \longrightarrow \mathbb{C}$ denote the following:

$$(u_1,u_2) \in \left(L^{m_1}\left(R_+; W_0^{1,2}(\Omega)\right)\right)^2 \equiv \left(L^{m_1}\left(R_+; W_0^{1,2}(\Omega)\right),\ L^{m_1}\left(R_+; W_0^{1,2}(\Omega)\right)\right)$$

holds for $m_1 \in \left[2,m\right]$, where $u(t,x) \equiv u_1(t,x) + iu_2(t,x)$, $m \geq 2^*$ consequently we can set $u(t,x) \equiv (u_1(t,x), u_2(t,x))$, i.e., $u\colon Q \longrightarrow R^2$.

Everywhere later $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ denote the dual form for the pair (X,X^*) of the Banach space X and its dual space X^*, for example, in the case when $X \equiv W_0^{1,2}(\Omega)$ and $X \equiv L^{m_1}\left(R_+; W_0^{1,2}(\Omega)\right)$ we have

$$(X,X^*) \equiv \left(\left(W_0^{1,2}(\Omega)\right)^2, \left(W_0^{1,2}(\Omega)\right)^2\right)$$

[1]Let $f\colon \Omega \times R^m \longrightarrow R$ be a given function, where Ω is a nonempty measurable set in R^n and $n,m \geq 1$. Then f is Carathéodory function if the following hold: $x \longrightarrow f(x,\eta)$ is measurable on Ω for all $\eta \in R^m$, and $\eta \longrightarrow f(x,\eta)$ is continuous on R^m for almost all $x \in \Omega$.

and

$$(X, X^*) \equiv \left(\left(L^{m_1} \left(R_+; W_0^{1,2}(\Omega) \right) \right)^2, \left(L^{m_1'} \left(R_+; W^{-1,2}(\Omega) \right) \right)^2 \right),$$

respectively, where $m_1' = \dfrac{m_1}{m_1 - 1}$. In the other words we will understand these expressions everywhere later as the following representations:

$$\langle g, w \rangle \equiv \int\limits_\Omega g(x)\overline{w}(x)dx, \qquad g_j \in W_0^{1,2}(\Omega), \quad w_j \in W^{-1,2}(\Omega), \quad g \equiv g_1 + ig_2,$$

and

$$[g, w] \equiv \int\limits_0^\infty \int\limits_\Omega g(t,x)\overline{w}(t,x)dxdt, \qquad g_j \in L^{m_1} \left(R_+; W_0^{1,2}(\Omega) \right),$$

$$w_j \in L^{m_1'} \left(R_+; W^{-1,2}(\Omega) \right)$$

respectively.

Assume the following conditions:

(i) let $\widetilde{p} < \dfrac{n+2}{n-2}$ if $n \geq 3$, $\widetilde{p} \in [2, \infty)$ if $n = 1, 2$ and $a \in L^\infty(\Omega)$;

(ii) there exist numbers $k_0 \geq 0$, $p_2 \geq 1$ and $k_1 \leq \min\left\{1; \dfrac{\widetilde{p}}{p}\right\}$ such that $1 \leq p_2 < \dfrac{2n}{n-2}$, if $n \geq 3$, $2 \leq p_2 < \infty$, if $n = 1, 2$ and

$$\left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle \geq -k_0 \left\| u \right\|_{p_2}^2 - k_1 \left\langle q(x)|u|^{p-2}u, \overline{u} \right\rangle \tag{1.2}$$

holds for any $u \in L^{(2,m)}\left(R_+; W_0^{1,2}(\Omega)\right)$ and a.e. $t \geq 0$, where $1 > C(2, p_2)^2 \cdot k_0$. Here $C(2; p_2)$ is the constant of the known inequality of embedding theorems for Sobolev spaces

$$\left\| \nabla u \right\|_2 \geq C(2; p_2) \left\| u \right\|_{p_2} \quad \forall u \in W_0^{1,2}(\Omega).$$

Definition 1.1. *A function*

$$u \in L^m \left(R_+; W_0^{1,2}(\Omega) \right) \cap L^2 \left(R_+; W_0^{1,2}(\Omega) \right) \cap \left\{ u \,\middle|\, \frac{\partial u}{\partial t} \in L^2 \left(R_+; L^2(\Omega) \right); \; u(0,x) = u_0 \right\}$$

is called a solution of the problem (0.1), (0.2) *if the following equation is fulfilled*:

$$\int\limits_0^\infty \int\limits_\Omega \left[i\frac{\partial u}{\partial t} - \Delta u + f(x,u) \right] \overline{\varphi}\, dxdt = \int\limits_0^\infty \int\limits_\Omega h\overline{\varphi}\, dxdt \tag{1.3}$$

for any $\varphi \in L^{(2,m)}\left(R_+; W_0^{1,2}(\Omega)\right)$.

It should be noted that the sense in which equation (1.3) is to be understood will be explained below (Section 2). We have proved the following result for the considered problem.

Theorem 1.1. *Let the function f have the representation* (1.1) *in the generalized sense, where $q \in W_{p_0}^{-1}(\Omega)$ is a nonnegative distribution* (*generalized function*[2]), $p_0 = \dfrac{2n}{2(n-1) - p(n-2)}$, $\dfrac{2(n-1)}{n-2} > p > 2$ *if $n \geq 3$, $p_0, p > 2$ are arbitrary if $n = 2$, and $p_0, p \geq 2$ are arbitrary if $n = 1$ $\left(\text{in particular, if } n = 3, \text{ then } 2 < p < 4 \text{ and } p_0 = \dfrac{6}{4-p}\right)$ and conditions* (i), (ii) *are fulfilled. Then for any $h \in L^2(Q)$ and $u_0 \in W_0^{1,2}(\Omega)$ the problem* (0.1), (0.2) *is solvable in* $L^{(2,m)}\Big(R_+; W_0^{1,2}(\Omega)\Big) \cap \left\{u \;\middle|\; \dfrac{\partial u}{\partial t} \in L^2\left(R_+; L^2(\Omega)\right);\ u(0,x) = u_0\right\}$.

For the investigation of the considered problem we used some general solvability theorems, which are conducted in Section 3. We begin with explanation of equation (1.3).

2. The solution concept and function spaces. So we will consider the case when the function $f(x,u)$ has the form (1.1) where functions a, q and u are the same as above. Consequently the function $q(x)$ is a generalized function, which has singularity of order 1. Therefore we must understand the equation (0.1) in the generalized function space sense, i.e.,

$$\int\limits_{\Omega} \left[i\frac{\partial u}{\partial t} - \Delta u + f(x,u)\right] \overline{\varphi}(x)dx \equiv$$

$$\equiv \int\limits_{\Omega} \left[i\frac{\partial u}{\partial t} - \Delta u(t,x) + q(x)\left|u(t,x)\right|^{p-2} u(t,x)\right] \overline{\varphi}(x)dx -$$

$$- \int\limits_{\Omega} a(x)\left|u(t,x)\right|^{\widetilde{p}-2} u(t,x)\, \overline{\varphi}(x)dx = \int\limits_{\Omega} h(t,x)\overline{\varphi}(x)dx \qquad (2.1)$$

for any $\varphi \equiv \varphi_1 + i\varphi_2$, $\varphi_j \in D(\Omega)$, $j = 1,2$, where $D(\Omega)$ is $C_0^\infty(\Omega)$ and $\operatorname{supp}\varphi_j \subset \Omega$ with corresponding topology. Here the equation (2.1) will be understood in the sense of the space $L_2(R_+)$.

In the beginning we need to define the expression $q|u|^{p-2}u$. It is known that (see, for example, [21]) in the case when $q \in W_{p_0}^{-1}(\Omega)$ we can represent it in the form $q(x) \equiv \sum_{k=0}^{n} D_k q_k(x)$, $D_k \equiv \dfrac{\partial}{\partial x_k}$, $D_0 \equiv I$, $q_k \in L_{p_0}(\Omega)$, $k = 0, \overline{1,n}$, in the generalized function space sense. From here it follows that if a solution of the considered problem belongs to the space which contains to $W_0^{1,\widetilde{p}_1}(\Omega)$ for some number $\widetilde{p}_1 > 1$ then we can understand the term $q|u|^{p-2}u$ in the following sense:

$$\left\langle q\,|u|^{p-2}u, \varphi\right\rangle \equiv \int\limits_{\Omega} q(x)|u(t,x)|^{p-2}u(t,x)\overline{\varphi}(x)dx \qquad (2.2)$$

for any $\varphi \equiv \varphi_1 + i\varphi_2$, $\varphi_j \in D(\Omega)$, $j = 1,2$, and a.e. $t > 0$. Therefore we must find the needed number $\widetilde{p}_1 \geq 2$. Namely we must find the relation between the numbers p_0 and $\widetilde{p}_1$. So, taking into account that for a function $u \in L^m\Big(R_+; W_0^{1,2}(\Omega)\Big)$, i.e., $\widetilde{p}_1 = 2$ (as $h \in L^2(Q)$ by the assumption) we have $u \in L^m\big(R_+; L^{\widetilde{p}_1^*}(\Omega)\big)$, where $\widetilde{p}_1^* = 2^* = \dfrac{2n}{n-2}$ for $n \geq 3$ by virtue of the embedding theorem, from (2.2) we get

