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THE D&P SHAPLEY VALUE: A WEIGHTED EXTENSION

D&P ЗНАЧЕННЯ ШЕПЛІ: ЗВАЖЕНЕ РОЗШИРЕННЯ

First, we propose a weighted extension of the D&P Shapley value and then study several equivalences among the potentializability and some properties. On the basis of these equivalences and consistency, two axiomatizations are also proposed.

Спочатку запропоновано зважене розширення D&P значення Шеплі, а потім вивчено кілька властивостей еквівалентності між потенціалізовністю та деякими іншими властивостями. На основі цих еквівалентностей та узгодженості також отримано дві аксіоматизації.

1. Introduction. For traditional games, *weights* are assigned to the "players" to modify the discriminations among players. Since players in *multichoice transferable-utility* (*TU*) *games* could be allowed to have more than one activity levels, it is reasonable that weights could be assigned to the "activity levels" to modify the discriminations among activity levels. The weights have different significance in different fields. For example, weights could be treated as parameters to modify the discriminations among different activity levels of investment strategies. Here we propose a weighted extension of the multichoice solution introduced by Derks and Peters [2], which we name the *weighted D&P value*.

In the framework of traditional games, Hart and Mas-Colell [3] proposed the *potential* to show that the Shapley value can be resulted as the marginal contributions vector of an unique potential. Hart and Mas-Colell [3] also defined the self-reduced game and related consistency to characterize the Shapley value. Subsequently, Ortmann [6, 7] and Calvo and Santos [1] propose some equivalent relations to characterize the collection of all traditional solutions that admit a potential.

Here we build on the results of Hart and Mas-Colell [3], Ortmann [6, 7] and Calvo and Santos [1] on multichoice TU games. Three main results are as follows.

1. We propose a weighted extension of the potential due to Hart and Mas-Colell [3] on multichoice TU games, and show that the weighted D&P value can be resulted as the marginal contributions vector of a weighted potential.

2. Inspired by the results due to Ortmann [6, 7] and Calvo and Santos [1], we characterize the collection of all multichoice solutions that admit a weighted potential. Here we provide some equivalences among the *potentializability* of a solution, the properties of the *weighted balanced contributions* and the *equal loss*. Further, we adopt the weighted potential to characterize the weighted D&P value of an *auxiliary game*.

3. By considering the players and the activity levels simultaneously, we propose an extended self-reduction and related consistency. Different from the potential approach of Hart and Mas-Colell [3], we show that the weighted D&P value satisfies consistency based on "*dividend*". Finally, we characterize the weighted D&P value by means of the result (2) and consistency respectively.

2. Preliminaries. Let U be the universe of players. Suppose each player $i \in U$ could be allowed to have $m_i \in \mathbb{N}$ actively levels. Also, we set $M_i = \{0, 1, \ldots, m_i\}$ as the actively level space of player i, where 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U, N \neq \varnothing$, let $M^N = \prod_{i \in N} M_i$

be the product set of the actively level spaces for players in N, and $M_+^S = \prod_{i\in S} M_+^i$ for all $S \subseteq N$. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multichoice TU game** is a triple (N, m, v), where N is a finite and nonempty set of players, $m = (m_i)_{i\in N}$ is the vector that describes the amount of activity levels for each player, and $v\colon M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_i)_{i\in N} \in M^N$ the worth when each player i participates at activity level $x_i \in M_i$ with $v(0_N) = 0$. Given a game (N, m, v) and $x \in M^N$, we write (N, x, v) for the multichoice TU subgame obtained by restricting v to $\{y \in M^N \mid y_i \le x_i \ \forall i \in N\}$. Denote the class of all multichoice TU games by MC.

Let $w\colon \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ be a nonnegative function such that $0 = w(0) < w(l) \le w(k)$ for all $l \le k$, then w could be called a **weight function**. Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, j) \mid i \in N, j \in M_i^+\}$.

Given a a weight function w for the actions. A **solution** on MC is a mapψ^w assigning to each $(N, m, v) \in MC$ an element

$$\psi^w(N, m, v) = \left(\psi^w_{i,j}(N, m, v)\right)_{(i,j)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}.$$

Here $\psi^w_{i,j}(N, m, v)$ is the value of the player i when he takes actively level j to participate in game (N, m, v). For convenience, we define $\psi^w_{i,0}(N, m, v) = 0$ for all $(N, m, v) \in MC$ and for all $i \in N$.

Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e^S_i(N)$ satisfies

$$e^S_i(N) = \begin{cases} 1 & \text{if} \quad i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $e^S(N)$ will be denoted by e^S if no confusion can arise.

Given $(N, m, v) \in MC$, $x \in M^N$ and $i \in N$, we define $\|x\|_w = \sum_{k=1}^n w(x_k)$, $\|x\| = \sum_{k\in N} x_k$ and $S(x) = \{k \in N \mid x_k \neq 0\}$.

