

**ON A TWO-WEIGHT CRITERIA FOR MULTIDIMENSIONAL HARDY TYPE OPERATOR IN  $p$ -CONVEX BANACH FUNCTION SPACES AND SOME APPLICATION \*****ПРО ДВОВАГОВИЙ КРИТЕРІЙ ДЛЯ БАГАТОВИМІРНОГО ОПЕРАТОРА ТИПУ ХАРДІ В  $p$ -ОПУКЛИХ БАНАХОВИХ ПРОСТОРАХ ФУНКЦІЙ ТА ДЕЯКІ ЗАСТОСУВАННЯ**

The main goal of this paper is to prove a two-weight criterion for multidimensional Hardy type operator from weighted Lebesgue spaces into  $p$ -convex weighted Banach function spaces. The problem for the dual operator is also considered. As an application, we prove a two-weight criterion for the boundedness of multidimensional geometric mean operator from weighted Lebesgue spaces into weighted Musielak–Orlicz spaces.

Головною метою даної статті є доведення двовагового критерію для багатовимірного оператора типу Харді із вагових лебегових просторів в  $p$ -опуклі вагові банахові простори функцій. Також розглянуто задачу для дуального оператора. Як застосування, встановлено двоваговий критерій обмеженості багатовимірного оператора геометричного середнього з вагових лебегових просторів у вагові простори Муселяка–Орліча.

**1. Introduction.** The investigation of Hardy operator in weighted Banach function spaces (BFS) have recently history. The goal of this investigations were closely connected with the found of criterion on the geometry and on the weights of BFS for validity of boundedness of Hardy operator in BFS. Characterization of the mapping properties such as boundedness and compactness were considered in the papers [10, 11, 14, 30] and etc. More precisely, in [10] and [11] were considered the boundedness of certain integral operator in ideal Banach spaces. In [14] was proved the boundedness of Hardy operator in Orlicz spaces. Also, in [30] the compactness and measure of noncompactness of Hardy type operator in BFS was proved. But in this paper we consider the boundedness of Hardy operator in  $p$ -convex BFS and find a new type criterion on the weights for validity of Hardy inequality. Note that the notion of BFS was introduced in [32]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak–Orlicz spaces and etc. is BFS.

In this paper, we establish an integral-type necessary and sufficient condition on weights, which provides the boundedness of the multidimensional Hardy type operator from weighted Lebesgue spaces into  $p$ -convex weighted BFS. We also investigate the corresponding problems for the dual operator. It is well known that the classical two-weight inequality for geometric mean operator is closely connected with the one-dimensional Hardy inequality (see [20]). Analogously, the Pólya–Knopp type inequalities with multidimensional geometric mean operator are connected with the multidimensional Hardy type operator. Therefore, in this paper, as an application of Hardy inequality we prove the boundedness of multidimensional geometric mean operator and boundedness of certain sublinear operator from weighted Lebesgue spaces into weighted Musielak–Orlicz spaces.

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**2. Preliminaries.** Let  $(\Omega, \mu)$  be a complete  $\sigma$ -finite measure space. By  $L_0 = L_0(\Omega, \mu)$  we denote the collection of all real-valued  $\mu$ -measurable functions on  $\Omega$ .

**Definition 1** [9, 29, 32]. We say that real normed space  $X$  is a BFS if:

(P<sub>1</sub>) the norm  $\|f\|_X$  is defined for every  $\mu$ -measurable function  $f$ , and  $f \in X$  if and only if  $\|f\|_X < \infty$ ;  $\|f\|_X = 0$  if and only if  $f = 0$  a.e.;

(P<sub>2</sub>)  $\|f\|_X = \|\|f\|\|_X$  for all  $f \in X$ ;

(P<sub>3</sub>) if  $0 \leq f_n \uparrow f \leq g$  a.e., then  $\|f_n\|_X \uparrow \|f\|_X$  (Fatou property);

(P<sub>4</sub>) if  $E$  is a measurable subset of  $\Omega$  such that  $\mu(E) < \infty$ , then  $\|\chi_E\|_X < \infty$ , where  $\chi_E$  is the characteristic function of the set  $E$ ;

(P<sub>5</sub>) for every measurable set  $E \subset \Omega$  with  $\mu(E) < \infty$ , there is a constant  $C_E > 0$  such that  $\int_E f(x) dx \leq C_E \|f\|_X$ .

Given a BFS  $X$  we can always consider its associate space  $X'$  consisting of those  $g \in L_0$  that  $f \cdot g \in L_1$  for every  $f \in X$  with the usual order and the norm  $\|g\|_{X'} = \sup \{\|f \cdot g\|_{L_1} : \|g\|_{X'} \leq 1\}$ . Note that  $X'$  is a BFS in  $(\Omega, \mu)$  and a closed norming subspaces.

