UDC 512.5

V. A. Faĭziev (Tver State Agricultural Academy, Russia),
P. K. Sahoo (Univ. Louisville, USA)

ON STABILITY OF CAUCHY EQUATION ON SOLVABLE GROUPS*

ПРО СТІЙКІСТЬ РІВНЯННЯ КОШІ НА РОЗВ'ЯЗНИХ ГРУПАХ

The notion of (ψ,γ)-stability was introduced in [*Faĭziev V. A., Rassias Th. M., Sahoo P. K.* The space of (ψ,γ)-additive mappings on semigroups // Trans. Amer. Math. Soc. – 2002. – **354**. – P. 4455 – 4472]. It was shown that the Cauchy equation $f(xy) = f(x) + f(y)$ is (ψ,γ)-stable on any Abelian group as well as any meta-Abelian group. In [*Faĭziev V. A., Sahoo P. K.* On (ψ,γ)-stability of Cauchy equation on some noncommutative groups // Publ. Math. Debrecen. – 2009. – **75**. – P. 67 – 83], it was proved that the Cauchy equation is (ψ,γ)-stable on step-two solvable groups and step-three nilpotent groups. In our paper, we prove a more general result and show that the Cauchy equation is (ψ,γ)-stable on solvable groups.

Поняття (ψ,γ)-стійкості введено в роботі [*Faĭziev V. A., Rassias Th. M., Sahoo P. K.* The space of (ψ,γ)-additive mappings on semigroups // Trans. Amer. Math. Soc. – 2002. – **354**. – P. 4455 – 4472]. Було показано, що рівняння Коші $f(xy) = f(x) + f(y)$ є (ψ,γ)-стійким як на довільній абелевій групі, так і на довільній метабелевій групі. В роботі [*Faĭziev V. A., Sahoo P. K.* On (ψ,γ)-stability of Cauchy equation on some noncommutative groups // Publ. Math. Debrecen. – 2009. – **75**. – P. 67 – 83] доведено, що рівняння Коші є (ψ,γ)-стійким як на двоступеневих розв'язних групах, так і на триступеневих нільпотентних групах. В нашій роботі доведено більш загальний результат і показано, що рівняння Коші є (ψ,γ)-стійким на розв'язних групах.

1. Introduction. In 1940, S.M. Ulam [11] posed the following fundamental problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f: G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T: G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? Interested reader should see S. M. Ulam [11] for a discussion of such problems, as well as D. H. Hyers [8], Th. M. Rassias [9], J. Aczél and J. Dhombres [1], G. L. Forti [7], and P. K. Sahoo and Pl. Kannappan [10]. The first affirmative answer to this problem was given by D. H. Hyers [8] in 1941.

On a group G, the Cauchy functional equation $f(x+y) = f(x) + f(y)$ takes the form $f(xy) = f(x) + f(y)$ for all $x, y \in G$. In connection with Hyers' result the following question arises. Let G be an arbitrary group and let a mapping $f: G \to \mathbb{R}$ (the set of reals) be such that the set, D, defined by $\{f(xy) - f(x) - f(y) \,|\, x, y \in G\}$ is bounded. Is it true that there is a mapping $T: G \to \mathbb{R}$ that satisfies $T(xy) - T(x) - T(y) = 0$ for all $x, y \in G$, and the set $\{T(x) - f(x) \,|\, x \in G\}$ is bounded. A negative answer was given in 1987 by G.L. Forti [6]. He constructed a real-valued function f on the free group $\mathcal{F}_2$ of rank two (and also on a free semigroup $\mathcal{S}_2$ of rank two) such that the set D is bounded but for any additive function T, the function $f(x) - T(x)$ is not bounded. It is worth pointing out that in 1987, Faĭziev in [2] gave a description of all such functions f on the free product of semigroups $A * B$. In [3], it was established that the Cauchy functional equation is not stable on the free product $A * B$ of groups A and B unless $A = B = \mathbb{Z}_2$.

* This paper was partially supported by an IRIG grant from the Office of the VP for Research, and an A&S grant from the College Arts and Sciences, University of Louisville.