[2] See Definition 2.1 of the Section 2.

$$\left\langle q\,|u|^{p-2}u,\varphi\right\rangle \equiv \int\limits_{\Omega} q(x)\,|u(t,x)|^{p-2}\,u(t,x)\overline{\varphi}(x)dx =$$

$$= \int\limits_{\Omega}\sum_{k=0}^{n}\frac{\partial}{\partial x_k}q_k(x)\,|u(t,x)|^{p-2}\,u(t,x)\overline{\varphi}(x)dx =$$

$$= -\int\limits_{\Omega}\sum_{k=1}^{n}q_k|u|^{p-2}u\frac{\partial\overline{\varphi}}{\partial x_k}dx - \int\limits_{\Omega}\sum_{k=1}^{n}q_k\frac{\partial}{\partial x_k}\left(|u|^{p-2}u\right)\overline{\varphi}dx + \int\limits_{\Omega} q_0\,|u|^{p-2}\,u\varphi dx =$$

$$= I_1 + I_2 + \int\limits_{\Omega} q_0|u|^{p-2}u\overline{\varphi}dx \tag{2.3}$$

by virtue of the generalized function theory.

***Remark* 2.1.** It should be noted that I_2 is estimated by following way. Firstly we note that

$$\frac{\partial}{\partial x_k}\left(|u|^{p-2}u\right) = \frac{\partial}{\partial x_k}\left(u,\overline{u}\right)^{\frac{p-2}{2}}u = (p-2)\left(u,\overline{u}\right)^{\frac{p-4}{2}}\left(\frac{\partial u}{\partial x_k},\overline{u}\right)u + \left(u,\overline{u}\right)^{\frac{p-2}{2}}\frac{\partial u}{\partial x_k} =$$

$$= (p-2)\left(u,\overline{u}\right)^{\frac{p-4}{2}}\left(\frac{\partial u}{\partial x_k},\overline{u}\right)u + |u|^{p-2}\frac{\partial u}{\partial x_k} =$$

$$= (p-2)\,|u|^{p-4}\left(\frac{\partial u}{\partial x_k},\overline{u}\right)u + |u|^{p-2}\frac{\partial u}{\partial x_k},$$

here as known $\left(u(t,x),\overline{u(t,x)}\right) = |u(t,x)|^2$ for $u(t,x)\in C$. Consequently we have

$$I_2 \equiv \int\limits_{\Omega}\sum_{k=1}^{n}q_k\frac{\partial}{\partial x_k}\left(|u|^{p-2}u\right)\overline{\varphi}\,dx =$$

$$= \int\limits_{\Omega}\sum_{k=1}^{n}q_k|u|^{p-2}\frac{\partial u}{\partial x_k}\overline{\varphi}dx + \int\limits_{\Omega}(p-2)\sum_{k=1}^{n}q_k|u|^{p-4}\left(\frac{\partial u}{\partial x_k},\overline{u}\right)u\overline{\varphi}\,dx \le$$

$$\le \int\limits_{\Omega}\sum_{k=1}^{n}q_k|u|^{p-2}\left|\frac{\partial u}{\partial x_k}\right||\overline{\varphi}|\,dx + (p-2)\int\limits_{\Omega}\sum_{k=1}^{n}q_k|u|^{p-2}\left|\frac{\partial u}{\partial x_k}\right||\overline{\varphi}|\,dx =$$

$$= (p-1)\int\limits_{\Omega}\sum_{k=1}^{n}q_k|u|^{p-2}\left|\frac{\partial u}{\partial x_k}\right||\overline{\varphi}|\,dx.$$

Here and in what follows we assume $n\ge 3$. Because if $n=1,2$ then we can choose arbitrary $p\ge 2$, as will be observed below. Let us take into account that $\varphi_j\in D(\Omega)$ and $n\ge 3$, then in order for the expression in the left part of (2.3) to have the meaning, it is enough for us to take $1\le p-1\le \dfrac{2n\,(p_0-1)}{p_0(n-2)}$ for the integral I_1 and $0\le p-2\le \dfrac{n\,(p_0-2)}{p_0(n-2)}$ for the integral I_2. Therefore if $2\le p\le \dfrac{3np_0-2\,(n+2p_0)}{p_0(n-2)}$ then the left part of (2.3) is defined. Now, let $\varphi_j\in W_0^{1,2}(\Omega)$, $j=1,2$. Then it is sufficient to study one of the I_1 and I_2. Let us consider I_1, from which we obtain, that $2 \leq p \leq \dfrac{2np_0 - 2(n+p_0)}{p_0(n-2)}$, moreover we can choose $p \geq 2$ only if $p_0 > n$. On the other hand, if we take into account that given p, we obtain $p_0 = \dfrac{2n}{2(n-1) - p(n-2)}$, and consequently in order for $p_0 < \infty$ we must choose $2(n-1) > p(n-2)$ or $p < \dfrac{2(n-1)}{n-2}$. In the case when $n = 3$ then $p < 4$ and $p_0 = \dfrac{6}{4-p}$.

Thus we determined under what conditions the left part of (2.3) is defined. Hence that implies the correctness of the statement.

Proposition 2.1. *Assume $\widetilde{f}$ be an operator defined by expression $\widetilde{f}(u) \equiv q|u|^{p-2}u$, where $q \in W^{-1,p_0}(\Omega)$, and $u \in L^{(2,m)}\left(R_+; W_0^{1,2}(\Omega)\right)$. If $2 \leq p < \dfrac{2(n-1)}{n-2}$ and $p_0 = \dfrac{2n}{2(n-1) - p(n-2)}$ if $n \geq 3$ $\left(\text{in particular, if } n = 3, \text{ then } 2 \leq p < 4 \text{ and } p_0 = \dfrac{6}{4-p}\right)$, then $\widetilde{f}\colon L^{(2,m)}\left(R_+; W_0^{1,2}(\Omega)\right) \longrightarrow L^2\left(R_+; W^{-1,2}(\Omega)\right)$ is a bounded operator.*

So that exactly explains Proposition 2.1 and the representation (2.2) for any $\varphi_j \in W_0^{1,2}(\Omega)$ we consider the following class of the functions $u\colon \Omega \longrightarrow C$:

$$M_{\eta, W^{1,\beta}(\Omega)} \equiv \left\{ u \in L(\Omega) \;\middle|\; \eta(u) \in W^{1,\beta}(\Omega),\ \eta(u) \equiv |u|^{\alpha/\beta}\, u \right\} \equiv S_{1,\alpha,\beta}(\Omega), \tag{2.4}$$

where $\alpha \geq 0$, $\beta > 1$ are certain numbers, $W^{1,\beta}(\Omega)$ is a Sobolev space, i.e., we consider a class of the pn-spaces[3].

It is not difficult to see that if $1 \leq \alpha_0 + \beta_0 \leq \alpha_1 + \beta_1$, $0 \leq \beta_0 < \beta_1$, $\alpha_1\beta_0 \leq \alpha_0\beta_1$, $1 \leq \beta_1$ then

$$\int\limits_\Omega |u|^{\alpha_0} \sum_{k=1}^n |D_k u|^{\beta_0}\, dx \leq c \int\limits_\Omega |u|^{\alpha_1} \sum_{k=1}^n |D_k u|^{\beta_1}\, dx + c_1 \tag{2.5}$$

holds for any $u \in C_0^1(\Omega)$, where constants $c, c_1 \geq 0$ are independent from u.

Furthermore if we will introduce the space $M_{\eta, W_0^{1,\beta}(\Omega)} \equiv \overset{\circ}{S}_{1,\alpha,\beta}(\Omega) \equiv S_{1,\alpha,\beta}(\Omega) \cap \{u(x) | u|_{\partial\Omega} = 0\}$ then we get the following lemma.