Let  $X$  be a BFS and  $\omega$  be a weight, that is, positive Lebesgue measurable and a.e. finite functions on  $\Omega$ . Let  $X_\omega = \{f \in L_0 : f\omega \in X\}$ . This space is a weighted BFS equipped with the norm  $\|f\|_{X_\omega} = \|f\omega\|_X$ . (For more detail and proofs of results about BFS we refer the reader to [9] and [29].)

Note that the notion of BFS was introduced by W. A. J. Luxemburg in [32].

Let us recall the notion of  $p$ -convexity and  $p$ -concavity of BFS's.

**Definition 2** [42]. Let  $X$  is a BFS. Then  $X$  is called  $p$ -convex for  $1 \leq p \leq \infty$  if there exists a constant  $M > 0$  such that for all  $f_1, \dots, f_n \in X$

$$\left\| \left( \sum_{k=1}^n |f_k|^p \right)^{\frac{1}{p}} \right\|_X \leq M \left( \sum_{k=1}^n \|f_k\|_X^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty,$$

or  $\|\sup_{1 \leq k \leq n} |f_k|\|_X \leq M \max_{1 \leq k \leq n} \|f_k\|_X$  if  $p = \infty$ . Similarly  $X$  is called  $p$ -concave for  $1 \leq p \leq \infty$  if there exists a constant  $M > 0$  such that for all  $f_1, \dots, f_n \in X$

$$\left( \sum_{k=1}^n \|f_k\|_X^p \right)^{\frac{1}{p}} \leq M \left\| \left( \sum_{k=1}^n |f_k|^p \right)^{\frac{1}{p}} \right\|_X \quad \text{if } 1 \leq p < \infty,$$

or  $\max_{1 \leq k \leq n} \|f_k\|_X \leq M \|\sup_{1 \leq k \leq n} |f_k|\|_X$  if  $p = \infty$ .

**Remark 1.** Note that the notions of  $p$ -convexity, respectively  $p$ -concavity are closely related to the notions of upper  $p$ -estimate (strong  $\ell_p$ -composition property), respectively lower  $p$ -estimate (strong  $\ell_p$ -decomposition property) as can be found in [29].

Now we reduce some examples of  $p$ -convex and respectively  $p$ -concave BFS. Let  $R^n$  be the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$  and  $\Omega$  be a Lebesgue measurable subset in  $R^n$  and  $|x| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}$ . The Lebesgue measure of a set  $\Omega$  will be denoted by  $|\Omega|$ . It is well known that  $|B(0, 1)| = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$ , where  $B(0, 1) = \{x : x \in R^n ; |x| < 1\}$ .

**Example 1.** Let  $1 \leq q \leq \infty$  and  $X = L_q$ . Then the space  $L_q$  is  $p$ -convex ( $p$ -concave) BFS if and only if  $1 \leq p \leq q \leq \infty$  ( $1 \leq q \leq p \leq \infty$ .)

The proof implies from usual Minkowski inequality in Lebesgue spaces.

**Example 2.** The following lemma shows that the variable Lebesgue spaces  $L_{q(y)}(\Omega)$  is  $p$ -convex BFS.

**Lemma 1** [1]. Let  $1 \leq p \leq q(x) \leq \bar{q} < \infty$  for all  $y \in \Omega_2 \subset R^m$ . Then the inequality

$$\| \|f\|_{L_p(\Omega_1)} \|_{L_{q(\cdot)}(\Omega_2)} \leq C_{p,q} \| \|f\|_{L_{q(\cdot)}(\Omega_2)} \|_{L_p(\Omega_1)}$$

is valid, where

$$C_{p,q} = \left( \|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty + p \left( \frac{1}{\underline{q}} - \frac{1}{\bar{q}} \right) \right) (\|\chi_{\Delta_1}\|_\infty + \|\chi_{\Delta_2}\|_\infty), \quad \underline{q} = \operatorname{ess\,inf}_{\Omega_2} q(x),$$

$$\bar{q} = \operatorname{ess\,sup}_{\Omega_2} q(x), \quad \Delta_1 = \{(x, y) \in \Omega_1 \times \Omega_2 : q(y) = p\}, \quad \Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1$$

and  $f : \Omega_1 \times \Omega_2 \rightarrow R$  is any measurable function such that

$$\| \|f\|_{L_p(\Omega_1)} \|_{L_{q(\cdot)}(\Omega_2)} = \inf \left\{ \mu > 0 : \int_{\Omega_2} \left( \frac{\|f(\cdot, y)\|_{L_p(\Omega_1)}}{\mu} \right)^{q(y)} dy \leq 1 \right\} < \infty$$

and  $\|f(\cdot, y)\|_{L_p(\Omega_1)} = \left( \int_{\Omega_1} |f(x, y)|^p dx \right)^{1/p}$ .

Analogously, if  $1 \leq q(x) \leq p < \infty$ , then  $L_{q(x)}(\Omega)$  is  $p$ -concave BFS.