The first paper to extend Hyers' result to a class nonabelian groups and semigroups was [2]. The notion of (ψ, γ)-stability of the Cauchy functional equation was introduced in [5]. In [5] among other results, it was proven that the Cauchy functional equation $f(xy) = f(x) + f(y)$ is (ψ, γ)-stable on any abelian group as well as any meta-Abelian (step-two nilpotent) group. It was also shown that any arbitrary group A can be embedded into a group G, where the Cauchy functional equation is (ψ, γ)-stable. In [4], the (ψ, γ)-stability of the Cauchy functional equation on step-two solvable groups and step-three nilpotent groups were treated. In this paper, we show that the Cauchy functional equation is (ψ, γ)-stable on solvable groups.

2. Definitions and notations. We introduce some relevant definitions and notations that will be useful for the proof of the main result.

Definition 1. *A quasicharacter of a semigroup G is a real-valued function f on G such that the set $\{f(xy) - f(x) - f(y) | x, y \in G\}$ is bounded.*

Definition 2. *By a pseudocharacter of a group G we mean its quasicharacter f that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$.*

The set of quasicharacters of a group G is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KX(G)$. The subspace of $KX(G)$ consisting of pseudocharacters will be denoted by $PX(G)$ and the subspace consisting of real additive characters of the group G, will be denoted by $X(G)$. We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$.

Next we recall some notions from [5] that we need for this paper. Let $\mathbb{R}_0^+ = [0, \infty)$ be the set of nonnegative numbers and $\mathbb{R}^+ = (0, \infty)$ be the set of positive numbers. Let G be an arbitrary group. Throughout this paper, the function $\psi \colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is considered to be an increasing function satisfying the following three additional conditions:

(1) $\psi(t_1 t_2) \leq \psi(t_1)\, \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$,

(2) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$,

(3) $\lim_{n\to\infty} \dfrac{\psi(n)}{n} = 0, \quad n \in \mathbb{N}$.

Throughout this paper, by γ we will mean a function $\gamma \colon G \to \mathbb{R}_0^+$ satisfying the inequality $\gamma(xy) \leq$ $\leq \gamma(x) + \gamma(y) + d$ for all $x, y \in G$ and some real nonnegative constant d. It is obvious that for any $x \in G$ and for any $m \in \mathbb{N}$, the function γ satisfies the inequality

$$\gamma(x^m) \leq m\,\gamma(x) + (m - 1)\, d. \tag{1}$$

Definition 3. *Let G be an arbitrary group and E a Banach space. Further, let $\psi \colon \mathbb{R}_0^+ \to \mathbb{R}^+$ and $\gamma \colon G \to \mathbb{R}_0^+$ be the functions as described above. The mapping $f \colon G \to E$ is said to be a (ψ, γ)- quasiadditive mapping if there exists a $\theta \in \mathbb{R}^+$ such that*

$$\|f(xy) - f(x) - f(y)\| \leq \theta\, [\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x, y \in G \tag{2}$$

holds.

It is clear that the set of all (ψ, γ)-quasiadditive mappings from G to E is a real linear space relative to the usual operations. Let us denote it by $KAM_{\psi,\gamma}(G; E)$.

Definition 4. *Let $\varphi \colon G \to E$ be a mapping from the group G to a Banach space E. The mapping φ is said to be a (ψ, γ)-pseudoadditive mapping if it is a (ψ, γ)-quasiadditive mapping satisfying $\varphi(x^n) = n\, \varphi(x)$ for all $x \in G$ and for each $n \in \mathbb{Z}$.*

We denote the space of all (ψ,γ)-pseudoadditive mappings from a group G to a Banach space E by $PAM_{\psi,\gamma}(G;E)$. By $HOM(G;E)$ we mean the set of all homomorphisms from G to E. By $B_{\psi,\gamma}(G;E)$ we denote the linear space of functions from G to E over reals satisfying the relation:

$$\|f(x)\| \le c\,\psi(\gamma(x)) \qquad \text{for some } \ c>0 \ \text{ and for all } \ x\in G.$$