For all $x, y \in \mathbb{R}^N$, we say $y \le x$ if $y_i \le x_i$ for all $i \in N$. The analogue of unanimity games for multichoice TU games are **minimal effort games** (N, m, u^x_N), where $x \in M^N \setminus \{0_N\}$, defined by for all $y \in M^N$,

$$u^x_N(y) = \begin{cases} 1 & \text{if} \quad y \ge x, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that for $(N, m, v) \in MC$, it holds that $v = \sum_{x\in M^N\setminus\{0_N\}} a^x(v) u^x_N$, where $a^x(v) = \sum_{S\subseteq S(x)} (-1)^{|S|} v(x - e^S)$ is called to be the **dividend** among the necessary levels in x.

Definition 1. *The weighted D&P value Θ^w is the solution on MC which associates with each $(N, m, v) \in MC$, each weight function w, each player $i \in N$ and each level $j \in M_i^+$ the value*

$$\Theta^w_{i,j}(N, m, v) = \sum_{\substack{x\in M^N \\ x_i \ge j}} \frac{w(x_i)}{\|x\|_w} \cdot a^x(v)\ .$$

By the definition of Θ^w, all players allocate the dvivdend based on weights proportionably. The weighted D&P value $\Theta^w_{i,j}$ is the "weighted-marginal accumulation" of player i from level j to m_i. The weight $w(j)$ could be treated as a prior reward of the activity level j.

***Remark* 1.** Derks and Peters [2] proposed the **D&P Shapley value** Θ. For each $(N, m, v) \in MC$, each player $i \in N$ and each level $j \in M_i^+$,

$$\Theta_{i,j}(N, m, v) = \sum_{\substack{x \in M^N \\ x_i \geq j}} \frac{a^x(v)}{\|x\|}.$$

Klijn et al. [5] and Hwang and Liao [4] provided several axiomatizations of the D&P Shapley value respectively.

3. Potentializability. Let $N \subseteq U$. For $x \in \mathbb{R}^N$ and $S \subseteq N$, we write x_S to be the restriction of x at S. For $(i, j), (k, l) \in L^{N,m}$, we introduce the substitution notation x_{-i} to stand for $x_{N\setminus\{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$. Moreover, x_{-ik} to stand for $x_{N\setminus\{i,k\}}$ and let $z = (x_{-ik}, j, l) \in \mathbb{R}^N$ be defined by $z_{-ik} = x_{-ik}$, $z_i = j$ and $z_k = l$.

Given a weight function w and $(N, m, v) \in MC$, we define a function $P_w : MC \longrightarrow IR$ which associates a real number $P_w(N, m, v)$. Moreover,

$$D^{i,j} P_w(N, m, v) = \sum_{k=j}^{m_i} w(k) \Big[P_w\big(N, (m_{-i}, k), v\big) - P_w\big(N, (m_{-i}, k-1), v\big) \Big].$$

Definition 2. *Let w be a weight function. A solution ψ^w admits a w-potential if there exists a function $P_w : MC \to \mathbb{R}$ satisfies for all $(N, m, v) \in MC$ with $N \neq \varnothing$ and for all $(i, j) \in L^{N,m}$,*

$$\psi^w_{i,j}(N, m, v) = D^{i,j} P_w(N, m, v).$$

Moreover, a function $P_w : MC \longrightarrow \mathbb{R}$ is **0-normalized** if $P_w(N, 0_N, v) = 0$ for each $N \subseteq U$. P_w is **efficient** if for all $(N, m, v) \in MC$,

$$\sum_{i \in N} \sum_{j=1}^{m_i} D^{i,j} P_w(N, m, v) = v(m). \tag{1}$$

Theorem 1. *Let w be a weight function. A solution ψ^w admits a uniquely 0-normalized and efficient w-potential P_w if and only if ψ^w is the solution Θ^w on MC. For all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,*

$$\Theta^w_{i,j}(N, m, v) = D^{i,j} P_w(N, m, v).$$

Proof. Given a weight function w and $(N, m, v) \in MC$. Formula (1) can be rewritten as

$$P_w(N, m, v) =$$

$$= \frac{1}{\|m\| \|m\|_w} \left[v(m) + \sum_{i \in S(m)} \sum_{j=0}^{m_i - 1} j \Big(w(j) - w(j+1) \Big) P_w\Big(N, (m_{-i}, j), 0), v\Big) \right]. \tag{2}$$

Starting with $P_w(N, 0_N, v)$, it determines $P_w(N, m, v)$ recursively. This shows the existence of the weighted potential P_w, and moreover that $P_w(N, m, v)$ is uniquely determined by (1) (or (2)) applied to (N, x, v) for all $x \in M^N$. Let

$$P_w(N,m,v) = \sum_{x\in M^N\setminus\{0_N\}} \frac{1}{\|x\|_w}\cdot a^x(v)\ . \tag{3}$$

It is easy to check that (1) is satisfied by this P_w; hence (3) defines the uniquely 0-normalized and efficient weighted potential. The result now follows since for all $(i,j)\in L^{N,m}$,

$$\Theta^w_{i,j}(N,m,v) = D^{i,j}P_w(N,m,v) = \sum_{\substack{x\in M^N\\ x_i\geq j}} \frac{w(j)}{\|x\|_w}\cdot a^x(v).$$

4. Equivalences and axiomatization. Here we provide some equivalences to characterize the weighted D&P value. Let w be a weight function and ψ^w be a solution on MC.