**Definition 3** [18, 37]. Let  $\Omega \subset R^n$  be a Lebesgue measurable set. A real function  $\varphi : \Omega \times [0, \infty) \mapsto [0, \infty)$  is called a generalized  $\varphi$ -function if it satisfies:

(a)  $\varphi(x, \cdot)$  is a  $\varphi$ -function for all  $x \in \Omega$ , i.e.,  $\varphi(x, \cdot) : [0, \infty) \mapsto [0, \infty)$  is convex and satisfies  $\varphi(x, 0) = 0, \lim_{t \rightarrow +0} \varphi(x, t) = 0$ ;

(b)  $\psi : x \mapsto \varphi(x, t)$  is measurable for all  $t \geq 0$ .

If  $\varphi$  is a generalized  $\varphi$ -function on  $\Omega$ , we shortly write  $\varphi \in \Phi$ .

**Definition 4** [18, 37]. Let  $\varphi \in \Phi$  and be  $\rho_\varphi$  defined by the expression

$$\rho_\varphi(f) := \int_{\Omega} \varphi(y, |f(y)|) dy \quad \text{for all } f \in L_0(\Omega).$$

We put  $L_\varphi = \{f \in L_0(\Omega) : \rho_\varphi(\lambda_0 f) < \infty \text{ for some } \lambda_0 > 0\}$  and

$$\|f\|_{L_\varphi} = \inf \left\{ \lambda > 0 : \rho_\varphi \left( \frac{f}{\lambda} \right) \leq 1 \right\}.$$

The space  $L_\varphi$  is called Musielak–Orlicz space.

Let  $\omega$  be a weight function on  $\Omega$ , i.e.,  $\omega$  is a nonnegative, almost everywhere positive function on  $\Omega$ . In this work we considered the weighted Musielak–Orlicz spaces. We denote

$$L_{\varphi, \omega} = \{f \in L_0(\Omega) : f\omega \in L_\varphi\}.$$

It is obvious that the norm in this spaces is given by

$$\|f\|_{L_{\varphi, \omega}} = \|f\omega\|_{L_\varphi}.$$

**Remark 2.** Let  $\varphi(x, t) = t^{q(x)}$  in the Definition 4, where  $1 \leq q(x) < \infty$  and  $x \in \Omega$ . Then we have the definition of variable exponent weighted Lebesgue spaces  $L_{q(x)}(\Omega)$  (see [18]).

**Example 3.** The following lemma shows that the Musielak–Orlicz spaces  $L_\varphi$  is  $p$ -convex BFS.

**Lemma 2** [6]. Let  $\Omega_1 \subset R^n$  and  $\Omega_2 \subset R^m$ . Let  $(x, t) \in \Omega_1 \times [0, \infty)$  and  $\varphi(x, t^{1/p}) \in \Phi$  for some  $1 \leq p < \infty$ . Suppose  $f: \Omega_1 \times \Omega_2 \mapsto R$ . Then the inequality

$$\| \|f(x, \cdot)\|_{L_p(\Omega_2)} \|_{L_\varphi} \leq 2^{1/p} \| \|f(\cdot, y)\|_{L_\varphi} \|_{L_p(\Omega_2)}$$

is valid.

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces and etc. is  $p$ -convex ( $p$ -concave) BFS. Now we reduce more general result connected with Minkowski's integral inequality.

Let  $X$  and  $Y$  be BFS's on  $(\Omega_1, \mu)$  and  $(\Omega_2, \nu)$  respectively. By  $X[Y]$  and  $Y[X]$  we denote the spaces with mixed norm and consisting of all functions  $g \in L_0(\Omega_1 \times \Omega_2, \mu \times \nu)$  such that  $\|g(x, \cdot)\|_Y \in X$  and  $\|g(\cdot, y)\|_X \in Y$ . The norms in this spaces is defined as

$$\|g\|_{X[Y]} = \| \|g(x, \cdot)\|_Y \|_X, \quad \|g\|_{Y[X]} = \| \|g(\cdot, y)\|_X \|_Y.$$

**Theorem 1** [42]. Let  $X$  and  $Y$  be BFS's with the Fatou property. Then the generalized Minkowski integral inequality

$$\|f\|_{X[Y]} \leq M \|f\|_{Y[X]}$$

holds for all measurable functions  $f(x, y)$  if and only if there exists  $1 \leq p \leq \infty$  such that  $X$  is  $p$ -convex and  $Y$  is  $p$ -concave.

It is known that  $X[Y]$  and  $Y[X]$  are BFS's on  $\Omega_1 \times \Omega_2$  (see [29]).

**3. Main results.** We consider the multidimensional Hardy type operator and its dual operator

$$Hf(x) = \int_{|y|<|x|} f(y) dy \quad \text{and} \quad H^*f(x) = \int_{|y|>|x|} f(y) dy,$$

where  $f \geq 0$  and  $x \in R^n$ .

Now we prove a two-weight criterion for multidimensional Hardy type operator acting from the  $p$ -concave weighted BFS to weighted Lebesgue spaces.