Definition 5. *The Cauchy functional equation*

$$f(xy) = f(x) + f(y) \quad \forall\, x,y \in G \tag{3}$$

is said to be (ψ,γ)-stable for the pair $(G;E)$ if for any $f: G \to E$ satisfying the functional inequality

$$\|\, f(xy) - f(x) - f(y) \,\| \le \theta\,[\psi(\gamma(x)) + \psi(\gamma(y))] \quad \forall x,y \in G, \tag{4}$$

there is a solution $g\colon G\to E$ of functional equation (3) *such that the function $f(x) - g(x)$ belongs to the space $B_{\psi,\gamma}(G;E)$.*

It was shown in [5] that the equation (3) is (ψ,γ)-stable for the pair $(G;E)$ if and only if $PAM_{\psi,\gamma}(G;E) = HOM(G;E)$. The following result is one of the main results in [5]. Let E_1 and E_2 be Banach spaces over reals. Then the equation (3) is (ψ,γ)-stable for the pair (G,E_1) if and only if it is (ψ,γ)-stable for the pair (G,E_2). In view of this result it is not important which Banach space is used on the range. Thus one may consider the (ψ,γ)-stability of the functional equation (3) on the pair $(G,\mathbb{R})$. Let us simplify the following notations: In the case $E=\mathbb{R}$ the spaces $KAM_{\psi,\gamma}(G;\mathbb{R})$, $PAM_{\psi,\gamma}(G;\mathbb{R})$, and $HOM(G;\mathbb{R})$ will be denoted by $KX_{\psi,\gamma}(G)$, $PX_{\psi,\gamma}(G)$, $X(G)$, respectively. Further, we will call a (ψ,γ)-additive map a (ψ,γ)-quasicharacter, and a (ψ,γ)-pseudoadditive map a (ψ,γ)-pseudocharacter. It is clear that if γ is a constant function then $PX_{\psi,\gamma}(G) = PX(G)$. If $f\in PX_{\psi,\gamma}(G)$, then it known (see [5]) that the (ψ,γ)-pseudocharacter satisfies (i) $f(xy)=f(yx)$ for any $x,y\in G$, and (ii) $f(xy) = f(x)+f(y)$ if $xy=yx$.

3. Some preliminary results.

Theorem 1. *Suppose G is a group, H a normal subgroup of G and A a subgroup of G. Let G be the semidirect product of A and H, that is $G = A\ltimes H$. If f belongs to $PX_{\psi,\gamma}(G)$, $f\big|_A \in X(A)$, and $f\big|_H \in X(H)$, then $f\in X(G)$.*

Proof. Let $a\in A$ and $v\in H$, then for any $n\in\mathbb{N}$ the following relation:

$$(av)^n = a^n v^{a^{n-1}} v^{a^{n-2}} \ldots v^a v, \tag{5}$$

holds, where v^{a^i} denotes the element $a^{-i}va^i$. The element $v^{a^{n-1}}v^{a^{n-1}}\ldots v^a v$ belongs to H and

$$v^{a^{n-1}}v^{a^{n-2}}\ldots v^a v = a^{-(n-1)}va^{(n-1)}a^{-(n-2)}va^{(n-2)}\ldots a^{-1}vav =$$

$$= a^{-(n-1)}vava\ldots vav. \tag{6}$$

Therefore

$$\gamma(v^{a^{n-1}}v^{a^{n-2}}\ldots v^a v) = \gamma(a^{-n}(av)^n) \le$$

$$\le \gamma(a^{-n}) + \gamma((av)^n) + d \le$$

$$\le n\,\gamma(a^{-1}) + (n-1)\,d + n\gamma(av) + (n-1)d + d \le$$

$$\leq n\,\gamma(a^{-1}) + (n-1)\,d + n\,[\,\gamma(a) + \gamma(v) + d\,] + (n-1)\,d + d \leq$$

$$\leq n\,\left[\gamma(a^{-1}) + \gamma(a) + \gamma(v)\right] + (3n-1)\,d$$

and

$$\psi\big(\gamma(v^{a^{n-1}} v^{a^{n-2}} \ldots v^a v)\big) \leq$$

$$\leq \psi\big(n\,[\gamma(a^{-1}) + \gamma(a) + \gamma(v)] + (3n-1)\,d\,\big) \leq$$

$$\leq \psi(n)\,\psi(\,\gamma(a^{-1}) + \gamma(a) + \gamma(v)\,) + \psi(n)\,\psi(3d).$$