Lemma 2.1. *Let $u \in W_0^{1,2}(\Omega)$ and the number p satisfy the inequation $2 < p < \dfrac{2(n-1)}{n-2}$, $n \geq 3$. Then the function $v(x) \equiv \eta(u(x)) \equiv |u(x)|^p$ belongs to $W_0^{1,\beta}(\Omega)$ for any $\beta \in [1, p_0']$, where $p_0 = \dfrac{2n}{2(n-1) - p(n-2)}$ and $p_0' = \dfrac{p_0}{p_0 - 1} = \dfrac{2n}{p(n-2)+2}$. (It is obvious: $u \in W_0^{1,2}(\Omega) \Longrightarrow v \equiv |u|^p \in W_0^{1,\beta}(\Omega)$ for any $\beta \in [1,2)$ if $n = 2$, and for any $\beta \in [1,2]$ if $n = 1$.)*

Proof. We have

$$\int\limits_\Omega |u|^{(p-1)\beta} |D_k u|^\beta\, dx \leq k(\varepsilon) \int\limits_\Omega |D_k u|^2\, dx + \varepsilon \int\limits_\Omega |u|^{(p-1)\beta \frac{2}{2-\beta}}\, dx$$

for any $u \in W^{1,2}(\Omega)$ and $\beta \in [1, p_0']$. It is enough to consider the case $\beta = p_0' = \dfrac{2n}{p(n-2)+2}$, because $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. So, from here we get

[3] These are a complete metric spaces; about their properties see, for example, [30] and references therein.

$$\int\limits_{\Omega} |u|^{(p-1)\beta} |D_k u|^{\beta} \, dx \le c(\varepsilon) \int\limits_{\Omega} |D_k u|^2 \, dx + \varepsilon \int\limits_{\Omega} |u|^{(p-1)p_0' \frac{2}{2-p_0'}} dx + c_0.$$

Then $(p-1)\, p_0' \dfrac{2}{2-p_0'} = \dfrac{2n}{n-2}$ holds under the conditions of the Lemma 2.1. Consequently, we obtain $u \in W_0^{1,2}(\Omega) \Longrightarrow v \equiv |u|^p \in W_0^{1,\beta}(\Omega)$ for any $\beta \in [1, p_0']$, with choosing $\varepsilon > 0$ sufficiently small and with use the embedding theorem for Sobolev spaces.

Lemma 2.1 is proved.

***Corollary* 2.1.** *Let $u, w \in W_0^{1,2}(\Omega)$ and the number p is such that $2 < p < \dfrac{2(n-1)}{n-2}$, $n \ge 3$. Then the function $v(x) \equiv |u(x)|^{p-2} u(x) w(x)$ belongs to $W_0^{1,\beta}(\Omega)$ (i.e., $v \in W_0^{1,\beta}(\Omega)$) for any $\beta \in [1, p_0']$, where $p_0 = \dfrac{2n}{2(n-1) - p(n-2)}$ and $p_0' = \dfrac{p_0}{p_0 - 1}$.*

Now we introduce a concept of the nonnegative generalized function.

Definition 2.1. *A generalized function $q(x)$ is called a nonnegative distribution* ("$q \ge 0$") *iff $\langle q, \varphi\rangle \ge 0$ holds for any nonnegative test function $\varphi \in D(\Omega)$.*

3. General solvability results. Let X, Y be reflexive Banach spaces and X^*, Y^* their dual spaces, moreover Y is a reflexive Banach space with strictly convex norm together with Y^* (see, for example, references of [29]). Let $f\colon D(f) \subseteq X \longrightarrow Y$ be an operator. So we conduct variant of the main result of [29] (the more general cases can be seen in [31]). Consider the following conditions:

(a) X, Y be Banach spaces such as above and $f\colon D(f) \subseteq X \longrightarrow Y$ be a continuous mapping, moreover there is the closed ball $B_{r_0}^X(x_0) \subset X$ of an element x_0 of $D(f)$ that belongs to $D(f)$ $(B_{r_0}(x_0) \subseteq D(f))$[4].

Let the following conditions are fulfilled on the closed ball $B_{r_0}^X(x_0) \subseteq D(f)$:

(b) f is a bounded mapping on the ball $B_{r_0}^X(x_0)$, i.e., $\|f(x)\|_Y \le \mu(\|x\|_X)$ holds for any $x \in B_{r_0}^X(x_0)$ where $\mu\colon R_+^1 \longrightarrow R_+^1$ is a continuous function;

(c) there is a mapping $g\colon D(g) \subseteq X \longrightarrow Y^*$, and a continuous function $\nu\colon R_+^1 \longrightarrow R^1$ nondecreasing for $\tau \ge \tau_0$ such that $D(f) \subseteq D(g)$, and for any $S_r^X(x_0) \subset B_{r_0}^X(x_0)$, $0 < r \le r_0$, closure of $g\left(S_r^X(x_0)\right) \equiv S_r^{Y^*}(0)$, $S_r^X(x_0) \subseteq g^{-1}\left(S_r^{Y^*}(0)\right)$

$$\langle f(x) - f(x_0), g(x)\rangle \ge \nu\left(\|x - x_0\|_X\right) \|x - x_0\|_X\,, \tag{3.1}$$
$$\text{a.e.} \quad x \in B_{r_0}^X(x_0) \quad \text{and} \quad \nu(r_0) \ge \delta_0 > 0$$

holds, here $\delta_0 > 0$, $\tau_0 \ge 0$ are constants;

(d) almost each $\widetilde{x} \in \operatorname{int} B_{r_0}^X(x_0)$ possesses a neighborhood $V_\varepsilon(\widetilde{x})$, $\varepsilon \ge \varepsilon_0 > 0$, such that the inequation

$$\|f(x_2) - f(x_1)\|_Y \ge \Phi\left(\|x_2 - x_1\|_X, \widetilde{x}, \varepsilon\right) + \psi\left(\|x_1 - x_2\|_Z, \widetilde{x}, \varepsilon\right) \tag{3.2}$$

holds for any $x_1, x_2 \in V_\varepsilon(\widetilde{x}) \cap B_{r_0}^X(x_0)$, where $\Phi(\tau, \widetilde{x}, \varepsilon) \ge 0$ is a continuous function of τ and $\Phi(\tau, \widetilde{x}, \varepsilon) = 0 \Leftrightarrow \tau = 0$ (in particular, maybe $\widetilde{x} = x_0$, $\varepsilon = \varepsilon_0 = r_0$ and $V_\varepsilon(\widetilde{x}) = V_{r_0}(x_0) \equiv B_{r_0}^X(x_0)$, consequently $\Phi(\tau, \widetilde{x}, \varepsilon) \equiv \Phi(\tau, x_0, r_0)$ on $B_{r_0}^X(x_0)$), Z is a Banach space and the inclusion $X \subset Z$ is compact, and $\psi(\cdot, \widetilde{x}, \varepsilon)\colon R_+^1 \longrightarrow R^1$ is a continuous function at τ and $\psi(0, \widetilde{x}, \varepsilon) = 0$;

[4] Here it is enough assume: there is the closed neighborhood $U_\delta(x_0) \subset X$ of an element x_0 of $D(f)$ that belongs to $D(f)$ $(U_\delta(x_0) \subseteq D(f))$ and $U_\delta(x_0)$ is equivalence to $B_{r_0}^X(x_0)$ for some numbers $\delta, r_0 > 0$. Consequently, it is enough account that $U_{r_0}(x_0) \equiv B_{r_0}^X(x_0)$.

(d′) f possesses the P-property on the ball $B^X_{r_0}(x_0)$, i.e., for any precompact subset $M \subseteq \operatorname{Im} f$ of Y there exists a (general) subsequence $M_0 \subset M$ such that there exists a precompact subset G of $B^X_{r_0}(x_0) \subset X$ that satisfies the inclusions $f^{-1}(M_0) \subseteq G$ and $f(G \cap D(f)) \supseteq M_0$.

Theorem 3.1. *Let the conditions* (a), (b), (c) *be fulfilled. Then if the image* $f\left(B^X_{r_0}(x_0)\right)$ *of the ball* $B^X_{r_0}(x_0)$ *is closed (or is fulfilled the condition* (d) *or* (d′)), *then* $f\left(B^X_{r_0}(x_0)\right)$ *is a bodily subset* (*i.e., with nonempty interior*) *of* Y, *moreover* $f\left(B^X_{r_0}(x_0)\right)$ *contains a bodily subset* M *that has the form*

$$M \equiv \left\{ y \in Y \mid \langle y, g(x)\rangle \le \langle f(x), g(x)\rangle \quad \forall x \in S^X_{r_0}(x_0)\right\}.$$

Now we lead a solvability theorem for the nonlinear equation in Banach spaces, which is proved using Theorem 3.1. Let $F_0\colon D(F) \subseteq X \longrightarrow Y$ and $F_1\colon D(F_1) \subseteq X \longrightarrow Y$ be some nonlinear mappings such that $D(F_0) \cap D(F_1) = G \subseteq X$ and $G \neq \varnothing$. Consider the equation

$$F(x) \equiv F_0(x) + F_1(x) = y, \quad y \in Y, \tag{3.3}$$

where y is an arbitrary element of Y.