**Theorem 2.** Let  $v(x)$  and  $w(x)$  are weights on  $R^n$ . Suppose that  $X_w$  be a  $p$ -convex weighted BFS's for  $1 \leq p < \infty$  on  $R^n$ . Then the inequality

$$\|Hf\|_{X_w} \leq C \|f\|_{L_{p,v}} \tag{1}$$

holds for every  $f \geq 0$  and for all  $\alpha \in (0, 1)$  if and only if

$$A(\alpha) = \sup_{t>0} \left( \int_{|y|<t} [v(y)]^{-p'} dy \right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}}(\cdot) \left( \int_{|y|<|\cdot|} [v(y)]^{-p'} dy \right)^{\frac{1-\alpha}{p'}} \right\|_{X_w} < \infty. \tag{2}$$

Moreover, if  $C > 0$  is the best possible constant in (1), then

$$\sup_{0<\alpha<1} \frac{p' A(\alpha)}{(1-\alpha) \left[ \left( \frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p}} \leq C \leq M \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

**Proof. Sufficiency.** Passing to the polar coordinates, we have

$$h(y) = \left( \int_{|z|<|y|} [v(z)]^{-p'} dz \right)^{\frac{\alpha}{p'}} = \left( \int_0^{|y|} s^{n-1} \left( \int_{|\xi|=1} [v(s\xi)]^{-p'} d\xi \right) ds \right)^{\frac{\alpha}{p'}}$$

where  $d\xi$  is the surface element on the unit sphere. Obviously,  $h(y) = h(|y|)$ , i.e.,  $h(y)$  is a radial function.

Applying Hölder's inequality for  $L_p(\mathbb{R}^n)$  spaces and after some standard transformations, we obtain

$$\begin{aligned} \|Hf\|_{X_w} &= \left\| w(\cdot) \int_{|y|<|\cdot|} f(y) dy \right\|_X = \left\| w(\cdot) \int_{|y|<|\cdot|} [f(y)h(y)v(y)] [h(y)v(y)]^{-1} dy \right\|_X \leq \\ &\leq \left\| w(\cdot) \|f h v\|_{L_p(|y|<|\cdot|)} \left\| [h v]^{-1} \right\|_{L_{p'}(|y|<|\cdot|)} \right\|_X = \\ &= \left\| \left\| w(\cdot) f h v \chi_{\{|y|<|\cdot|\}}(y) \right\|_{L_p(|y|<|\cdot|)} \left\| [h v]^{-1} \right\|_{L_{p'}(|y|<|\cdot|)} \right\|_X = \\ &= \left\| w f h v \chi_{\{|\cdot|<|x|\}}(\cdot) \left\| [h v]^{-1} \right\|_{L_{p'}(|\cdot|<|x|)} \right\|_{X[L_p]} . \end{aligned}$$

Applying Theorem 1, we get

$$\begin{aligned} &\left\| w f h v \chi_{\{|\cdot|<|x|\}}(\cdot) \left\| [h v]^{-1} \right\|_{L_{p'}(|\cdot|<|x|)} \right\|_{X[L_p]} \leq \\ &\leq M \left\| w f h v \chi_{\{|\cdot|<|x|\}}(\cdot) \left\| [h v]^{-1} \right\|_{L_{p'}(|\cdot|<|x|)} \right\|_{L_p[X]} = \\ &= M \left\| \left\| w(\cdot) f h v \chi_{\{|y|<|\cdot|\}}(y) \right\|_{L_p(|y|<|\cdot|)} \left\| [h v]^{-1} \right\|_{L_{p'}(|y|<|\cdot|)} \right\|_X = \\ &= M \left\| f h v \left\| w(\cdot) \chi_{\{|y|<|\cdot|\}}(y) \right\|_{L_p(|y|<|\cdot|)} \left\| [h v]^{-1} \right\|_{L_{p'}(|y|<|\cdot|)} \right\|_X . \end{aligned}$$

By switching to polar coordinates and after some calculations, we have

$$\begin{aligned} \left\| [h v]^{-1} \right\|_{L_{p'}(|y|<|x|)} &= \left( \int_{|y|<|x|} [h(|y|) v(y)]^{-p'} dy \right)^{1/p'} = \\ &= \left( \int_0^{|x|} r^{n-1} [h(r)]^{-p'} \left[ \int_{|\xi|=1} [v(r\xi)]^{-p'} d\xi \right] dr \right)^{1/p'} = \end{aligned}$$

