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SLANT LIGHTLIKE SUBMERSIONS FROM AN INDEFINITE ALMOST HERMITIAN MANIFOLD ONTO A LIGHTLIKE MANIFOLD

ПОХИЛІ СВІТЛОПОДІБНІ ЗАНУРЕННЯ З НЕВИЗНАЧЕНОГО МАЙЖЕ ЕРМІТОВОГО МНОГОВИДУ В СВІТЛОПОДІБНИЙ МНОГОВИД

We introduce slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold. We establish existence theorems for these submersions. We also investigate the necessary and sufficient conditions for the leaves of the distributions to be totally geodesic foliations in indefinite almost Hermitian manifolds.

Введено поняття похилих світлоподібних занурень з невизначеного майже ермітового многовиду в світлоподібний многовид. Доведено теореми існування для таких занурень. Вивчено необхідні та достатні умови для того, щоб листки розподілів були повністю геодезичними розшаруваннями в невизначених майже ермітових многовидах.

1. Introduction. Riemannian submersions between Riemannian manifolds were studied by O'Neill [6] and Gray [5]. Later Watson [14] defined almost Hermitian submersions between almost Hermitian manifolds. Semi-Riemannian submersions were introduced by O'Neill in [7]. It is known that when M and B are Riemannian manifolds, then the fibers are always Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds, then the fibers may not be semi-Riemannian manifolds. Therefore in [10], Sahin introduced a screen lightlike submersion from a lightlike manifold onto a semi-Riemannian manifold and in [11], Sahin and Gündüzalp introduced a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold. As a generalization of almost Hermitian submersions, Sahin [12] introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds. The geometry of lightlike submanifolds has extensive uses in mathematical physics, particularly in general relativity [3]. Also, it is well-known that semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang – Mills theory, Kaluza – Klein theory, supergravity and superstring theories [1, 2, 4, 13]. Moreover, we obtained nonexistence of totally contact umbilical proper slant lightlike submanifolds of indefinite Sasakian manifolds in [8]. Thus all these motivated us to club the theory of lightlike submersions with slant submersions. In this paper, we introduce slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold. We establish existence theorems for such submersions. We also investigate the necessary and sufficient conditions for the leaves of the distributions to be totally geodesic foliation in indefinite almost Hermitian manifolds.

2. Lightlike submersions. The used notations and fundamental equations for lightlike submersions are refereed from [11].

Let (M, g) be a real n-dimensional smooth manifold where g is a symmetric tensor field of type $(0, 2)$. The radical space $\mathrm{Rad}\, T_xM$ of T_xM is defined by

$$\mathrm{Rad}\, T_xM = \big\{\xi \in T_xM : g(\xi, X) = 0 \quad \forall X \in T_xM\big\}.$$

The dimension of $\mathrm{Rad}\, T_xM$ is called the nullity degree of g. If the mapping $\mathrm{Rad}\, TM : x \in M \rightarrow$ $\rightarrow \mathrm{Rad}\, T_xM$ defines a smooth distribution on M of rank $r > 0$, then $\mathrm{Rad}\, TM$ is called the radical distribution of M and the manifold M is called an r-lightlike manifold if $0 < r \leq n,$ for detail see [3].

Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. Consider a smooth submersion $f\colon M_1 \to M_2$, then $f^{-1}(p)$ is a submanifold of M_1 of dimension dimM_1 - dimM_2 for $p \in M_2$. The kernel of f_* at the point p is given by

$$\mathrm{Ker}\, f_* = \big\{X \in T_p(M_1)\colon f_*(X) = 0\big\},$$

and $(\mathrm{Ker}\, f_*)^{\perp}$ is given by

$$(\mathrm{Ker}\, f_*)^{\perp} = \big\{Y \in T_p(M_1)\colon g_1(Y, X) = 0 \ \ \forall X \in \mathrm{Ker}\, f_*\big\}.$$

Since $T_p(M_1)$ is a semi-Riemannian vector space therefore $(\mathrm{Ker}\, f_*)^{\perp}$ may not be a complementary to $\mathrm{Ker}\, f_*$ so assume $\triangle = \mathrm{Ker}\, f_* \cap (\mathrm{Ker}\, f_*)^{\perp} \neq \{0\}$. Thus we have the following four cases of submersions.