Let $B^X_r(x_0) \subseteq D(F_0) \cap D(F_1) \subseteq X$ be the closed ball, $r > 0$ be a number. Consider the following conditions:

1) $F_0\colon B^X_r(x_0) \longrightarrow Y$ is a bounded continuous operator together with its inverse operator F_0^{-1} (as $F_0^{-1}\colon D\left(F_0^{-1}\right) \subseteq Y \longrightarrow X$);

2) $F_1\colon B^X_r(x_0) \longrightarrow Y$ is a nonlinear continuous operator;

3) there are continuous functions $\mu_i\colon R^1_+ \longrightarrow R^1_+$, $i = 1, 2$ and $\nu\colon R^1_+ \longrightarrow R^1$ such that the inequations

$$\|F_0(x) - F_0(x_0)\|_Y \le \mu_1(\|x - x_0\|_X) \qquad \text{and} \qquad \|F_1(x) - F_1(x_0)\|_Y \le \mu_2(\|x - x_0\|_X),$$
$$\big\langle F(x) - F(x_0), g(x)\big\rangle \ge c\big\langle F_0(x) - F_0(x_0),\, g(x)\big\rangle \ge \nu\left(\|x - x_0\|_X\right)\|x - x_0\|_X$$

hold for any $x \in B^X_r(x_0)$, moreover $\nu(r) \ge \delta_0$ holds for some number $\delta_0 > 0$, where the mapping $g\colon B^X_r(x_0) \subseteq D(g) \subseteq X \longrightarrow Y^*$ fulfills the conditions of Theorem 3.1, $c > 0$ is some number;

4) almost each $\widetilde{x} \in \operatorname{int} B^X_r(x_0)$ possesses a neighborhood $B^X_\varepsilon(\widetilde{x})$, $\varepsilon \ge \varepsilon_0 > 0$, such that the inequation

$$\|F(x_1) - F(x_2)\|_Y \ge c_1 \|F_0(x_1) - F_0(x_2)\|_Y \ge$$
$$\ge k_0\big(\|x_1 - x_2\|_X, \widetilde{x}, \varepsilon\big) - k_1\big(\|x_1 - x_2\|_Z, \widetilde{x}, \varepsilon\big), \quad X \Subset Z,$$

holds for any $x_1, x_2 \in B^X_\varepsilon(\widetilde{x})$ and some number $\varepsilon_0 > 0$, where $k_i(\tau, \widetilde{x}, \varepsilon) \ge 0$, $i = 0, 1$, are continuous functions of τ for any given $\widetilde{x}$, and such that $k_0(\tau, \widetilde{x}, \varepsilon) = 0 \Longleftrightarrow \tau = 0$, $k_1(0, \widetilde{x}, \varepsilon) = 0$, and $X \Subset Z$ (i.e., $X \subset Z$ is compact).

Then the following statement is true, which follows from Theorem 3.1.

Theorem 3.2. *Let the conditions* 1 – 3 *be fulfilled. Then if* $F\left(B^X_r(x_0)\right)$ *is closed* (*or is fulfilled the condition* 4 *or* (d′)), *then the equation* (3.3) *has a solution in the ball* $B^X_r(x_0)$ *for any* $y \in Y$ *satisfying the inequation*

$$\langle y - F(x_0), g(x)\rangle \le \nu\left(\|x - x_0\|_X\right)\|x - x_0\|_X \quad \forall x \in S^X_r(x_0).$$

4. Proof of existence theorem of problem (0.1), (0.2). It should be noted that here we continue the investigation of the problem studied in the article [32] where this problem is studied in the case $Q \equiv Q_T \equiv (0,T) \times \Omega$, when $T < \infty$ is some number. Here we will study the global existence of this problem, i.e., in the case when $Q \equiv R_+ \times \Omega$. So in the beginning we set a space and explain the way of the investigation. As the solutions $u(t,x)$ of the considered problem will seek in the form $u(t,x) \equiv u_1(t,x) + iu_2(t,x)$, where $u_j : Q \longrightarrow R$, $j = 1,2$ we can set this function $u(t,x)$ as the vector function, i.e., $\overrightarrow{u(t,x)} \equiv (u_1(t,x); u_2(t,x))$ and $\overrightarrow{u} : Q \longrightarrow R^2$. Consequently if we write $u \in X$ $\big($for example, $X \equiv L^m\big(R_+; W_0^{1,2}(\Omega)\big) \cap W^{1,2}\big(R_+; L^2(\Omega)\big)$, $m \geq \max\{p, \widetilde{p}\}\big)$, then this we understand as $u_j \in X$, $j = 1,2$ or $\overrightarrow{u} \in X \times X$. Now we can define a solution of the problem (0.1), (0.2) more exactly.

Definition 4.1. *We say that the function* $u \in L^m\Big(R_+; W_0^{1,2}(\Omega)\Big) \cap W^{1,2}\big(R_+; L^2(\Omega)\big) \cap \big\{w(t,x) \mid w(0,x) = u_0(x)\big\} \equiv X$ (*as complex function*) *is a solution of the problem* (0.1), (0.2) *if it satisfies the equation*

$$i\left\langle \frac{\partial u}{\partial t}, \overline{v}\right\rangle + \langle \nabla u, \nabla \overline{v}\rangle + \big\langle q(x)|u|^{p-2}u, \overline{v}\big\rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{v}\right\rangle = \langle h, \overline{v}\rangle$$

for any $v \in L^{(2,m)}\Big(R_+; W_0^{1,2}(\Omega)\Big)$ *and a.e.* $t > 0$.

Let us $f : X \longrightarrow Y$ is the operator generated by the problem (0.1), (0.2), where X is the denoted above space, and

$$Y \equiv L^2\left(R_+; W^{-1,2}(\Omega)\right) + L^{m'}\big(R_+; W^{-1,2}(\Omega)\big) + L^{\widetilde{q}}(Q),$$

where $\widetilde{q} = \dfrac{\widetilde{p}}{\widetilde{p}-1}$. We will show that for this operator are fulfilled all conditions of the main theorem. We make this by sequence of steps.

1. Clearly that the conditions (a) and (b) fulfilled. Indeed, the explanations conducted in the previous sections shows that $f : X \longrightarrow Y$ is the continuous bounded operator. It should be noted that the calculation of the function μ not is difficult, therefore we not will conduct this computation here (see below Proposition 4.3).

Proposition 4.1. *Let all conditions of Theorem* 1.1 *are fulfilled, then the operator* f *satisfies the condition* (c) *with the operator* $\dfrac{\partial}{\partial t} + I$ *on the space* $W^{1,2}\Big(R_+; W_0^{1,2}(\Omega)\Big) \cap L^m\Big(R_+; W_0^{1,2}(\Omega)\Big) \cap \cap \{v(t,x) \mid v(0,x) = u_0(x)\}$. *Moreover takes place the following inequations*:

$$\int\limits_0^t \operatorname{Im}\left\langle f(u), \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle ds \equiv \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u_0\|_2^2 + \int\limits_0^t \left\|\frac{\partial u}{\partial s}\right\|_{L_2(\Omega)}^2 ds,$$

$$\int\limits_0^t \operatorname{Re}\left\langle f(u), \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle ds \geq \eta\|\nabla u(t)\|_2^2 + \eta\int\limits_0^t \|\nabla u(s)\|_2^2 ds +$$

$$+\delta_1\int\limits_0^t \big\langle q(x)|u|^{p-2}u, \overline{u}\big\rangle(s)\, ds + \delta_2\big\langle q(x)|u|^{p-2}u, \overline{u}\big\rangle(t) -$$

$$-C\big(\|\nabla u_0\|, \|q\|_{W^{-1,2}}, \|u_0\|_{2^*}, \|a\|_m, p, \widetilde{p}\big),$$

the constants of these inequations are determined in (4.6).

Proof. Consider the expression $\left\langle f(u), \dfrac{\partial \overline{u}}{\partial t} + \overline{u} \right\rangle$, which we can write as

$$\left\langle f(u), \frac{d\overline{u}}{dt} \right\rangle = i \left\langle \frac{\partial u}{\partial t}, \frac{\partial \overline{u}}{\partial t} \right\rangle + \frac{1}{2}\frac{d}{dt}\langle \nabla u, \nabla \overline{u} \rangle + \left\langle q(x)|u|^{p-2}u, \frac{\partial \overline{u}}{\partial t} \right\rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \frac{d\overline{u}}{dt} \right\rangle, \tag{4.1}$$

$$\langle f(u), \overline{u} \rangle = i \left\langle \frac{\partial u}{\partial t}, \overline{u} \right\rangle + \langle \nabla u, \nabla \overline{u} \rangle + \left\langle q(x)\,|u|^{p-2}\,u, \overline{u} \right\rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle, \tag{4.2}$$

for any $u \in W^{1,2}\Big(R_+; W_0^{1,2}(\Omega)\Big)$. We begin by (4.2), then we get

$$\langle f(u), \overline{u} \rangle = \frac{i}{2}\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u\|_2^2 + \langle q(x)|u|^{p-2}u, \overline{u} \rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle.$$

Consequently, we have

$$\operatorname{Im}\langle f(u), \overline{u} \rangle = \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2$$

and

$$\operatorname{Re}\langle f(u), \overline{u} \rangle = \|\nabla u\|_2^2 + \langle q(x)|u|^{p-2}u, \overline{u} \rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle,$$

as $\operatorname{Im} q(x) = 0$ and $\operatorname{Im} a(x) = 0$. Whence follows that

$$\int\limits_0^t \operatorname{Im}\langle f(u), \overline{u} \rangle\, ds = \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u_0\|_2^2$$

and

$$\operatorname{Re}\langle f(u), \overline{u} \rangle \geq \|\nabla u\|_2^2 + \langle q(x)|u|^{p-2}u, \overline{u} \rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle \Longrightarrow$$

$$\Longrightarrow \operatorname{Re}\langle f(u), \overline{u} \rangle \geq \|\nabla u\|_2^2 + (1 - k_1)\langle q(x), |u|^p \rangle - k_0\|u\|_{p_2}^2 \Longrightarrow$$

$$\Longrightarrow \operatorname{Re}\langle f(u), \overline{u} \rangle \geq \big(1 - k_0 C^2(p_2)\big)\|\nabla u(t)\|_2^2,$$

for a.e. $t > 0$, as $k_1 \leq 1$ and $k_0 C^2(p_2) < 1$ by virtue of the condition (ii).