Efficiency (EFF): For all $(N,m,v)\in MC$, $\sum_{i\in N}\sum_{j=1}^{m_i}\psi^w_{i,j}(N,m,v)=v(m)$. ψ^w is said to be **weak efficiency (WEFF)** if for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ^w satisfies EFF.

Weighted balanced contributions (WBC): For all $(N,m,v)\in MC$ and all $i,k\in N$, $i\neq k$,

$$\frac{1}{w(m_i)}\Big[\psi^w_{i,m_i}\big(N,m,v\big)-\psi^w_{i,m_i}\big(N,m-e^{\{k\}},v\big)\Big]=$$

$$=\frac{1}{w(m_k)}\Big[\psi^w_{k,m_k}\big(N,m,v\big)-\psi^w_{k,m_k}\big(N,m-e^{\{i\}},v\big)\Big].$$

Equal loss (EL)[1]: For all $(N,m,v)\in MC$ and all $(i,j)\in L^{N,m}$, $j\neq m_i$,

$$\psi_{i,j}\big(N,m,v\big)-\psi_{i,j}\big(N,m-e^{\{i\}},v\big)=\psi_{i,m_i}\big(N,m,v\big).$$

ψ^w is said to be **weak equal loss (WEL)** if for all $(N,m,v)\in MC$ with $|S(m)|=1$, ψ^w satisfies EL.

Definition 3. *Given $(N,m,v)\in MC$ and ψ^w be a solution. The auxiliary multichoice TU game (N,m,v_{ψ^w}) is defined by*

$$v_{\psi^w}(x)=\sum_{i\in S(x)}\sum_{j=1}^{x_i}\psi^w_{i,j}(N,x,v)$$

for all $x\in M^N$. Note that $v=v_{\psi^w}$ if ψ^w satisfies efficiency.

Theorem 2. *Let w be a weight function and ψ^w be a solution. The following are equivalent*:

(a) *ψ^w admits a w-potential*;

(b) *ψ^w satisfies WBC and EL*;

(c) *$\psi^w(N,m,v)=\Theta^w(N,m,v_{\psi^w})$ for all $(N,m,v)\in MC$.*

Proof. Let w be a weight function and ψ^w be a solution. To verify (a) ⇒ (b), suppose ψ^w admits a w-potential P_w. For all $(N,m,v)\in MC$ and for all $i,k\in N$, $i\neq k$,

$$\frac{1}{w(m_i)}\left[\psi^w_{i,m_i}\big(N,m,v\big)-\psi^w_{i,m_i}\big(N,m-e^{\{k\}},v\big)\right]=$$

$$=\frac{1}{w(m_i)}w(m_i)\left[P_w\big(N,m,v\big)-P_w\big(N,m-e^{\{i\}},v\big)\right]-$$

[1]This axiom was introduced by Klijn, Slikker and Zazuelo [5].

$$-\frac{1}{w(m_i)}w(m_i)\big[P_w\big(N,m-e^{\{k\}},v\big)-P_w\big(N,m-e^{\{i\}}-e^{\{k\}},v\big)\big]=$$

$$=\Big[P_w\big(N,m,v\big)-P_w\big(N,m-e^{\{k\}},v\big)\Big]-$$

$$-\Big[P_w\big(N,m-e^{\{i\}},v\big)-P_w\big(N,m-e^{\{i\}}-e^{\{k\}},v\big)\Big]=$$

$$=\frac{1}{w(m_k)}w(m_k)\big[P_w\big(N,m,v\big)-P_w\big(N,m-e^{\{k\}},v\big)\big]-$$

$$-\frac{1}{w(m_k)}w(m_k)\Big[P_w\big(N,m-e^{\{i\}},v\big)-P_w\big(N,m-e^{\{i\}}-e^{\{k\}},v\big)\Big]=$$

$$=\frac{1}{w(m_k)}\Big[\psi^w_{j,m_k}\big(N,m,v\big)-\psi^w_{j,m_k}\big(N,m-e^{\{i\}},v\big)\Big].$$

Hence, ψ^w satisfies WBC. Next, we show that ψ^w satisfies EL. For all $(N,m,v)\in MC$ and for all $(i,j)\in L^{N,m}$, $j\neq m_i$,

$$\psi^w_{i,j}\Big(N,m,v\big)-\psi^w_{i,j}\big(N,m-e^{\{i\}},v\Big)=$$

$$=\sum_{t=j}^{m_i}w(t)\Big[P_w\big(N,(m_{-i},t),v\big)-P_w\big(N,(m_{-i},t-1),v\big)\Big]-$$

$$-\sum_{t=j}^{m_i-1}w(t)\Big[P_w\big(N,(m_{-i},t),v\big)-P_w\big(N,(m_{-i},t-1),v\big)\Big]=$$

$$=w(m_i)\big[P_w\Big(N,m,v\big)-P_w\big(N,m-e^{\{i\}},v\Big)\Big]=\psi^w_{i,j}(N,m,v).$$

That is, ψ^w satisfies EL.