Case 1: $0 < \dim \triangle < \min\big\{\dim(\mathrm{Ker}\, f_*), \dim(\mathrm{Ker}\, f_*)^{\perp}\big\}$. Then $\triangle$ is the radical subspace of $T_p(M_1)$. Since $\mathrm{Ker}\, f_*$ is a real lightlike vector space therefore complementary nondegenerate subspace $\triangle$ in $\mathrm{Ker}\, f_*$ is $S(\mathrm{Ker}\, f_*)$ and we obtain

$$\mathrm{Ker}\, f_* = \triangle \perp S(\mathrm{Ker}\, f_*).$$

Similarly

$$(\mathrm{Ker}\, f_*)^{\perp} = \triangle \perp S(\mathrm{Ker}\, f_*)^{\perp},$$

where $S(\mathrm{Ker}\, f_*)^{\perp}$ is a complementary subspace of $\triangle$ in $(\mathrm{Ker}\, f_*)^{\perp}$. Since $S(\mathrm{Ker}\, f_*)^{\perp}$ is nondegenerate in $T_p(M_1)$, therefore we get

$$T_p(M_1) = S(\mathrm{Ker}\, f_*) \perp (S(\mathrm{Ker}\, f_*))^{\perp},$$

where $(S(\mathrm{Ker}\, f_*))^{\perp}$ is the complementary subspace of $S(\mathrm{Ker}\, f_*)$ in $T_p(M_1)$. Since $S(\mathrm{Ker}\, f_*)$ and $(S(\mathrm{Ker}\, f_*))^{\perp}$ are nondegenerate therefore we have

$$(S(\mathrm{Ker}\, f_*))^{\perp} = S(\mathrm{Ker}\, f_*)^{\perp} \perp (S(\mathrm{Ker}\, f_*)^{\perp})^{\perp}.$$

Then from [3], we can construct a quasiorthonormal basis of M_1 along $\mathrm{Ker}\, f_*$ therefore we obtain

$$\begin{gathered} g(\xi_i, \xi_j) = g(N_i, N_j) = 0, \qquad g(\xi_i, N_j) = \delta_{ij}, \\ g(W_\alpha, \xi_j) = g(W_\alpha, N_j) = 0, \qquad g(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta}, \end{gathered} \tag{1}$$

where $\{N_i\}$ are smooth lightlike vector fields of $(S(\mathrm{Ker}\, f_*)^{\perp})^{\perp}$, $\{\xi_i\}$ is a basis of $\triangle$ and $\{W_\alpha\}$ is a basis of $S(\mathrm{Ker}\, f_*)^{\perp}$. Denote the set of vector fields $\{N_i\}$ by $\mathrm{ltr}(\mathrm{Ker}\, f_*)$ and consider

$$\mathrm{tr}(\mathrm{Ker}\, f_*) = \mathrm{ltr}(\mathrm{Ker}\, f_*) \perp S(\mathrm{Ker}\, f_*)^{\perp}.$$

Using (1), it is clear that $\mathrm{ltr}(\mathrm{Ker}\, f_*)$ and $\mathrm{Ker}(f_*)$ are not orthogonal to each other. Denote $\mathcal{V} = \mathrm{Ker}\, f_*$, the vertical space of $T_p(M_1)$ and $\mathcal{H} = \mathrm{tr}(\mathrm{Ker}\, f_*)$, the horizontal space then we have

$$T_p(M_1) = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Definition 2.1 [11]. *Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. Let $f: M_1 \to M_2$ be a submersion such that*:

(a) $\dim \triangle = \dim\{(\operatorname{Ker} f_*) \cap (\operatorname{Ker} f_*)^\perp\} = r,\ 0 < r < \min\big\{\dim(\operatorname{Ker} f_*), \dim(\operatorname{Ker} f_*)^\perp\big\}$;

(b) *f_* preserves the length of horizontal vectors, that is, $g_1(X, Y) = g_2(f_*X, f_*Y)$ for $X, Y \in \Gamma(\mathcal{H})$.*

Then f is called an r-lightlike submersion.