Thereby if we examine now (4.1) and (4.2) together then we will obtain

$$\left\langle f(u), \frac{\partial \overline{u}}{\partial t} + \overline{u} \right\rangle \equiv \frac{i}{2}\frac{d}{dt}\||u(t)|\|_2^2 + i\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2 +$$

$$+ \|\nabla u\|_2^2 + \langle q(x)|u|^{p-2}u, \overline{u} \rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u} \right\rangle +$$

$$+ \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u(t)\|_{L_2(\Omega)}^2 + \frac{1}{p}\frac{\partial}{\partial t}\langle q(x), |u|^p \rangle + \frac{1}{\widetilde{p}}\frac{\partial}{\partial t}\left\langle a, |u|^{\widetilde{p}} \right\rangle,$$

as far as

$$\frac{\partial}{\partial t}|u|^p=\frac{\partial}{\partial t}\left(u,\overline{u}\right)^{\frac{p}{2}}=\frac{p}{2}\left(u,\overline{u}\right)^{\frac{p}{2}-1}\frac{\partial}{\partial t}\left(u,\overline{u}\right)=\frac{p}{2}\left(u,\overline{u}\right)^{\frac{p}{2}-1}2\left(u,\frac{\partial}{\partial t}\overline{u}\right)=p\left(|u|^{p-2}u,\frac{\partial\overline{u}}{\partial t}\right),$$

in other words we have

$$\operatorname{Im}\left\langle f(u),\frac{\partial\overline{u}}{\partial t}+\overline{u}\right\rangle\equiv\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_2^2+\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^2 \tag{4.3}$$

and

$$\operatorname{Re}\left\langle f(u),\frac{\partial\overline{u}}{\partial t}+\overline{u}\right\rangle\equiv\frac{1}{2}\frac{\partial}{\partial t}\left\|\nabla u(t)\right\|_{L_2(\Omega)}^2+\left\|\nabla u(t)\right\|_2^2+$$

$$+\frac{1}{p}\frac{\partial}{\partial t}\left\langle q(x),|u|^p\right\rangle+\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle+\frac{1}{\widetilde{p}}\frac{\partial}{\partial t}\left\langle a,|u|^{\widetilde{p}}\right\rangle+\left\langle a(x)|u|^{\widetilde{p}-2}u,\overline{u}\right\rangle. \tag{4.4}$$

If we integrate with respect to t these equation then we get

$$\int_0^t\operatorname{Im}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds\equiv\int_0^t\left[\frac{1}{2}\frac{d}{ds}\left\|u(s)\right\|_2^2+\left\|\frac{\partial u}{\partial s}\right\|_{L_2(\Omega)}^2\right]ds$$

and

$$\int_0^t\operatorname{Re}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds\equiv\int_0^t\left[\frac{1}{2}\frac{\partial}{\partial s}\left\|\nabla u(s)\right\|_2^2+\left\|\nabla u(s)\right\|_2^2\right]ds+$$

$$+\int_0^t\left[\frac{1}{p}\frac{\partial}{\partial s}\left\langle q(x),|u|^p\right\rangle+\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle\right]ds+\int_0^t\left[\frac{1}{\widetilde{p}}\frac{\partial}{\partial s}\left\langle a,|u|^{\widetilde{p}}\right\rangle+\left\langle a(x)|u|^{\widetilde{p}-2}u,\overline{u}\right\rangle\right]ds.$$

Thence follow

$$\int_0^t\operatorname{Im}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds\equiv\frac{1}{2}\left\|u(t)\right\|_2^2-\frac{1}{2}\left\|u_0\right\|_2^2+\int_0^t\left\|\frac{\partial u}{\partial s}\right\|_{L_2(\Omega)}^2ds,$$

$$\tag{4.5}$$

$$\int_0^t\operatorname{Re}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds\equiv\frac{1}{2}\left\|\nabla u(t)\right\|_2^2-\frac{1}{2}\left\|\nabla u_0\right\|_2^2+$$

$$+\int_0^t\left\|\nabla u(s)\right\|_2^2ds+\int_0^t\left[\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle+\left\langle a(x)|u|^{\widetilde{p}-2}u,\overline{u}\right\rangle\right]ds+$$

$$+\left[\frac{1}{p}\left\langle q(x),|u|^p\right\rangle+\frac{1}{\widetilde{p}}\left\langle a(x),|u|^{\widetilde{p}}\right\rangle\right](t)-\frac{1}{p}\left\langle q(x),|u_0|^p\right\rangle-\frac{1}{\widetilde{p}}\left\langle a,|u_0|^{\widetilde{p}}\right\rangle.$$

For estimate the $\int_0^t\operatorname{Re}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds$ we use (1.2) (i.e., the condition (ii))

$$\left\langle a(x)|u|^{\widetilde{p}-2}u,\overline{u}\right\rangle \geq -k_0\left\|u\right\|_{p_2}^2 - k_1\left\langle q(x),|u|^p\right\rangle$$

then we obtain

$$\int\limits_0^t \operatorname{Re}\left\langle f(u),\frac{\partial\overline{u}}{\partial s}+\overline{u}\right\rangle ds \geq \frac{1}{2}\left\|\nabla u(t)\right\|_2^2 + \int\limits_0^t \left\|\nabla u(s)\right\|_2^2 ds +$$

$$+\delta_1\int\limits_0^t \left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(s)ds - k_0\int\limits_0^t \left\|u\right\|_{p_2}^2(s)ds + \delta_2\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(t) -$$

$$-k_0\left\|u(t)\right\|_{p_2}^2 - \frac{1}{2}\left\|\nabla u_0\right\|_2^2 - \frac{1}{p}\left\langle q(x),|u_0|^p\right\rangle - \frac{1}{\widetilde{p}}\left\langle a,|u_0|^{\widetilde{p}}\right\rangle \geq$$

$$\geq \eta\left\|\nabla u(t)\right\|_2^2 + \eta\int\limits_0^t \left\|\nabla u(s)\right\|_2^2 ds + \delta_1\int\limits_0^t \left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(s)ds +$$

$$+\delta_2\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(t) - C\big(\left\|\nabla u_0\right\|,\left\|q\right\|_{W^{-1,2}},\left\|u_0\right\|_{2^*},\left\|a\right\|_m,p,\widetilde{p}\big) \tag{4.6}$$

for a.e. $t>0$, where $\delta_1 = 1-k_1\geq 0$, $\delta_2 = p^{-1}-\widetilde{p}^{-1}k_1\geq 0$, $\eta = 1-C\left(2,p_2\right)^2 k_0>0$. The expressions (4.5) and (4.6) shows that the condition (c) of the main theorem takes place for the operator $f: X\longrightarrow Y$ generated by posed problem.

Proposition 4.1 is proved.

2. Now we prove an inequation used in the proof of the fulfilment of the condition (d′).

Proposition 4.2. *Let all conditions of Theorem* 1.1 *are fulfilled, then the inequality*

$$\left\|f(u)-f(v)\right\|_Y \geq \left\|u-v\right\|_2(t) + \left\|\nabla(u-v)\right\|_2 - M\max\left\{\left\|u\right\|_{\widetilde{p}}^{\widetilde{p}-2};\left\|v\right\|_{\widetilde{p}}^{\widetilde{p}-2}\right\}\left\|u-v\right\|_{\widetilde{p}},$$

holds for any $u,v\in X\cap\{u\mid u(0,x)=u_0(x)\}$.