Let $g = av$. Then for any $n \in \mathbb{N}$, since $f \in PX_{\psi,\gamma}(G)$, the following relation:

$$|f(g^n) - f(a^n) - f(v^{a^{n-1}} v^{a^{n-1}} \ldots v^a v)| \leq$$

$$\leq \theta\left[\psi(\gamma(a^n)) + \psi(\gamma(v^{a^{n-1}} v^{a^{n-1}} \ldots v^a v))\right] \leq$$

$$\leq \theta\left[\psi(n)\psi(\gamma(a) + d) + \psi(\gamma(v^{a^{n-1}} v^{a^{n-1}} \ldots v^a v))\right] \leq$$

$$\leq \theta\,\psi(n)\,\left[\psi(\gamma(a) + d) + \psi(\gamma(a^{-1}) + \gamma(a) + \gamma(v)) + \psi(3d)\right] \tag{7}$$

holds. Moreover, since, $f \in PX_{\psi,\gamma}(G)$, $f(xy) = f(yx)$ for any $x, y \in G$. Hence it follows that, for any $x, y \in G$, the relation $f(y^{-1}xy) = f(x)$ holds. This implies that f is invariant under inner automorphisms of group G. Therefore, since $f\big|_H \in X(H)$ and function f is invariant with respect to inner automorphisms of the group G, we get

$$f(v^{a^{n-1}} v^{a^{n-1}} \ldots v^a v) =$$

$$= f\left(v^{a^{n-1}}\right) + f\left(v^{a^{n-1}}\right) + \ldots + f\left(v^a\right) + f(v) = n\,f(v).$$

Therefore from (7) it follows

$$n\,|f(g) - f(a) - f(v)| \leq$$

$$\leq \theta\,\psi(n)\left[\psi(\gamma(a) + d) + \psi(\gamma(a^{-1}) + \gamma(a) + \gamma(v)) + \psi(3d)\right].$$

The latter relation implies

$$|f(g) - f(a) - f(v)| \leq$$

$$\leq \theta\,\frac{\psi(n)}{n}\,\left[\psi(\gamma(a) + d) + \psi(\gamma(a^{-1}) + \gamma(a) + \gamma(v)) + \psi(3d)\right] \tag{8}$$

for any $n \in \mathbb{N}$. This implies that

$$f(av) = f(a) + f(v) \quad \forall a \in A\ \forall v \in H. \tag{9}$$

Now if $a, b \in A$ and $u, v \in H$, then from (9) it follows that

$$f(avbu) = f(abv^b u) = f(ab) + f(v^b u) =$$

$$= f(a) + f(b) + f(v^b) + f(u) =$$

$$= f(a) + f(b) + f(v) + f(u) =$$

$$= f(a) + f(v) + f(b) + f(u) = f(av) + f(bu).$$

Hence $f \in X(G)$ and the theorem is proved.

The next theorem is a corollary of Theorem 1.

Theorem 2. *Let G be a group. Suppose that G is a semidirect product $G = A \ltimes H$ of its subgroups A and H, where H is normal. Suppose that the Cauchy functional equation is (ψ, γ)-stable on A and H. Then it is (ψ, γ)-stable on G.*

Proof. To prove this theorem it is enough to show that $PX_{\psi,\gamma}(G) = X(G)$. Since $X(G) \subseteq$ $\subseteq PX_{\psi,\gamma}(G)$, we only need to show that $PX_{\psi,\gamma}(G) \subseteq X(G)$. Hence, let $f \in PX_{\psi,\gamma}(G)$. Since by hypothesis the Cauchy functional equation is (ψ, γ)-stable on A and H, therefore $f\big|_A \in X(A)$, and $f\big|_H \in X(H)$. Hence by Theorem 1 we have $f \in X(G)$. This proves that $PX_{\psi,\gamma}(G) \subseteq X(G)$ and the Cauchy functional equation is (ψ, γ)-stable on G.

Theorem 2 is proved.

4. Stability on solvable groups. In this section, we prove the main result of this paper using Theorem 2.

Theorem 3. *The Cauchy equation is (ψ, γ)-stable on any solvable group.*

Proof. We prove this theorem by induction on the degree of solvability. The solvable groups of degree one are abelian groups. We know that for abelian groups the theorem is true (see [5]). Suppose that the theorem is true for all solvable groups having degree of solvability no more then k and let us prove it for solvable groups with degree $k + 1$.