Case 2: $\dim \triangle = \dim(\operatorname{Ker} f_*) < \dim(\operatorname{Ker} f_*)^\perp$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = S(\operatorname{Ker} f_*)^\perp \perp \operatorname{ltr}(\operatorname{Ker} f_*)$ and f is called an isotropic submersion.

Case 3: $\dim \triangle = \dim(\operatorname{Ker} f_*)^\perp < \dim(\operatorname{Ker} f_*)$. Then $\mathcal{V} = S(\operatorname{Ker} f_*) \perp \triangle$ and $\mathcal{H} = \operatorname{ltr}(\operatorname{Ker} f_*)$ and f is called a coisotropic submersion.

Case 4: $\dim \triangle = \dim(\operatorname{Ker} f_*) = \dim(\operatorname{Ker} f_*)^\perp$. Then $\mathcal{V} = \triangle$ and $\mathcal{H} = \operatorname{ltr}(\operatorname{Ker} f_*)$ and f is called a totally lightlike submersion.

We need the following theorem to define slant lightlike submersion from an indefinite almost Hermitian manifold onto a lightlike manifold.

Theorem 2.1. *Let $f: M_1 \to M_2$ be an r-lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J_1), where g_1 is a semi-Riemannian metric of index $2r$ to an r-lightlike manifold (M_2, g_2). Let $J\triangle$ be a distribution on M such that $\triangle \cap J\triangle = 0$. Then any complementary distribution to $J\triangle \oplus J \operatorname{ltr}(\operatorname{Ker} f_*)$ in $S(\operatorname{Ker} f_*)$ is Riemannian.*

Proof. Assume that $J \operatorname{ltr}(\operatorname{Ker} f_*)$ is invariant with respect to J therefore $1 = g(\xi, N) = g(J\xi, JN) = 0$, for any $\xi \in \Gamma(\operatorname{Rad}(\operatorname{Ker} f_*))$ and $N \in \Gamma(\operatorname{ltr}(\operatorname{Ker} f_*))$, which leads to a contradiction. Also $J \operatorname{ltr}(\operatorname{Ker} f_*)$ does not belong to $S(\operatorname{Ker} f_*)^\perp$, since $S(\operatorname{Ker} f_*)^\perp$ is orthogonal to $S(\operatorname{Ker} f_*)$, therefore $0 = g(J\xi, JN) = g(\xi, N) = 1$. Thus $J \operatorname{ltr}(\operatorname{Ker} f_*)$ is distribution on M. Moreover $J \operatorname{ltr}(\operatorname{Ker} f_*)$ does not belong to $\triangle$, if $JN \in \Gamma(\triangle)$ then $J^2N = -N \in \Gamma(J\triangle)$ which is a contradiction. Similarly $J \operatorname{ltr}(\operatorname{Ker} f_*)$ does not belong to $J\triangle$. Hence $J \operatorname{ltr}(\operatorname{Ker} f_*) \subset S(\operatorname{Ker} f_*)$ such that $J\triangle \cap J \operatorname{ltr}(\operatorname{Ker} f_*) = \{0\}$.

Denote the complementary distribution to $J\triangle \oplus J \operatorname{ltr}(\operatorname{Ker} f_*)$ in $S(\operatorname{Ker} f_*)$ by D. Then for a local quasiorthonormal frames on M_1, $\{\xi_1, \ldots, \xi_r, J\xi_1, \ldots, J\xi_r, N_1, \ldots N_r, JN_1, \ldots, JN_r\}$ form an orthonormal basis of $\triangle \oplus J\triangle \oplus \operatorname{ltr}(\operatorname{Ker} f_*) \oplus J \operatorname{ltr}(\ker f_*)$. Now define $\{U_1, \ldots, U_{2r}, V_1, \ldots, V_{2r}\}$ as

$$U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1), \qquad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1),$$

$$U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2), \qquad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2),$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r), \qquad U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r),$$

$$V_1 = \frac{1}{\sqrt{2}}(J\xi_1 + JN_1), \qquad V_2 = \frac{1}{\sqrt{2}}(J\xi_1 - JN_1),$$

$$V_3 = \frac{1}{\sqrt{2}}(J\xi_2 + JN_2), \qquad V_4 = \frac{1}{\sqrt{2}}(J\xi_2 - JN_2),$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$V_{2r-1} = \frac{1}{\sqrt{2}}(J\xi_r + JN_r), \qquad V_{2r} = \frac{1}{\sqrt{2}}(J\xi_r - JN_r).$$