Proof. Let us $u,v\in X\cap\{u\mid u(0,x)=u_0(x)\}$ and consider $\left\|f(u)-f(v)\right\|_Y$ that we can estimate as

$$\left\|i\frac{\partial(u-v)}{\partial t} - \Delta(u-v) + q\left(|u|^{p-2}u-|v|^{p-2}v\right) + a\left(|u|^{\widetilde{p}-2}u-|v|^{\widetilde{p}-2}v\right)\right\|_Y \geq$$

$$\geq \left\|i\frac{\partial(u-v)}{\partial t} - \Delta(u-v) + q\left(|u|^{p-2}u-|v|^{p-2}v\right)\right\|_Y - \left\|a\left(|u|^{\widetilde{p}-2}u-|v|^{\widetilde{p}-2}v\right)\right\|_Y. \tag{4.7}$$

In order that to esimate of the first adding of right-hand side of the inequality (4.7) we act in the following way. In beginning we set

$$\left\langle i\frac{\partial(u-v)}{\partial t},\overline{(u-v)}\right\rangle - \left\langle\Delta(u-v),\overline{(u-v)}\right\rangle + \left\langle q\left(|u|^{p-2}u-|v|^{p-2}v\right),\overline{(u-v)}\right\rangle =$$

$$= \frac{i}{2}\frac{d}{dt}\left\langle(u-v),\overline{(u-v)}\right\rangle + \left\|\nabla(u-v)\right\|_2^2 + \left\langle q\left(|u|^{p-2}u-|v|^{p-2}v\right),\overline{(u-v)}\right\rangle \tag{4.8}$$

and study it.

Here for the last adding takes place the inequality

$$\left\langle q\left(|u|^{p-2}u-|v|^{p-2}v\right),\overline{(u-v)}\right\rangle=\langle q,|u|^{p}\rangle+\langle q,|v|^{p}\rangle-\left\langle q|u|^{p-2}u,\overline{v}\right\rangle-\left\langle q|v|^{p-2}v,\overline{u}\right\rangle.$$

As the expression $\left|\left\langle q|u|^{p-2}u,\overline{v}\right\rangle+\left\langle q|v|^{p-2}v,\overline{u}\right\rangle\right|$ has the following estimation:

$$\left|\left\langle q|u|^{p-2}u,\overline{v}\right\rangle+\left\langle q|v|^{p-2}v,\overline{u}\right\rangle\right|\leq\left\langle q|u|^{p-1},|v|\right\rangle+\left\langle q|v|^{p-1},|u|\right\rangle$$

therefore we can consider the right-hand side of (4.8) without of the last adding.

Consequently we get the following estimation for the first adding of the right-hand side of the inequality (4.7):

$$\left\|i\frac{\partial(u-v)}{\partial t}-\Delta\left(u-v\right)+q\left(|u|^{p-2}u-|v|^{p-2}v\right)\right\|_{Y}\geq K\left[\,\|u-v\|_{2}\left(t\right)+\|\nabla(u-v)\|_{2}\,\right],\quad K>0,$$

with taking into account the equation (4.8), the last reasons and the equation

$$\int\limits_{0}^{t}\left\langle\frac{\partial\left(u-v\right)}{\partial s},\overline{(u-v)}\right\rangle ds=\frac{1}{2}\left\|u-v\right\|_{2}^{2}(t),$$

whereas $u(0,x)=v(0,x)=u_0$ by choosingly, that we need make by virtue of the condition (d′) of the main theorem.

Now consider the second adding of right-hand side of the inequality (4.7), for which we have

$$\left|\left\langle a\left(|u|^{\widetilde{p}-2}u-|v|^{\widetilde{p}-2}v\right),\overline{(u-v)}\right\rangle\right|=\int\limits_{\Omega}a\left(|u|^{\widetilde{p}-2}u-|v|^{\widetilde{p}-2}v\right)\overline{(u-v)}dx\leq$$

$$\leq\int\limits_{\Omega}a\,\varphi\left(u,v\right)|u-v|^{2}\,dx,\quad 0\leq\varphi\left(u,v\right)\leq M\left(\max\left\{|u|,|v|\right\}\right)^{\widetilde{p}-2},$$

where $M>0$ be some number and $\varphi\left(u,v\right)$ be a continuous function.

Taking into account the last inequalities in (4.7) we obtain

$$\|f(u)-f(v)\|_{Y}\geq\|u-v\|_{2}\left(t\right)+\|\nabla(u-v)\|_{2}-M\max\left\{\|u\|_{\widetilde{p}}^{\widetilde{p}-2};\|v\|_{\widetilde{p}}^{\widetilde{p}-2}\right\}\|u-v\|_{\widetilde{p}}.\tag{4.9}$$

Proposition 4.2 is proved.

3. Now we will conduct a priori estimations for a solutions of the problem.

Proposition 4.3. *Let all conditions of Theorem* 1.1 *are fulfilled, then all solutions belong to bounded subset of the space*

$$X\equiv W^{1,2}\left(R_{+};L^{2}(\Omega)\right)\cap L^{m}\left(R_{+};W_{0}^{1,2}(\Omega)\right)\cap$$

$$\cap\left\{v\;\Big|\;|v|^{p}\in L^{\beta}\left(R_{+};W_{0}^{1,\beta}(\Omega)\right)\right\}\cap\left\{u\;\Big|\;u(0,x)=u_{0}(x)\right\},$$

i.e., there is constants $K\equiv K\left(\|h\|_{2,Q_T},\|u_0\|_{W^{1,2}},\|q\|_{W^{-1,2}},\|a\|,p,\;\widetilde{p}\right)$ *such that* $\|u\|_X\leq K$.

Proof. In the beginning we note that from examination of the expression $\langle f(u), \overline{u}\rangle$ in the proof of Proposition 4.1 we get the inequations

$$\frac{1}{2}\left\|u(t)\right\|_2^2 - \frac{1}{2}\left\|u_0\right\|_2^2 \leq \int\limits_0^t \left|\langle h, \overline{u}\rangle\right| ds \leq \int\limits_0^t \left\|h(s)\right\|_2 \left\|u(s)\right\|_2 ds$$

and

$$\|\nabla u\|_2^2 + \left\langle q(x)|u|^{p-2}u, \overline{u}\right\rangle + \left\langle a(x)|u|^{\widetilde{p}-2}u, \overline{u}\right\rangle \leq \left|\langle h, \overline{u}\rangle\right| \Longrightarrow$$

$$\Longrightarrow \|\nabla u\|_2^2 + (1-k_1)\left\langle q(x), |u|^p\right\rangle - k_0\left\|u\right\|_{p_2}^2 \leq \left|\langle h, \overline{u}\rangle\right| \Longrightarrow$$

$$\Longrightarrow \left(1 - k_0 C^2\left(p_2\right)\right)\left\|\nabla u\left(t\right)\right\|_2^2 \leq \left\|h(t)\right\|_2 \left\|u(t)\right\|_2,$$

as $k_1 \leq 1$ and $k_0 C^2\left(p_2\right) < 1$ by virtue of the condition (ii). Then we get that the following estimations are true:

$$\left(1 - k_0 C^2(p_2)\right)\|\nabla u(t)\|_2^2 - \varepsilon\|u(t)\|_2^2 \leq c(\varepsilon)\|h(t)\|_2^2$$

or

$$\left\|\nabla u(t)\right\|_2 \leq \widehat{c}(\varepsilon)\left\|h(t)\right\|_2 \quad \text{for a.e.} \quad t \geq 0,$$

as far as $\left\|u(t)\right\|_2 \leq C_1\left(\operatorname{mes}\Omega\right)\left\|\nabla u(t)\right\|_2$ for any $u(t) \in W_0^{1,2}(\Omega)$ by the embedding theorems, where $\widehat{c}(\varepsilon) \equiv \widehat{c}\left(\varepsilon, \operatorname{mes}\Omega, k_0 C\right)$, $C_1\left(\operatorname{mes}\Omega\right) > 0$ are constants, moreover,

$$\|u\|_{L^2\left(R_+; W_0^{1,2}(\Omega)\right)} \leq \widehat{c}_1(\varepsilon)\left\|h\right\|_{L^2\left(R_+; W_0^{1,2}(\Omega)\right)} + \widehat{c}_2\left\|u_0\right\|_2.$$

Thus we obtain that $u(t,x)$ belong to the bounded subset of $L^2\left(R_+; W_0^{1,2}(\Omega)\right)$ for given $h \in$ $\in L^2(Q)$.