To verify (b) $\Rightarrow$ (c), suppose ψ^w satisfies WBC and EL. Let $(N,m,v)\in MC$. The proof proceeds by induction on the number $\|m\|$. If $\|m\|=1$, let $S(m)=\{i\}$ and $m_i=1$, then by the definition of v_{ψ^w} and efficiency of Θ^w, $\psi^w_{i,1}(N,m,v)=v_{\psi^w}(m)=\Theta^w_{i,1}(N,m,v_{\psi^w})$. Suppose that $\psi^w(N,m,v)=$ $=\Theta^w(N,m,v_{\psi^w})$ for $\|m\|\leq k$, where $k\geq 1$.

Case $\|m\|=k+1$: Let $i\in S(m)$. By induction hypotheses and WBC of ψ^w and Θ^w, for all $k\in S(m)$ with $k\neq i$,

$$\frac{1}{w(m_i)}\psi^w_{i,m_i}(N,m,v)-\frac{1}{w(m_k)}\psi^w_{k,m_k}(N,m,v)=$$

$$=\frac{1}{w(m_i)}\psi^w_{i,m_i}(N,m-e^{\{k\}},v)-\frac{1}{w(m_k)}\psi^w_{k,m_k}(N,m-e^{\{i\}},v)=$$

$$=\frac{1}{w(m_i)}\Theta^w_{i,m_i}(N,m-e^{\{k\}},v_{\psi^w})-\frac{1}{w(m_k)}\Theta^w_{k,m_k}(N,m-e^{\{i\}},v_{\psi^w})=$$

$$= \frac{1}{w(m_i)}\Theta^w_{i,m_i}(N,m,v_{\psi^w}) - \frac{1}{w(m_k)}\Theta^w_{k,m_k}(N,m,v_{\psi^w}).$$

So we have that,

$$\frac{1}{w(m_i)}\big[\psi^w_{i,m_i}(N,m,v) - \Theta^w_{i,m_i}(N,m,v_{\psi^w})\big] =$$

$$= \frac{1}{w(m_k)}\big[\psi^w_{k,m_k}(N,m,v) - \Theta^w_{k,m_k}(N,m,v_{\psi^w})\big]. \tag{4}$$

By induction hypotheses and EL of ψ^w and Θ^w, for every $(h,l) \in L^{N,m}$, $l \neq m_h$,

$$\psi^w_{h,l}\big(N,m,v\big) - \Theta^w_{h,l}\big(N,m,v_{\psi^w}\big) =$$

$$= \Big[\psi^w_{h,m_h}\big(N,m,v\big) + \psi^w_{h,l}\big(N,m-e^{\{h\}},v\big)\Big] -$$

$$-\Big[\Theta^w_{h,m_h}\big(N,m,v_{\psi^w}\big) + \Theta^w_{h,l}\big(N,m-e^{\{h\}},v_{\psi^w}\big)\Big] =$$

$$= \psi^w_{h,m_h}\big(N,m,v\big) + \Theta^w_{h,l}\big(N,m-e^{\{h\}},v_{\psi}\big) -$$

$$-\Theta^w_{h,m_h}\big(N,m,v_{\psi^w}\big) - \Theta^w_{h,l}\big(N,m-e^{\{h\}},v_{\psi^w}\big) =$$

$$= \psi^w_{h,m_h}\big(N,m,v\big) - \Theta^w_{h,m_h}\big(N,m,v_{\psi^w}\big). \tag{5}$$

By definition of v_{ψ^w}, efficiency of Θ^w, equations (4), (5) and the induction hypotheses,

$$0 = v_{\psi^w}(m) - v_{\psi^w}(m) =$$

$$= \sum_{h\in S(m)}\sum_{l=1}^{m_h}\psi^w_{h,l}(N,m,v) - \sum_{h\in S(m)}\sum_{l=1}^{m_h}\Theta^w_{h,l}(N,m,v) =$$

$$= \sum_{h\in S(m)}\big[\psi^w_{h,m_h}(N,m,v) - \Theta^w_{h,m_h}(N,m,v_{\psi^w})\big] =$$

$$= \sum_{h\in S(m)}\frac{m_h w(m_h)}{w(m_i)}\big[\psi^w_{i,m_i}(N,m,v) - \Theta^w_{i,m_i}(N,m,v_{\psi^w})\big]. \tag{6}$$

Hence, $\psi^w_{i,m_i}(N,m,v) - \Theta^w_{i,m_i}(N,m,v_{\psi^w}) = 0$. By equations (4), (5) and (6), $\psi^w_{k,l}(N,m,v) = \Theta^w_{k,l}(N,m,v_{\psi^w})$ for all $(k,l) \in L^{N,m}$.