Hence $\mathrm{Span}\{\xi_i, N_i, J\xi_i, JN_i\}$ is a nondegenerate space of constant index $2r$, that is, $\triangle \oplus J\triangle \oplus \oplus \mathrm{ltr}(\mathrm{Ker}\, f_*) \oplus J\,\mathrm{ltr}(\mathrm{Ker}\, f_*)$ is nondegenerate and of constant index $2r$ on M_1. Since

$$\mathrm{index}(TM_1) =$$
$$= \mathrm{index}(\triangle \oplus \mathrm{ltr}(\mathrm{Ker}\, f_*)) + \mathrm{index}(J\triangle \oplus J\,\mathrm{ltr}(\mathrm{Ker}\, f_*)) + \mathrm{index}(D \perp S(\mathrm{Ker}\, f_*^{\perp})),$$

therefore we have $2r = 2r + \mathrm{index}(D \perp S(\mathrm{Ker}\, f_*^{\perp}))$, this implies that $D \perp S(\mathrm{Ker}\, f_*^{\perp})$ is Riemannian and hence D is Riemannian.

Theorem 2.1 is proved.

3. Slant lightlike submersion. Since geometry of lightlike submanifolds has extensive uses in mathematical physics therefore as a generalization of holomorphic and totally real submanifolds, slant lightlike submanifolds of indefinite Hermitian manifolds were introduced by Sahin in [9] as below.

Definition 3.1. *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Then M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied*:

(A) $\mathrm{Rad}(TM)$ *is a distribution on M such that* $J\,\mathrm{Rad}\,TM \cap \mathrm{Rad}(TM) = \{0\}$.

(B) *For each nonzero vector field tangent to D at $p \in U \subset M$, the angle $\theta(X)$ between JX and the vector space D_p is constant, that is, it is independent of the choice of $p \in U \subset M$ and $X \in D_p$, where D is complementary distribution to $J\,\mathrm{Rad}\,TM \oplus J\,\mathrm{ltr}(TM)$ in the screen distribution $S(TM)$. This constant angle $\theta(X)$ is called slant angle of the distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \dfrac{\pi}{2}$.*

Thus using Theorem 2.1 and the definition of slant lightlike submanifolds, we can define slant lightlike submersions from an indefinite almost Hermitian manifold onto a lightlike manifold as below.

Definition 3.2. *Let (M_1, g_1, J) be a real $2m$-dimensional indefinite almost Hermitian manifold, where g_1 is semi-Riemannian metric of index $2r$, $0 < r < m$ and (M_2, g_2) an r-lightlike manifold. Let $f \colon M_1 \to M_2$ be an r-lightlike submersion. We say that f is a slant lightlike submersion if the following conditions are satisfied*:

(C) *$J\triangle$ is a distribution in* $\mathrm{Ker}\, f_*$ *such that* $\triangle \cap J\triangle = \{0\}$.

(D) *For each nonzero vector field X tangent to D, the angle $\theta(X)$ between JX and D is constant, where D is complementary distribution to $J\triangle \oplus J\,\mathrm{ltr}(\mathrm{Ker}\, f_*)$ in $S(\mathrm{Ker}\, f_*)$.*

Hence we have

$$T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p =$$
$$= \big\{\triangle \perp (J\triangle \oplus J\,\mathrm{ltr}(\ker f_*) \perp D\} \oplus \{f(D) \perp \mu \perp \mathrm{ltr}(\mathrm{Ker}\, f_*)\big\},$$

where μ is the orthogonal complementary subbundle to $f(D)$ in $S(\mathrm{Ker}\, f_*)$. Let f be a slant lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J) onto an r-lightlike manifold (M_2, g_2), then any $X \in \mathcal{V}_p$ can be written as

$$JX = \phi X + \omega X, \tag{2}$$

where ϕX and ωX are the tangential and the transversal components of JX, respectively. Similarly for any $V \in \mathcal{H}_p$, we get