$$\begin{aligned}
&= \left( \int_0^{|x|} \left[ \int_0^r s^{n-1} \left( \int_{|\xi|=1} [v(s\xi)]^{-p'} d\xi \right) ds \right]^{-\alpha} \left( \int_{|\xi|=1} [v(r\xi)]^{-p'} d\xi \right) r^{n-1} dr \right)^{1/p'} \\
&= \frac{1}{(1-\alpha)^{1/p'}} \left( \int_0^{|x|} \frac{d}{dr} \left\{ \left( \int_0^r s^{n-1} \left( \int_{|\xi|=1} [v(s\xi)]^{-p'} d\xi \right) ds \right)^{1-\alpha} \right\} dr \right)^{1/p'} = \\
&= \frac{1}{(1-\alpha)^{1/p'}} \left( \int_0^{|x|} s^{n-1} \left( \int_{|\xi|=1} [v(s\xi)]^{-p'} d\xi \right) ds \right)^{\frac{1-\alpha}{p'}} = \\
&= \frac{1}{(1-\alpha)^{1/p'}} \left( \int_{|z|<|x|} [v(z)]^{-p'} dz \right)^{\frac{1-\alpha}{p'}}.
\end{aligned}$$

Therefore from condition (2), we obtain

$$\begin{aligned}
&\|fhv\|_{w(\cdot)\chi_{\{|y|<|\cdot|\}}(y)} \| [hv]^{-1} \|_{L_{p'}(|y|<|\cdot|)} \| \|_{X} \| \|_{L_p} = \\
&= \frac{1}{(1-\alpha)^{1/p'}} \left\| f v \left[ h \left\| \chi_{\{|\cdot|>|y|\}} \left( \int_{|z|<|\cdot|} [v(z)]^{-p'} dz \right)^{\frac{1-\alpha}{p'}} \right\| \right] \right\|_{X_w} \| \|_{L_p} \leq \frac{A(\alpha)}{(1-\alpha)^{1/p'}} \|f v\|_{L_p}.
\end{aligned}$$

Thus

$$\|Hf\|_{X_w} \leq M \frac{A(\alpha)}{(1-\alpha)^{1/p'}} \|f\|_{L_{p,v}} \quad \text{for all } \alpha \in (0, 1).$$

*Necessity.* Let  $f \in L_{p,v}(R^n)$ ,  $f \geq 0$ , and inequality (1) is valid. We choose the test function as

$$f(x) = \frac{p'}{1-\alpha} [g(t)]^{-\frac{\alpha}{p'} - \frac{1}{p}} v^{-p'}(x) \chi_{\{|x|<t\}}(x) + [g(|x|)]^{-\frac{\alpha}{p'} - \frac{1}{p}} v^{-p'}(x) \chi_{\{|x|>t\}}(x),$$

where  $t > 0$  is a fixed number and

$$g(t) = \int_{|y|<t} v^{-p'}(y) dy = \int_0^t s^{n-1} \left( \int_{|\eta|=1} v^{-p'}(s\eta) d\eta \right) ds.$$

It is obvious that  $\frac{dg}{dt} = t^{n-1} \int_{|\eta|=1} v^{-p'}(t\eta) d\eta$ . Again by switching to polar coordinates, from the right-hand side of inequality (1) we get

$$\begin{aligned}
\|f\|_{L_{p,v}} &= \left[ \int_{|x|<t} \left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{-\alpha(p-1)-1} v^{-p'}(x) dx + \int_{|x|>t} [g(|x|)]^{-\alpha(p-1)-1} v^{-p'}(x) dx \right]^{1/p} = \\
&= \left[ \left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} + \int_t^\infty r^{n-1} [g(r)]^{-\alpha(p-1)-1} \left( \int_{|\xi|=1} v^{-p'}(r\xi) d\xi \right) dr \right]^{1/p} = \\
&= \left[ \left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} - \frac{1}{\alpha(p-1)} \int_t^\infty \frac{d}{dr} [g(r)]^{-\alpha(p-1)} dr \right]^{1/p} = \\
&= \left[ \left(\frac{p'}{1-\alpha}\right)^p [g(t)]^{\alpha(1-p)} + \frac{1}{\alpha(p-1)} \left\{ [g(t)]^{-\alpha(p-1)} - \left[ \int_{R^n} v^{-p'}(y) dy \right]^{-\alpha(p-1)} \right\} \right]^{1/p} \leq \\
&\leq \left[ \left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} [g(t)]^{-\alpha/p'} = \left[ \left(\frac{p'}{1-\alpha}\right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p} [h(t)]^{-1}.
\end{aligned}$$