To verify (c) $\Rightarrow$ (a), suppose that $\psi^w(N,m,v) = \Theta^w(N,m,v_{\psi^w})$ for all $(N,m,v) \in MC$. Since the weighted D&P value Θ^w admits a unique w-potential P_{Θ^w}, we define a w-potential of ψ^w as $P_{\psi^w}(N,m,v) = P_{\Theta^w}(N,m,v_{\psi^w})$ for all $(N,m,v) \in MC$. Then for every $(i,j) \in L^{N,m}$,

$$D^{i,j}P_{\psi^w}(N,m,v) =$$

$$= \sum_{k=j}^{m_i} w(k)\big[P_{\psi^w}\big(N,(m_{-i},k),v\big) - P_{\psi^w}\big(N,(m_{-i},k-1),v\big)\big] =$$

$$= \sum_{k=j}^{m_i} w(k)\big[P_{\Theta^w}\big(N,(m_{-i},k),v_{\psi^w}\big) - P_{\Theta^w}\big(N,(m_{-i},k-1),v_{\psi^w}\big)\big] =$$

$$= \Theta^w_{i,j}(N,m,v_{\psi^w}) = \psi^w_{i,j}(N,m,v).$$

Hence, P_{ψ^w} is a w-potential of ψ^w.

Theorem 3. *A solution ψ^w satisfies EFF, EL and WBC if and only if $\psi^w = \Theta^w$.*

Proof. Given a weight function w. By equation (1) and Theorem 1, it easy to check that Θ^w satisfies EFF. Since Θ^w admits a w-potential, Θ^w satisfies WBC and EL by Theorem 2.

By Definition 3 and EFF of Θ^w, $v_{\Theta^w} = v$. By Theorem 2, the proof is completed.

5. Player-action reduction and axiomatization. Here we propose the player-action reduction and related consistency to characterize the weighted D&P value.

Given $(N,m,v) \in MC$, a solution ψ^w on MC, $S \subseteq N, S \neq \varnothing$ and $\gamma \in M_+^{N\setminus S}$. The **player-action reduced game** $(S, m_S, v^{\psi^w}_{S,\gamma})$ with respect to S, m, γ and ψ^w is defined as follows. For all $\alpha \in M^S$,

$$v^{\psi^w}_{S,m,\gamma}(\alpha) = v(\alpha,\gamma) - \sum_{k\in N\setminus S}\sum_{t=1}^{\gamma_k} \psi^w_{k,t}(N,(\alpha,\gamma),v).$$

The player-action reduction asserts that, when reapportioning the payoff allotment within S, all members in $N \setminus S$ take nonzero levels based on the action vector γ to cooperate. Then in the player-action reduction, the coalition S takes activity level α to cooperate with the coalition $N \setminus S$ with activity level γ. A solution ψ^w satisfies **player-action consistency (PACON)** if for all $S \subseteq N$, for all $(i,j) \in L^{S,m_S}$ and for all $\gamma \in M_+^{N\setminus S}$, $\psi^w_{i,j}\big(S, m_S, v^{\psi^w}_{S,m,\gamma}\big) = \psi^w_{i,j}\big(N,(m_S,\gamma),v\big)$.

Lemma 1. *Let $(N,m,v) \in MC$ and $(S, m_S, v^{\Theta^w}_{S,m,\gamma})$ be a player-action reduced game. If $v = \sum_{\alpha\in M^N\setminus\{0_N\}} a^\alpha(v)\cdot u^\alpha_N$, then $v^{\Theta^w}_{S,\gamma}$ can be expressed to be $v^{\Theta^w}_{S,m,\gamma} = \sum_{\alpha\in M^S\setminus\{0_S\}} a^\alpha(v^{\Theta^w}_{S,m,\gamma})u^\alpha_S$, where for all $\alpha \in M^S$,*

$$a^\alpha(v^{\Theta^w}_{S,m,\gamma}) = \sum_{\beta\leq\gamma} \frac{\|\alpha\|_w}{\|\alpha\|_w + \|t\|_w}\cdot a^{(\alpha,t)}(v).$$

Proof. Let $(N,m,v) \in MC$, $S \subseteq N$ and $\gamma \in M_+^{N\setminus S}$. For all $\alpha \in M^S$,

$$v^{\Theta^w}_{S,m,\gamma}(\alpha) = v(\alpha,\gamma) - \sum_{k\in N\setminus S}\sum_{t=1}^{\gamma_k} \Theta^w_{k,t}\big(N,(\alpha,\gamma),v\big). \tag{7}$$