$$JV = BV + CV, \tag{3}$$

where BV and CV are the tangential and the transversal components of JV, respectively. Denote by P_1, P_2, Q_1 and Q_2 the projections on the distributions $\triangle$, $J\triangle$, $J\,\mathrm{ltr}(\ker f_*)$ and D, respectively. Then we can write

$$X = P_1X + P_2X + Q_1X + Q_2X \tag{4}$$

for any $X \in \mathcal{V}_p$. Applying J to (4) we obtain

$$JX = JP_1X + JP_2X + \phi Q_2X + \omega Q_2X + \omega Q_1X \tag{5}$$

for any $X \in \mathcal{V}_p$. Then clearly

$$JP_1X = \phi P_1X \in \Gamma(J\triangle), \qquad JP_2X = \phi P_2X \in \Gamma(\triangle), \quad \omega P_1X = 0, \qquad \omega P_2X = 0,$$
$$\phi Q_2X \in \Gamma(D), \quad \omega Q_2X \in \Gamma(f(D)), \quad \phi Q_1X = 0, \quad \omega Q_1X \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*)).$$

Therefore we can write

$$\phi X = \phi P_1X + \phi P_2X + \phi Q_2X.$$

Since the geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$. Therefore Sahin [11] defined these tensors for lightlike submersions as

$$\mathcal{T}_XY = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY, \tag{6}$$

$$\mathcal{A}_XY = \nu\nabla_{hX}hY + h\nabla_{hX}\nu Y, \tag{7}$$

where ∇ is the Levi–Civita connection of g_1. It should be noted that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric in Riemannian submersions but not in lightlike submersions because the horizontal and vertical subspaces are not orthogonal to each other. $\mathcal{T}$ and $\mathcal{A}$ both reverses the horizontal and vertical subspaces and moreover $\mathcal{T}$ has symmetry property, that is

$$\mathcal{T}_XY = \mathcal{T}_YX. \tag{8}$$

Using (6) and (7), we have the following lemma.

Lemma 3.1. *Let f be a slant lightlike submersion form an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then*

(i) $\nabla_UV = \mathcal{T}_UV + \nu\nabla_UV$,
(ii) $\nabla_VX = h\nabla_VX + \mathcal{T}_VX$,
(iii) $\nabla_XV = \mathcal{A}_XV + \nu\nabla_XV$,
(iv) $\nabla_XY = h\nabla_XY + \mathcal{A}_XY$

for any $X, Y \in \Gamma(\mathrm{tr}(\mathrm{Ker}\, f_*))$ and $U, V \in \Gamma(\mathrm{Ker}\, f_*)$.

Using (2) and (3) with Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Let f be a slant lightlike submersion form an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then*

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \phi Y, \qquad (\nabla_X \phi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y, \tag{9}$$

where

$$(\nabla_X \omega)Y = h\nabla_X \omega Y - \omega\nabla_X Y, \qquad (\nabla_X \phi)Y = \mathcal{V}\nabla_X \phi Y - \phi\mathcal{V}\nabla_X Y$$

for any $X, Y \in (\ker f_*)$.

Theorem 3.1. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then f is a proper slant lightlike submersion if and only if*

(i) $J(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ *is a distribution on* M_1;

(ii) *for any* $X \in \Gamma(\mathrm{Ker}\, f_*)$ *there exists a constant* $\lambda \in [-1, 0]$ *such that*

$$\phi^2 Q_2 X = \lambda Q_2 X. \tag{10}$$

Moreover, in this case, $\lambda = -\cos^2\theta$.

Proof. Let f be a slant lightlike submersion then $J\triangle$ is a distribution on $S(TM)$. Hence using Theorem 2.1, $J(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ is a distribution on M_1. Next the slant angle between JQ_2X and D_p is constant and given by

$$\cos\theta(Q_2X) = \frac{g(JQ_2X, \phi Q_2X)}{|JQ_2X||\phi Q_2X|} = -\frac{g(Q_2X, \phi^2 Q_2X)}{|Q_2X||\phi Q_2X|}. \tag{11}$$