After some calculations, from the left-hand side of inequality (1), we have

$$\begin{aligned}
\|Hf\|_{X_w} &= \left\| \int_{|y|<|\cdot|} f(y) dy \right\|_{X_w} \geq \left\| \chi_{\{|\cdot|>t\}} \int_{|y|<|\cdot|} f(y) dy \right\|_{X_w} = \\
&= \left\| \chi_{\{|\cdot|>t\}} \left( \frac{p'}{1-\alpha} \int_{|y|<t} [g(t)]^{-\frac{\alpha}{p'}-\frac{1}{p}} v^{-p'}(y) dy + \int_{t<|y|<|\cdot|} [g(|y|)]^{-\frac{\alpha}{p'}-\frac{1}{p}} v^{-p'}(y) dy \right) \right\|_{X_w} = \\
&= \left\| \chi_{\{|\cdot|>t\}} \left( \frac{p'}{1-\alpha} [g(t)]^{\frac{1-\alpha}{p'}} + \int_t^{|\cdot|} r^{n-1} [g(r)]^{-\frac{\alpha}{p'}-\frac{1}{p}} \left( \int_{|\eta|=1} v^{-p'}(r\eta) d\eta \right) dr \right) \right\|_{X_w} = \\
&= \left\| \chi_{\{|\cdot|>t\}} \left( \frac{p'}{1-\alpha} [g(t)]^{\frac{1-\alpha}{p'}} + \frac{p'}{1-\alpha} \int_t^{|\cdot|} \frac{d}{dr} [g(r)]^{\frac{1-\alpha}{p'}} dr \right) \right\|_{X_w} = \\
&= \left\| \chi_{\{|\cdot|>t\}} \left[ \frac{p'}{1-\alpha} [g(t)]^{\frac{1-\alpha}{p'}} + \frac{p'}{1-\alpha} \left( [g(|\cdot|)]^{\frac{1-\alpha}{p'}} - [g(t)]^{\frac{1-\alpha}{p'}} \right) \right] \right\|_{X_w} = \\
&= \frac{p'}{1-\alpha} \left\| \chi_{\{|\cdot|>t\}} [g(\cdot)]^{\frac{1-\alpha}{p'}} \right\|_{X_w}.
\end{aligned}$$

Hence, this implies that

$$\frac{p'}{1-\alpha} \left[ \left( \frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{-1/p} [g(t)]^{\alpha/p'} \left\| \chi_{\{|\cdot|>t\}} [g(\cdot)]^{\frac{1-\alpha}{p'}} \right\|_{X_w} \leq C,$$

i.e.,

$$\frac{p' A(\alpha)}{(1-\alpha) \left[ \left( \frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p}} \leq C \quad \text{for all } \alpha \in (0, 1).$$

Theorem 2 is proved.

For the dual operator, the below stated theorem is proved analogously.

**Theorem 3.** *Let  $v(x)$  and  $w(x)$  are weights on  $R^n$ . Suppose that  $X_w$  be a  $p$ -convex weighted BFS's for  $1 \leq p < \infty$  on  $R^n$ . Then the inequality*

$$\|H^* f\|_{X_w} \leq C \|f\|_{L_{p,v}} \quad (3)$$

holds for every  $f \geq 0$  and for all  $\gamma \in (0, 1)$  if and only if

$$B(\gamma) = \sup_{t>0} \left( \int_{|y|>t} [v(y)]^{-p'} dy \right)^{\frac{\gamma}{p'}} \left\| \chi_{\{|z|<t\}}(\cdot) \left( \int_{|y|>|\cdot|} [v(y)]^{-p'} dy \right)^{\frac{1-\gamma}{p'}} \right\|_{X_w} < \infty.$$

Moreover, if  $C > 0$  is the best possible constant in (3), then

$$\sup_{0<\gamma<1} \frac{p' B(\gamma)}{(1-\gamma) \left[ \left( \frac{p'}{1-\gamma} \right)^p + \frac{1}{\gamma(p-1)} \right]^{1/p}} \leq C \leq M \inf_{0<\gamma<1} \frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

**Corollary 1.** *Note that Theorems 2 and 3 in the case  $X_w = L_{\varphi,w}$ ,  $\varphi(x, t^{1/p}) \in \Phi$  for some  $1 \leq p < \infty$ ,  $x \in R^n$  was proved in [6]. In the case  $X_w = L_{q,w}$ ,  $1 < p \leq q < \infty$ , for  $x \in (0, \infty)$ ,  $\alpha = \frac{s-1}{p-1}$  and  $s \in (1, p)$  Theorems 2 and 3 was proved in [44]. For  $x \in R^n$  in the case  $X_w = L_{q(x),w}$  and  $1 < p \leq q(x) \leq \text{ess sup}_{x \in R^n} q(x) < \infty$  Theorems 2 and 3 was proved in [3] (see also [2]).*

**Remark 3.** In the case  $n = 1$ ,  $X_w = L_{q,w}$ ,  $1 < p \leq q \leq \infty$ , at  $x \in (0, \infty)$ , for classical Lebesgue spaces the various variants of Theorems 2 and 3 were proved in [13, 20, 27, 28, 35, 36, 43] and etc. In particular, in the Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [16, 17, 19, 21, 26, 33, 34] and etc. For  $X_w = L_{q(x),w}$ ,  $1 < p \leq q(x) \leq \text{ess sup}_{x \in [0,1]} q(x) < \infty$  and  $x \in [0, 1]$  the two-weighted criterion for one-dimensional Hardy operator was proved in [26]. Also, other type two-weighted criterion for multidimensional Hardy type operator in the case  $X_w = L_{q(x),w}$ ,  $1 < p \leq q(x) \leq \text{ess sup}_{x \in R^n} q(x) < \infty$  and  $x \in R^n$  was proved in [33] (see also [34]). In the papers [12] and [41] the inequalities of modular type for more general operators was proved. Also, in [14] the Hardy type inequalities with special power-type weights in Orlicz spaces was proved.