By EFF of Θ^w, $v^{\Theta^w}_{S,\gamma}(0_S) = 0$ and for all $\alpha \in M^S \setminus \{0_S\}$,

$$(7) = \sum_{k\in S(\alpha)}\sum_{t=1}^{\alpha_k} \Theta^w_{k,t}(N,(\alpha,\gamma),v) =$$

$$= \sum_{k\in S(\alpha)} \sum_{t=1}^{\alpha_k} \sum_{\substack{\mu\le(\alpha,\gamma)\\ \mu_k\ge t}} \frac{w(\mu_k)a^{\mu}(v)}{\|\mu\|_w} =$$

$$= \sum_{k\in S(\alpha)} \left[\sum_{\substack{\mu\le(\alpha,\gamma)\\ \mu_k\ge 1}} \frac{w(\mu_k)a^{\mu}(v)}{\|\mu\|_w} + \ldots + \sum_{\substack{\mu\le(\alpha,\gamma)\\ \mu_k\ge \alpha_k}} \frac{w(\mu_k)a^{\mu}(v)}{\|\mu\|_w} \right] =$$

$$= \sum_{k\in S(\alpha)} \left[\sum_{\substack{p\le\alpha\\ p_k\ge 1}} \sum_{\beta\le\gamma} \frac{w(p_k)a^{(p,\beta)}(v)}{\|p\|_w+\|\beta\|_w} + \ldots + \sum_{\substack{p\le\alpha\\ p_k\ge \alpha_k}} \sum_{\beta\le\gamma} \frac{w(p_k)a^{(p,\beta)}(v)}{\|p\|_w+\|\beta\|_w} \right] =$$

$$= \sum_{p\le\alpha} \sum_{\beta\le\gamma} \frac{\|p\|_w}{\|p\|_w+\|\beta\|_w} a^{(p,\beta)}(v). \tag{8}$$

Set

$$a^p(v^{\Theta^w}_{S,m,\gamma}) = \sum_{\beta\le\gamma} \frac{\|p\|_w}{\|p\|_w+\|\beta\|_w} \cdot a^{(p,\beta)}(v).$$

By equation (8), for all $\alpha \in M^S$,

$$v^{\Theta^w}_{S,m,\gamma}(\alpha) = \sum_{p\le\alpha} \sum_{\beta\le\gamma} \frac{\|p\|_w}{\|p\|_w+\|\beta\|_w} \cdot a^{(p,\beta)}(v) = \sum_{p\le\alpha} a^p(v^{\Theta^w}_{S,m,\gamma}).$$

Hence $v^{\Theta^w}_{S,m,\gamma}$ can be expressed to be $v^{\Theta^w}_{S,m,\gamma} = \sum_{\alpha\in M^S\setminus\{0_S\}} a^{\alpha}(v^{\Theta^w}_{S,m,\gamma}) \cdot u^{\alpha}_S$.

Different from the potential approach of Hart and Mas-Colell [3], we investigate the player-action consistency of the weighted D&P value by applying dividend.

Lemma 2. *The solution Θ^w satisfies PACON.*

Proof. Let $(N,m,v) \in MC$, $S \subseteq N$ and $\gamma \in M^{N\setminus S}_+$. By Definition 1 and Lemma 1, for all $(i,j) \in L^{S,m_S}$,

$$\Theta^w_{i,j}(S, m_S, v^{\Theta^w}_{S,m,\gamma}) = \sum_{\substack{\alpha\in M^S\\ \alpha_i\ge j}} \frac{w(\alpha_i)a^{\alpha}(v^{\Theta^w}_{S,m,\gamma})}{\|\alpha\|_w} =$$

$$= \sum_{\substack{\alpha\in M^S\\ \alpha_i\ge j}} \frac{w(\alpha_i)}{\|\alpha\|_w} \sum_{t\le\gamma} \frac{\|\alpha\|_w}{\|\alpha\|_w+\|t\|_w} \cdot a^{(\alpha,t)}(v) =$$

$$= \sum_{\substack{\beta\le(m_S,\gamma)\\ \beta_i\ge j}} \frac{w(\alpha_i)a^{\beta}(v)}{\|\beta\|_w} = \Theta^w_{i,j}(N,(m_S,\gamma),v).$$

Hence, the solution Θ^w satisfies PACON.

Lemma 3. *If a solution ψ^w satisfies PACON and WEFF, then ψ^w satisfies EFF.*

Proof. Let w be a weight function and ψ^w be a solution. Assume that ψ^w satisfies WEFF and PACON. Let $(N, m, v) \in MC$. It is trivial for $|S(m)| = 1$ by WEFF. Assume that $|S(m)| \geq 2$. Let $k \in S(m)$. By the definition of the reduction,

$$v^{\psi^w}_{\{k\},m,m_{N\setminus\{k\}}}(m_k) = v(m) - \sum_{i\in N\setminus\{k\}} \sum_{j=1}^{m_i} \psi^w_{i,j}(N, m, v).$$

Since ψ^w satisfies PACON, for all $j \in M_k^+$,

$$\psi^w_{k,j}(N, m, v) = \psi^w_{k,j}\Big(N, m_k, v^{\psi^w}_{\{k\},m,m_{N\setminus\{k\}}}\Big).$$

By WEFF of ψ^w,

$$v^{\psi^w}_{\{k\},m,m_{N\setminus\{k\}}}(m_k) = \sum_{j=1}^{m_k} \psi^w_{k,j}\Big(N, m_k, v^{\psi^w}_{\{k\},m,m_{N\setminus\{k\}}}\Big) = \sum_{j=1}^{m_k} \psi^w_{k,j}(N, m, v).$$

Hence $\sum_{i\in N} \sum_{j=1}^{m_i} \psi^w_{i,j}(N, m, v) = v(m)$, i.e., ψ^w satisfies EFF.