Using the equations (4.3) and (4.4), and also the estimates (4.5) and (4.6) we get

$$\left|\int\limits_0^t \operatorname{Re}\left\langle h, \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle ds\right| \geq \int\limits_0^t \operatorname{Re}\left\langle f(u), \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle ds \geq$$

$$\geq \eta_2\left\|\nabla u(t)\right\|_2^2 + \eta_1 \int\limits_0^t \left\|\nabla u(s)\right\|_2^2 ds + \delta_1 \int\limits_0^t \left\langle q(x)|u|^{p-2}u, \overline{u}\right\rangle(s) ds +$$

$$+\delta_2\left\langle q(x)|u|^{p-2}u, \overline{u}\right\rangle(t) - C\left(\left\|u_0\right\|_{W^{1,2}}, \left\|q\right\|_{W^{-1,2}}, \left\|a\right\|_m, p, \widetilde{p}\right) \tag{4.10}$$

and

$$\left|\int\limits_0^t \operatorname{Im}\left\langle h, \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle ds\right| \geq \int\limits_0^t \operatorname{Im}\left\langle f(u), \frac{\partial \overline{u}}{\partial s} + \overline{u}\right\rangle \equiv$$

$$\equiv \frac{1}{2}\left\|u(t)\right\|_2^2 - \frac{1}{2}\left\|u_0\right\|_2^2 + \int\limits_0^t \left\|\frac{\partial u}{\partial s}\right\|_{L_2(\Omega)}^2 ds. \tag{4.11}$$

Then from (4.10) we get the estimation

$$\eta_2 \left\|\nabla u(t)\right\|_2^2+\eta_1\int\limits_0^t \left\|\nabla u(s)\right\|_2^2 ds+\delta_1\int\limits_0^t \left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(s)ds+$$

$$+\delta_2\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(t)-C\left(\|u_0\|_{W^{1,2}},\|q\|_{W^{-1,2}},\|a\|_{m'},p,\widetilde{p}\,\right)\le$$

$$\le\int\limits_0^t \|h\|_2\left(\left\|\frac{\partial u}{\partial s}\right\|_2+\|u\|_2\right)ds \quad \text{a.e.} \quad t>0,$$

and from (4.11) we obtain

$$\frac{1}{2}\left\|u(t)\right\|_2^2-\frac{1}{2}\left\|u_0\right\|_2^2+\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_{L_2(\Omega)}^2 ds\le\int\limits_0^t\|h\|_2\left(\left\|\frac{\partial u}{\partial s}\right\|_2+\|u\|_2\right)ds \quad \text{for a.e.} \quad t>0,$$

then with combine of last two inequations we get

$$\eta\left\|\nabla u(t)\right\|_2^2+\eta\int\limits_0^t\left\|\nabla u(s)\right\|_2^2 ds+\delta_1\int\limits_0^t\left\langle q\left(x\right)\left|u\right|^{p-2}u,\overline{u}\right\rangle\left(s\right)ds+$$

$$+\delta_2\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(t)-C\left(\|u_0\|_{W^{1,2}},\|q\|_{W^{-1,2}},\|a\|,p,\widetilde{p}\right)+$$

$$+\frac{1}{2}\left\|u(t)\right\|_2^2-\frac{1}{2}\left\|u_0\right\|_2^2+\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_2^2 ds\le$$

$$\le\varepsilon_1\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_2^2 ds+\varepsilon_2\int\limits_0^t\|u\|_2^2 ds+C\left(\varepsilon_1,\varepsilon_2\right)\int\limits_0^t\|h\|_2^2 ds \quad \text{for a.e.} \quad t>0,$$

or

$$\eta\left\|\nabla u(t)\right\|_2^2+\widetilde{\eta}\int\limits_0^t\left\|\nabla u(s)\right\|_2^2 ds+\delta_1\int\limits_0^t\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(s)\,ds+$$

$$+\delta_2\left\langle q(x)|u|^{p-2}u,\overline{u}\right\rangle(t)+\frac{1}{2}\left\|u\left(t\right)\right\|_2^2+\left(1-\varepsilon_1\right)\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_2^2 ds\le$$

$$\le C\left(\varepsilon_1,\varepsilon_2\right)\int\limits_0^\infty\|h\|_2^2 ds+C_1\left(\|u_0\|_{W^{1,2}},\|q\|_{W^{-1,2}},\|a\|,p,\widetilde{p}\right).$$

Moreover, from here follows

$$\frac{1}{2}\left\|\nabla u(t)\right\|_2^2-\frac{1}{2}\left\|\nabla u(0)\right\|_2^2+\int\limits_0^t\left\|\nabla u(s)\right\|_2^2 ds+$$

$$+\int\limits_0^t \left\langle q(x)\,|u|^{p-2}\,u,\overline{u}\right\rangle(s)ds+\int\limits_0^t \left\langle a(x)|u|^{\widetilde{p}-2}u,\overline{u}\right\rangle(s)ds+$$

$$+\frac{1}{p}\left\langle q(x),|u|^p\right\rangle(t)-\frac{1}{p}\left\langle q(x),|u(0)|^p\right\rangle+\frac{1}{\widetilde{p}}\left\langle a,|u|^{\widetilde{p}}\right\rangle(t)-\frac{1}{\widetilde{p}}\left\langle a,|u(0)|^{\widetilde{p}}\right\rangle\le$$

$$\le\left(\varepsilon^{-1}+\frac{1}{2}\right)\int\limits_0^t\|h(s)\|_2^2\,ds+\varepsilon\int\limits_0^t\|u(s)\|_2^2\,ds+\frac{1}{2}\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_2^2ds+$$

$$+C\big(\|\nabla u_0\|,\|q\|_{W^{-1,2}},\|u_0\|_{2^*},\|a\|_m,p,\widetilde{p}\big),$$

by virtue of the condition (1.2).

Consequently we get the following inequation:

$$\eta\,\|\nabla u(t)\|_2^2+\eta_1(\varepsilon)\int\limits_0^t\|\nabla u(s)\|_2^2\,ds\le$$

$$\le\varepsilon^{-1}\int\limits_0^t\|h(s)\|_2^2\,ds+C\big(\|\nabla u_0\|_2,\|q\|_{W^{-1,2}},\|u_0\|_{2^*},\|h\|_2,\|a\|_m,p,\widetilde{p}\big)$$

in the other words we obtain

$$\int\limits_0^t\big\|\nabla u(s)\big\|_2^2ds\le\widetilde{D}\big(\varepsilon^{-1},\ldots\big)\big(\widetilde{\eta}_1(\varepsilon)\big)^{-1}\Big(1-e^{-\widetilde{\eta}_1(\varepsilon)t}\Big),\tag{4.12}$$

where $\widetilde{D}\big(\varepsilon^{-1},\ldots\big)=\widetilde{D}\big(\varepsilon^{-1},\|u_0\|_{W^{1,2}},\|q\|_{W^{-1,2}},\|h\|_2,\|a\|_m,p,\widetilde{p}\big)$.

Consequently we obtain, that any solution of the considered problem under the posed conditions satisfies the following inclusion:

$$u\in W^{1,2}\left(R_+;L^2(\Omega)\right)\cap L^\infty\Big(R_+;W_0^{1,2}(\Omega)\Big)\cap$$

$$\cap\left\{v\mid|v|^p\in L^\beta\left(R_+;W_0^{1,\beta}(\Omega)\right)\right\}\cap\{u\mid u(0,x)=u_0(x)\}\equiv X,\tag{4.13}$$

in addition the preimage of each bounded neighborhood of zero from $L^2(Q)\times W_0^{1,2}(\Omega)$ under operator f is the bounded neighborhood of zero of the space determined by (4.13), where $m\ge 2^*$ and $\beta>1$ is denoted by Lemma 2.1. We note that here is used the inclusion given in next remark.

Proposition 4.3 is proved.

***Remark* 4.1.** Let Z is a Banach space, then

$$L^2\left(R;Z\right)\cap L^\infty\left(R;Z\right)\subset L^m\left(R;Z\right),\quad 2\le m<\infty,$$

holds.

From here we get that the condition (c) is fulfilled for the operator f generated by the posed problem and the operator $g(v)\equiv\dfrac{\partial v}{\partial t}+v$ for any $v\in W^{1,2}\Big(R_+;W_0^{1,2}(\Omega)\Big)\cap L^m\Big(R_+;W_0^{1,2}(\Omega)\Big)$ that is dense in the requisit space defined in (4.13).

4. Now we can show that the operator f satisfies the condition (d′).

Proposition 4.4. *Let all conditions of Theorem* 1.1 *are fulfilled, then the operator f satisfies the condition* (d′).

Proof. More exactly, we will prove that the image $f(X)$ is the closed subset of Y. As the conditions (a), (b) and (c) of the main theorem are fulfilled for the operator $f : X \to Y$ then we get, that $f(X)$ contains a dense subset of the space Y by virtue of the first statement of this theorem.

So, let us the sequence $\{h_k\}_{k=1}^{\infty} \subset f(X)$ is the fundamental sequence in Y that converge to an element $h_0 \in Y$, since $f(X)$ contains a dense subset of the space Y therefore for any $h_0 \in Y$ there exists a sequence of such type. As the sequence $\{h_k\}_{k=1}^{\infty}$ is a bounded subset of Y, and consequently $f^{-1}(\{h_k\}_{k=1}^{\infty})$ belong to the bounded subset M_0 of X by virtue of the condition (c), which is proved in the step 3. It is known that X is the reflexive space therefore we can choose a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset M_0$ of $f^{-1}\left(\{h_k\}_{k=1}^{\infty}\right)$ such that $u_{k_j} \in f^{-1}(h_{k_j})$, $k_j \nearrow \infty$, and $\{u_{k_j}\}_{j=1}^{\infty}$ weakly converge in X, i.e., $u_{k_j} \rightharpoonup u_0 \in X$. Moreover it is known that $X \Subset L^m\left(R_+; L^{\ell}(\Omega)\right)$ is compact, where $1 < \ell < \dfrac{2n}{n-2}$ if $n \geq 3$ (see, for example, [31] and its references). Then the sequence $\{u_{k_j}\}_{j=1}^{\infty}$ have a subsequence, that strongly converge in the space $L^m\left(R_+; L^{\ell}(\Omega)\right)$, which for simplicity we denote also by $\{u_{k_j}\}_{j=1}^{\infty}$, i.e., we assume that $\{u_{k_j}\}_{j=1}^{\infty}$ is the subsequence of such type.