**4. Application.** Now we consider the multidimensional geometric mean operator defined as

$$Gf(x) = \exp \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} \ln f(y) dy \right),$$

where  $f > 0$  and  $|B(0, |x|)| = |B(0, 1)| |x|^n$ . It is obvious that  $G(f_1 \cdot f_2)(x) = Gf_1(x) \cdot Gf_2(x)$ .

We formulate a two-weighted criterion on boundedness of multidimensional geometric mean operator in weighted Musielak–Orlicz spaces.

**Theorem 4.** Let  $\varphi(x, t^{1/p}) \in \Phi$  for some  $0 < p < \infty$  and  $x \in R^n$ . Suppose that  $v(x)$  and  $w(x)$  are weight functions on  $R^n$ . Then the inequality

$$\|Gf\|_{L_{\varphi, w}} \leq C \|f\|_{L_{p, v}} \tag{4}$$

holds for every  $f > 0$  and for all  $s \in (1, p)$  if and only if

$$D(s) = \sup_{t>0} |B(0, t)|^{\frac{s-1}{p}} \left\| \frac{\chi_{\{|z|>t\}}(\cdot)}{|B(0, |\cdot|)|^{s/p}} \exp \left( \frac{1}{|B(0, |\cdot|)|} \int_{B(0, |\cdot|)} \ln \frac{1}{v(y)} dy \right) \right\|_{L_{\varphi, w}} < \infty. \tag{5}$$

Moreover, if  $C > 0$  is the best possible constant in (4), then

$$\sup_{s>1} \frac{e^{s/p}}{\left(e^s + \frac{1}{s-1}\right)^{1/p}} D(s) \leq C \leq 2^{1/p} \inf_{s>1} e^{\frac{s-1}{p}} D(s).$$

**Proof.** Let  $\alpha = \frac{s-1}{p-1}$ , where  $1 < s < p$ . We replace  $f$  with  $f^\beta$ ,  $v$  with  $v^\beta$ ,  $w$  with  $\frac{w^\beta(x)}{|B(0, |x|)|}$ ,  $0 < \beta < p$ , and  $p$  with  $\frac{p}{\beta}$  and  $\varphi(x, t)$  with  $\varphi(x, t^{1/\beta})$  in (1), (2), we find that for  $1 < s < \frac{p}{\beta}$

$$\begin{aligned} \left\| \frac{w^\beta}{|B(0, |\cdot|)|} H(f^\beta) \right\|_{L_{\varphi(\cdot, t^{1/\beta})}} &= \left\| \left( \frac{1}{|B(0, |\cdot|)|} \int_{B(0, |\cdot|)} f^\beta(y) dy \right)^{1/\beta} \right\|_{L_{\varphi, w}(R^n)}^\beta \leq \\ &\leq C_\beta \left( \int_{R^n} [f(y)v(y)]^p dy \right)^{\beta/p}. \end{aligned}$$

Then the inequality

$$\left\| \left( \frac{1}{|B(0, |\cdot|)|} \int_{B(0, |\cdot|)} f^\beta(y) dy \right)^{1/\beta} \right\|_{L_{\varphi, w}(R^n)} \leq C_\beta^{1/\beta} \left( \int_{R^n} [f(y)v(y)]^p dy \right)^{1/p} \tag{6}$$

holds if and only if

$$\begin{aligned}
& A\left(\frac{s-1}{p-1}\right) = \\
& = \left[ \sup_{t>0} \left( \int_{|y|<t} [v(y)]^{-\frac{\beta p}{p-\beta}} dy \right)^{\frac{s-1}{p}} \left\| \left( \frac{\chi_{\{|z|>t\}}(\cdot)}{|B(0,|\cdot|)|^{p-\beta s}} \int_{|y|<|\cdot|} [v(y)]^{-\frac{\beta p}{p-\beta}} dy \right)^{\frac{p-\beta s}{\beta p}} \right\|_{L_{\varphi,w}} \right]^{\beta} = \\
& = B^{\beta}(s, \beta) < \infty
\end{aligned}$$

and

$$\sup_{1<s<p/\beta} \left[ \frac{\left(\frac{p}{p-s\beta}\right)^{p/\beta}}{\left(\frac{p}{p-s\beta}\right)^{p/\beta} + \frac{1}{s-1}} \right]^{\beta/p} B^{\beta}(s, \beta) \leq C_{\beta} \leq 2^{\beta/p} \inf_{1<s<p/\beta} \left(\frac{p-\beta}{p-s\beta}\right)^{\frac{p-\beta}{p}} B^{\beta}(s, \beta). \quad (7)$$