Lemma 4. *Given a weight function w, a solution ψ^w, $(N, m, v) \in MC$, $S \subseteq N$, and $y \in$ $\in M^S \setminus \{0_S\}$. Then*

$$\Big(S, y, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) = \Big(S, y, v^{\psi^w}_{S,(y,m_{N\setminus S}),m_{N\setminus S})}\Big).$$

Proof. It is easy to derive this result by the definitions of a subgame and a reduced game, we omit it.

Lemma 5. *If a solution ψ^w satisfies WEL and PACON, then it also satisfies EL.*

Proof. Let w be a weight function and ψ^w be a solution on MC. Suppose that a solution ψ^w on MC satisfies WEL and PACON. Let $(N, m, v) \in MC$, $i \in N$ and $j \in M_i^+ \setminus \{m_i\}$. Let $y = m - e^{\{i\}}$, consider the reduction $\Big(\{i\}, (m_i - 1), v^{\psi}_{\{i\},y,m_{N\setminus S}}\Big)$ of the subgame (N, y, v) of (N, m, v) with respect to $\{i\}$, y, $m_{N\setminus S}$ and ψ^w, and the reduction $\Big(\{i\}, m_i, v^{\psi^w}_{\{i\},m,m_{N\setminus S}}\Big)$ of (N, m, v) with respect to $\{i\}$, m, $m_{N\setminus S}$ and ψ^w, respectively. By Lemma 4, it is easy to see that $\Big(\{i\}, (m_i - 1), v^{\psi^w}_{\{i\},y,m_{N\setminus S}}\Big)$ is the subgame of $\Big(\{i\}, m_i, v^{\psi^w}_{\{i\},m,m_{N\setminus S}}\Big)$, i.e.,

$$\Big(\{i\}, (m_i - 1), v^{\psi^w}_{\{i\},y,m_{N\setminus S}}\Big) = \Big(\{i\}, (m_i - 1), v^{\psi^w}_{\{i\},m,m_{N\setminus S}}\Big).$$

Hence

$$\psi^w_{i,j}(N, m, v) - \psi^w_{i,j}(N, m - e^{\{i\}}, v) =$$

$$= \psi^w_{i,j}(N, m, v) - \psi^w_{i,j}(N, y, v) \quad (\text{by } y = m - e^{\{i\}}) =$$

$$= \psi_{i,j}\Big(\{i\}, m_i, v^{\psi}_{\{i\},m,m_{N\setminus S}}\Big) - \psi_{i,j}\Big(\{i\}, (m_i - 1), v^{\psi}_{\{i\},y,m_{N\setminus S}}\Big) \quad (\text{by PACON}) =$$

$$= \psi_{i,j}\Big(\{i\}, m_i, v^{\psi}_{\{i\},m,m_{N\setminus S}}\Big) - \psi_{i,j}\Big(\{i\}, (m_i - 1), v^{\psi}_{\{i\},m,m_{N\setminus S}}\Big) \text{ (by Lemma 4)} =$$

$$= \psi_{i,m_i}\Big(\{i\}, m_i, v^{\psi}_{\{i\},m,m_{N\setminus S}}\Big) \quad \text{(by WEL)} = \psi_{i,m_i}(N,m,v) \quad \text{(by PACON)}.$$

So, ψ satisfies EL.

Subsequently, we characterize the weighted D&P value by means of player-action consistency. A solution ψ^w satisfies **standard for two-person games (ST)** if for all $(N,m,v) \in MC$ with $|S(m)| = 2$, $\psi^w(N,m,v) = \Theta^w(N,m,v)$.

***Remark* 2.** By adding a "dummy" player to one-person games, it is easy to show that if a solution ψ^w satisfies PACON and ST, then $\psi^w(N,m,v) = \Theta^w(N,m,v)$ for all $(N,m,v) \in MC$ with $|S(m)| = 1$. Hence, If ψ^w satisfies PACON and ST, it satisfies WEFF and WEL.

Theorem 4. 1. *A solution ψ^w satisfies WEFF, WEL, WBC and PACON if and only if $\psi^w = \Theta^w$.*

2. *A solution ψ^w satisfies ST and PACON if and only if $\psi^w = \Theta^w$.*

Proof. Given a weight function w. The proof of Proposition 1 of this theorem follows from Remark 2, Lemmas 2, 3, 5 and Theorem 3.