Let G_2 be free solvable group of degree $k + 1$ with free generators a, b. Denote by A and B subgroups of G, generated by a and b, respectively and let G_2' denote the commutator subgroup of G_2. It is well known that G_2 is a semidirect product $G_2 = A \ltimes (B \ltimes G_2')$. The commutator subgroup G_2' is a solvable group of degree k and B is a cyclic group. By Theorem 2, the Cauchy functional equation is (ψ, γ)-stable on subgroup $H = B \ltimes G_2'$. Now the group G_2 is a semidirect product $G_2 = A \ltimes H$ again by Theorem 2, the Cauchy equation is (ψ, γ)-stable on $G_2 = A \ltimes H$.

Now let K be an arbitrary solvable group of degree $k + 1$. Suppose that the Cauchy equation is not (ψ, γ)-stable on the solvable group K. It means that the set $PX_{\psi,\gamma}(K) \setminus X(K)$ is not empty. Let $\varphi \in PX_{\psi,\gamma}(K) \setminus X(K)$. Hence, there exist $x, y \in K$ such that $\varphi(xy) \neq \varphi(x) + \varphi(y)$. Let L be a subgroup of K generated by elements x and y. Since G_2 is free solvable group of degree $k+1$, the mapping $\pi(a) = x$, $\pi(b) = y$ can be uniquely extended to epimorphism $\pi : G_2 \to L$. Let $\gamma^* = \gamma \circ \pi$. Then function $f = \varphi \circ \pi$ belongs to the space $PX_{\psi,\gamma^*}(G_2)$. Since, $PX_{\psi,\gamma^*}(G_2) = X(G_2)$ we see that $f \in X(G_2)$. However, $f(ab) = \varphi(\pi(ab)) = \varphi(xy) \neq \varphi(x) + \varphi(y) = f(a) + f(b)$ which is a contradiction to the fact $f \in X(G_2)$. Hence $PX_{\psi,\gamma}(K) \setminus X(K)$ is not nonempty. Therefore $PX_{\psi,\gamma}(K) = X(K)$ and the Cauchy equation is (ψ, γ)-stable on G.

Theorem 3 is proved.

1. *Aczél J., Dhombres J.* Functional equations in several variables // Encycl. Math. and its Appl. – Cambridge: Cambridge Univ. Press, 1989.
2. *Faĭziev V. A.* Pseudocharacters on free product of semigroups // Funkts. analiz i ego pril. – 1987. – **21**. – S. 86 – 87.
3. *Faĭziev V. A.* On the stability of a functional equation on groups // Uspechi Mat. Nauk. – 1993. – **48**. – S. 193 – 194.
4. *Faĭziev V. A., Sahoo P. K.* On (ψ, γ)-stability of Cauchy equation on some noncommutative groups // Publ. Math. Debrecen. – 2009. – **75**. – P. 67 – 83.
5. *Faĭziev V. A., Rassias Th. M., Sahoo P. K.* The space of (ψ, γ)-additive mappings on semigroups // Trans. Amer. Math. Soc. – 2002. – **354**. – P. 4455 – 4472.
6. *Forti G. L.* Remark 11 // Rept of 22nd Int. Symp. Funct. Equat.: Aequat. Math. – 1985. – **29**. – P. 90 – 91.
7. *Forti G. L.* Hyers – Ulam stability of functional equations in several variables // Aequat. Math. – 1995. – **50**. – P. 143 – 190.
8. *Hyers D. H.* On the stability of the linear functional equation // Proc. Nat. Acad. Sci. USA. – 1941. – **27**. – P. 222 – 224.
9. *Rassias Th. M.* On the stability of the linear mapping in Banach spaces // Proc. Amer. Math. Soc. – 1978. – **72**. – P. 297 – 300.
10. *Sahoo P. K., Kannappan Pl.* Introduction to functional equations. – Boca Raton: CRC Press, 2011.
11. *Ulam S. M.* A collection of mathematical problems. – New York: Intersci. Publ., 1960.

Received 21.09.12,
after revision — 07.01.14