By the L'Hospital rule, we get

$$\begin{aligned}
& \lim_{\beta \rightarrow +0} \left( \frac{1}{|B(0,|x|)|^{\frac{p}{p-\beta s}}} \int_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}} dy \right)^{\frac{p-\beta s}{\beta p}} = \\
& = \lim_{\beta \rightarrow +0} \exp \left[ \frac{p \ln \frac{1}{|B(0,|x|)|} + (p-\beta s) \ln \left( \int_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}} dy \right)}{p\beta} \right] = \\
& = \lim_{\beta \rightarrow +0} \exp \left[ -\frac{s}{p} \ln \left( \int_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}} dy \right) + \right. \\
& \left. + \frac{(p-\beta s) \left(\frac{p}{p-\beta}\right)^2 \int_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}} \ln \frac{1}{v(y)} dy}{p \int_{|y|<|x|} [v(y)]^{-\frac{\beta p}{p-\beta}} dy} \right] = \\
& = \exp \left[ \frac{s}{p} \ln \frac{1}{|B(0,|x|)|} + \frac{\int_{|y|<|x|} \ln \frac{1}{v(y)} dy}{|B(0,|x|)|} \right] =
\end{aligned}$$

$$= \frac{1}{|B(0, |x|)|^{s/p}} \exp \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} \ln \frac{1}{v(y)} dy \right).$$

Therefore

$$\begin{aligned} \lim_{\beta \rightarrow +0} B(s, \beta) &= \sup_{t > 0} |B(0, t)|^{\frac{s-1}{p}} \left\| \frac{\chi_{\{|z| > t\}}(\cdot)}{|B(0, |\cdot|)|^{s/p}} \exp \left( \frac{1}{|B(0, |\cdot|)|} \int_{B(0, |\cdot|)} \ln \frac{1}{v(y)} dy \right) \right\|_{L_{\varphi, w}} = \\ &= D(s) < \infty \end{aligned}$$

and

$$\sup_{s > 1} \frac{e^{s/p}}{\left(e^s + \frac{1}{s-1}\right)^{1/p}} D(s) \leq \lim_{\beta \rightarrow +0} C_{\beta}^{1/\beta} \leq 2^{1/p} \inf_{s > 1} e^{(s-1)/p} D(s). \tag{8}$$

Further, we have

$$\lim_{\beta \rightarrow +0} \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f^{\beta}(y) dy \right)^{1/\beta} = \exp \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} \ln f(y) dy \right) = Gf(x).$$

From (7) it follows that  $\lim_{\beta \rightarrow +0} C_{\beta} = 1$ , and according to (5) and (8)  $\lim_{\beta \rightarrow +0} C_{\beta}^{1/\beta} = C < \infty$ . Therefore the inequality (8) is valid. Moreover, from (6) for  $\beta \rightarrow +0$  we obtain that

$$\|Gf\|_{L_{q(\cdot), w}(R^n)} \leq C \|f\|_{L_{p, v}(R^n)}$$

and by (8)

$$\sup_{s > 1} \frac{e^{s/p}}{\left(e^s + \frac{1}{s-1}\right)^{1/p}} D(s) \leq C \leq 2^{1/p} \inf_{s > 1} e^{(s-1)/p} D(s).$$

Theorem 4 is proved.

**Remark 4.** Let  $\varphi(x, t) = t^q$  and  $n = 1$ . Note that the simplest case of the condition  $\varphi(y, Ct) \leq C^q \varphi(y, t)$  with  $v = w = 1$  and  $p = q = 1$  was considered in [20] and in [25]. Later this inequality was generalized in various ways by many authors in [15, 22–24, 31, 38–40, 44] and etc.

**Corollary 2.** Let  $\varphi(x, t) = t^q$ ,  $0 < p \leq q < \infty$  and  $f$  be a positive function on  $R^n$ . Then

$$\left( \int_{R^n} [Gf(x)]^q |x|^{\delta q} dx \right)^{1/q} \leq C \left( \int_{R^n} f^p(x) |x|^{\mu p} dx \right)^{1/p} \tag{9}$$

holds with a finite constant  $C$  if and only if

$$\delta + \frac{n}{q} = \frac{\mu}{n} + \frac{n}{p}$$

and the best constant  $C$  has the following condition:

$$\sqrt[q]{\frac{p}{nq}} e^{\mu/n^2} |B(0, 1)|^{1/q-1/p} \sup_{s>1} \frac{e^{s/p} (s-1)^{1/p-1/q}}{[(s-1)e^s + 1]^{1/p}} \leq C \leq \frac{|B(0, 1)|^{1/q-1/p} e^{\mu/n^2+1/q}}{\sqrt[q]{n}}.$$

**Remark 5.** Let  $\varphi(x, t) = t^q$  and  $q = p$ . Then inequality (9) is sharp with the constant

$$C = \frac{e^{\mu/n^2+1/p}}{\sqrt[q]{n}}.$$

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