Thence use previous reasons and the (4.9) from step 2 we get $u_{k_j} \Longrightarrow u_0$ in $L^2\big(R_+; W_0^{1,2}(\Omega)\big)$ and in $L^m\big(R_+; L^2(\Omega)\big)$. Furthermore if take into account that in this problem first two adding are linear continuous operator then we obtain that $h_{k_j} = f\big(u_{k_j}\big) \longrightarrow f(u_0) \equiv h_0$, which show that $f(X)$ is the closed subset of Y.

Proposition 4.4 is proved.

Thus we can complete of the proof of Theorem 1.1. From Propositions 4.1 – 4.4 we get, that the operator $f : X \longrightarrow Y$ generated by the posed problem satisfies all conditions of the main theorem (Theorem 3.1 and also Theorem 3.2). Then using Theorem 3.1 we obtain, that the operator f satisfies the statement of Theorem 3.1, therefore the statement of Theorem 1.1 is correct. Consequently, the existence theorem for the problem (0.1), (0.2) (i.e., Theorem 1.1) is proved.

5. Behavior of solutions of problem (0.1), (0.2). We will study the behavior of the solution of problem (0.1), (0.2) in the sense of the space $W^{1,2}(Q) \cap L^m\big(R_+; W_0^{1,2}(\Omega)\big)$. Consider the following functional on the space $W^{1,2}(Q) \cap L^m\big(R_+; W_0^{1,2}(\Omega)\big)$:

$$I\left(v(t)\right) = \|\nabla v(t)\|_{L^2}^2 \equiv \|\nabla v(t)\|_2^2 \equiv \int\limits_{\Omega} |\nabla v(t,x)|^2\, dx.$$

So we have

$$2^{-1}\frac{d}{dt} I\left(u(t)\right) = \left\langle \frac{\partial}{\partial t}\nabla u, \overline{\nabla u} \right\rangle = -\left\langle \frac{\partial}{\partial t} u, \overline{\Delta u} \right\rangle.$$

Here if to take account $u(t,x)$ is the solution of the considered problem then we get

$$-i\left\langle \frac{\partial u}{\partial t}, \frac{\partial \overline{u}}{\partial t} \right\rangle - \left\langle \frac{\partial u}{\partial t}, q|u|^{p-2}\overline{u} \right\rangle - \left\langle \frac{\partial u}{\partial t}, a|u|^{\widetilde{p}-2}\overline{u} \right\rangle + \left\langle \frac{\partial u}{\partial t}, \overline{h} \right\rangle =$$

$$= i\left\| \frac{\partial u}{\partial t} \right\|_2^2 - p^{-1}\frac{d}{dt}\left\langle q, |u|^p \right\rangle - \widetilde{p}^{-1}\frac{d}{dt}\left\langle a, |u|^{\widetilde{p}} \right\rangle + \left\langle \frac{\partial u}{\partial t}, \overline{h} \right\rangle$$

whence we have

$$2^{-1}\frac{d}{dt}I\left(u(t)\right)=-\left\langle\frac{\partial u}{\partial t},q|u|^{p-2}\overline{u}\right\rangle-\left\langle\frac{\partial u}{\partial t},a\left|u\right|^{\widetilde{p}-2}\overline{u}\right\rangle+\mathrm{Re}\left\langle\frac{\partial u}{\partial t},\overline{h}\right\rangle$$

and

$$-\left\|\frac{\partial u}{\partial t}\right\|_2^2=-\,\mathrm{Im}\left\langle\frac{\partial u}{\partial t},\overline{h}\right\rangle\Longrightarrow\left\|\frac{\partial u}{\partial t}\right\|_2\le\|h\|_2 \tag{5.1}$$

for a.e. $t>0$, as $I\left(u(t)\right)$ is a real function.

Thus we obtain

$$2^{-1}I\left(u(t)\right)-2^{-1}I\left(u_0\right)\le-p^{-1}\int\limits_0^t\frac{d}{ds}\left\langle q,|u|^p\right\rangle ds-\widetilde{p}^{-1}\int\limits_0^t\frac{d}{ds}\left\langle a,|u|^{\widetilde{p}}\right\rangle ds+$$

$$+\int\limits_0^t\left|\left\langle\frac{\partial u}{\partial s},\overline{h}\right\rangle\right|ds\le-p^{-1}\left\langle q,|u|^p\right\rangle(t)+p^{-1}\left\langle q,|u_0|^p\right\rangle-$$

$$-\widetilde{p}^{-1}\left\langle a,|u|^{\widetilde{p}}\right\rangle(t)+\widetilde{p}^{-1}\left\langle a,|u_0|^{\widetilde{p}}\right\rangle+\int\limits_0^t\left\|\frac{\partial u}{\partial s}\right\|_2\|h\|_2(s)ds\le$$

$$\le k_0\left\|u(t)\right\|_{p_2}^2-\delta\left\langle q,|u|^p\right\rangle(t)+\int\limits_0^t\|h\|_2^2(s)ds$$

whence we get using the condition (1.2) (i.e., the inequality $2^{-1}>C(2,p_2)^2\cdot k_0$) and (5.1)

$$I\left(u(t)\right)\equiv\left\|\nabla u(t)\right\|_2^2\le 2k_0\left\|u(t)\right\|_{p_2}^2+2\int\limits_0^t\|h\|_2^2(s)ds+I\left(u_0\right)\Longrightarrow$$

$$\Longrightarrow\left(1-2C\left(2,p_2\right)^2k_0\right)\left\|\nabla u\left(t\right)\right\|_2^2\le 2\int\limits_0^t\|h\|_2^2(s)ds+I\left(u_0\right)$$

for a.e. $t\ge 0$.

Moreover in the previous section we obtained the following equations:

$$I_0\left(u(t)\right)\equiv\|u\|_2^2\Longrightarrow\frac{1}{2}\frac{d}{dt}I_0\left(u(t)\right)=\left\langle u_t,\overline{u}\right\rangle=$$

$$=i\left\langle-\Delta u+q|u|^{p-2}u+a|u|^{\widetilde{p}-2}u-h,\overline{u}\right\rangle=$$

$$=i\left\|\nabla u(t)\right\|_2^2+i\left\langle q(x),|u|^p\right\rangle+i\left\langle a(x),|u|^{\widetilde{p}}\right\rangle-i\left\langle h,\overline{u}\right\rangle\Longrightarrow$$

$$\Longrightarrow\left\|\nabla u(t)\right\|_2^2+\left\langle q(x),|u|^p\right\rangle+\left\langle a(x),|u|^{\widetilde{p}}\right\rangle-\mathrm{Re}\left\langle h,\overline{u}\right\rangle=0,$$

$$\frac{1}{2}\frac{d}{dt}I_0\left(u(t)\right) = -\operatorname{Im}\langle h, \overline{u}\rangle$$

whence follows

$$\|\nabla u(t)\|_2^2 \le 4\,\|h(t)\|_2^2 \quad \text{for a.e.} \quad t \ge 0.$$

Consequently is proved the following result.

Theorem 5.1. *Let the solution $u(t,x)$ of the problem* (0.1), (0.2) *is a sufficiently smooth function, i.e., $u \in W^{1,2}\big(R_+; W_0^{1,2}(\Omega)\big) \cap L^m\big(R_+; W_0^{1,2}(\Omega)\big)$ and all conditions of Theorem* 1.1 *are fulfilled. Then if the coefficient k_0 of the condition* (1.2) *satisfy the inequality $2^{-1} > C\left(2, p_2\right)^2 \cdot k_0$ then the following inequalities are true*:

$$\|\nabla u(t)\|_2, \ \left\|\frac{\partial u}{\partial t}\right\|_2 \le b\,\|h(t)\|_2 \Longrightarrow \|u(t)\|_2 \le b_1\,\|h(t)\|_2$$

for a.e. $t \ge 0$.

Remark **5.1.** It should be noted that the next inequality follows from (4.12) in the conditions of Theorem 5.1

$$\|\nabla u(t)\|_2^2 \le \widetilde{D}\left(\varepsilon^{-1}, \ldots\right) \exp\left\{-\widetilde{\eta}_1(\varepsilon)t\right\} + \|\nabla u_0\|_2^2 .$$

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