To prove Proposition 2 of this theorem. By Definition 1, it is easy to see that Θ^w satisfies ST. By Lemma 2, Θ^w satisfies PACON. To prove uniqueness of Proposition 2 of this theorem, suppose that the solution ψ^w on MC satisfies ST and PACON. By Remark 2, Lemmas 3 and 5, ψ^w satisfies EFF and EL. By Proposition 1 of this theorem, it remains to show that ψ^w satisfies WBC. Given $(N,m,v) \in MC$. The proof proceeds by induction on the number $\|m\|$. Assume that $\|m\| = 1$ and $S(m) = \{i\}$. By EFF of ψ^w and Θ^w, $\psi^w_{i,1}(N,m,v) = v(m) = \Theta^w_{i,1}(N,m,v)$. Assume that $\psi^w(N,m,v) = \Theta^w(N,m,v)$ if $\|m\| \le l-1$, where $l \ge 2$.

Case $\|m\| = l$: Two cases may be distinguish:

Case 1: Assume that $|S(m)| \le 2$. Since ψ^w satisfies ST, $\psi^w(N,m,v) = \Theta^w(N,m,v)$.

Case 2: Assume that $|S(m)| \ge 3$. Let $i,k \in S(m)$ and $S = \{i,k\}$. By PACON of ψ^w and Lemma 4,

$$\frac{1}{w(m_i)}\Big[\psi^w_{i,m_i}(N,m,v) - \psi^w_{i,m_i}(N, m - e^{\{k\}}, v)\Big] =$$

$$= \frac{1}{w(m_i)}\Big[\psi^w_{i,m_i}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \psi^w_{i,m_i}\Big(S, m_S - e^{\{k\}}, v^{\psi^w}_{S,m-e^{\{k\}},m_{N\setminus S}}\Big)\Big] =$$

$$= \frac{1}{w(m_i)}\Big[\psi^w_{i,m_i}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \psi^w_{i,m_i}\Big(S, m_S - e^{\{k\}}, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big)\Big]. \tag{9}$$

By ST of ψ^w, PACON and WBC of Θ^w and Lemma 4,

$$(9) = \frac{1}{w(m_i)}\Big[\Theta^w_{i,m_i}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \Theta^w_{i,m_i}\Big(S, m_S - e^{\{k\}}, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big)\Big] =$$

$$= \frac{1}{w(m_k)}\Big[\Theta^w_{k,m_k}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \Theta^w_{k,m_k}\Big(S, m_S - e^{\{i\}}, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big)\Big] =$$

$$= \frac{1}{w(m_k)}\Big[\psi^w_{k,m_k}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \psi^w_{k,m_k}\Big(S, m_S - e^{\{i\}}, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big)\Big] =$$

$$= \frac{1}{w(m_k)}\Big[\psi^w_{k,m_k}\Big(S, m_S, v^{\psi^w}_{S,m,m_{N\setminus S}}\Big) - \psi^w_{k,m_k}\Big(S, m_S - e^{\{i\}}, v^{\psi^w}_{S,m-e^{\{i\}},m_{N\setminus S}}\Big)\Big] =$$

$$= \frac{1}{w(m_k)} \left[\psi^w_{k,m_k}(N, m, v) - \psi^w_{k,m_k}(N, m - e^{\{i\}}, v) \right].$$

Hence, ψ^w satisfies WBC.

The following examples show that each of the axioms used in Theorems 3 and 4 is logically independent of the remaining axioms.

***Example* 1.** Define a solution ψ^w on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}(N, m, v) = 0.$$

Clearly, ψ^w satisfies EL(WEL), WBC and PACON, but it violates EFF(WEFF) and ST.

***Example* 2.** Define a solution ψ^w on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}(N, m, v) = \begin{cases} \Theta^w_{i,j}(N, m, v) + \varepsilon & \text{if} \quad j = m_i, \\ \Theta^w_{i,j}(N, m, v) - \dfrac{\varepsilon}{m_i - 1} & \text{if} \quad j \neq m_i, \end{cases}$$

where $\varepsilon \in \mathbb{R} \setminus \{0\}$. Clearly, ψ^wsatisfies EFF(WEFF), WBC and PACON, but it violates EL(WEL).

***Example* 3.** Define a solution ψ^w on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}(N, m, v) = \sum_{\substack{x \in M^N \\ x_i \geq j}} \frac{a^x(v)}{\|x\|}.$$

Clearly, ψ^w satisfies EFF(WEFF), EL(WEL) and PACON, but it violates WBC.

***Example* 4.** Define a solution ψ^w on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}(N, m, v) = \Theta^w_{i,j}(N, m, v)$$

if $|S(m)| = 1$ or $m_i = 1$; otherwise

$$\psi^w_{i,j}(N, m, v) = \Theta^w_{i,j}(N, m, v) + \varepsilon,$$

where $\varepsilon > 0$. Clearly, ψ^w satisfies WEFF, WBC and EL(WEL), but it violates PACON.

***Example* 5.** Define a solution ψ^w on MC by for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,

$$\psi^w_{i,j}(N, m, v) = \begin{cases} \Theta^w_{i,j}(N, m, v) & \text{if} \quad |S(m)| \leq 2, \\ \Theta^w_{i,j}(N, m, v) - \varepsilon & \text{otherwise}, \end{cases}$$

where $\varepsilon \in \mathbb{R} \setminus \{0\}$. Clearly, ψ^w satisfies ST, but it violates PACON.

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