On the other hand, $\cos\theta(Q_2X)$ is also given by

$$\cos\theta(Q_2X) = \frac{|\phi Q_2X|}{|JQ_2X|}. \tag{12}$$

Hence using (11) and (12), we obtain

$$\cos^2\theta(Q_2X) = -\frac{g(Q_2X, \phi^2 Q_2X)}{|Q_2X|^2}.$$

Since the angle $\theta(Q_2X)$ is constant on D therefore we have $\phi^2 Q_2X = \lambda Q_2X$, where $\lambda = -\cos^2\theta$. Conversely (i) implies that $J\triangle$ is a distribution on $S(\mathrm{Ker}\, f_*))$. Hence using Theorem 2.1, any complementary distribution to $J\triangle \oplus J\,\mathrm{ltr}(\mathrm{Ker}\, f_*)$ in $S(\mathrm{Ker}\, f_*)$ is Riemannian.

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let f be a proper slant lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2) with slant angle θ. Then, for any $X, Y \in (\mathrm{Ker}\, f_*)$, we have*

$$g_1(\phi X, \phi Y) = \cos^2\theta\, g_1(X, Y), \tag{13}$$

$$g_1(\omega X, \omega Y) = \sin^2\theta\, g_1(X, Y). \tag{14}$$

Theorem 3.2. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then f is a proper slant lightlike submersion if and only if*

(i) *$J(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ is a distribution on M;*

(ii) *for any vector field X tangent to M_1, there exists a constant $\nu \in [-1, 0]$ such that*

$$B\omega Q_2 X = \nu Q_2 X, \tag{15}$$

where $\nu = -\sin^2\theta$.

Proof. Let f be a slant lightlike submersion, then $J(\mathrm{ltr}(\ker f_*))$ is a distribution on M_1. Next, applying J to (5) and using (2) to (4), we obtain

$$-X = -P_1X - P_2X + \phi^2 Q_2 X + \omega\phi Q_2 X + B\omega Q_1 X + B\omega Q_2 X,$$

comparing the components of the distribution D both sides of the above equation we get

$$-Q_2X = \phi^2 Q_2 X + B\omega Q_2 X, \tag{16}$$

hence using (10), we obtain (15). Conversely, using (15) and (16), we have $\phi^2 Q_2 X = -\cos^2\theta Q_2 X$. Hence proof follows from Theorem 3.1.

Further, we prove that the orthogonal complement subbundle μ of fD in $S(\mathrm{Ker}\, f_*)^\perp$ is holomorphic with respect to J and we obtain its dimension.

Theorem 3.3. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then μ is invariant with respect to J.*

Proof. Using (2), for any $V \in \Gamma(\mu)$ and $\omega X \in \Gamma(f(D))$, we have $g_1(JV, \omega X) = -g_1(JV, \phi X)$. By virtue of Theorem 3.1, we get $g_1(JV, \omega X) = -\cos^2\theta g_1(V, X) + g_1(V, \omega\phi X) = 0$. Similarly $g_1(JV, Y) = -g_1(V, JY) = 0$ for any $Y \in \Gamma(\mathrm{Ker}\, f_*)$. Also for any $N \in \Gamma(\mathrm{ltr}(\mathrm{Ker}\, f_*))$ we have $g_1(JV, N) = -g_1(V, JN) = 0$. Hence the proof follows.

Theorem 3.4. *Let f be a proper slant lightlike submersion from an almost Hermitian manifold (M_1^m, g_1, J) onto an r-lightlike manifold (M_2^n, g_2), where g_1 is a semi-Riemannian metric of index $2r$. Then $\dim(\mu) = 2n - m + 2r$. If $\mu = \{0\}$, then $n = \dfrac{m - 2r}{2}$.*

Proof. Since $\dim D = m - n - 3r$ and $\dim S(\mathrm{Ker}\, f_*^\perp) = n - r$. Therefore $\dim\mu = 2n - m + 2r$. Moreover M_1 is almost Hermitian manifold so its dimension m is even and hence dimension of μ is even.

Lemma 3.3. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1^m, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2^n, g_2). Let $\{e_1, \ldots, e_{m-n-3r}\}$ be a local orthonormal basis of D, then $\{\csc\theta\omega e_1, \ldots \ldots, \csc\theta\omega e_{m-n-3r}\}$ is a local orthonormal basis of fD.*

Proof. Since $\{e_1, \ldots, e_{m-n-3r}\}$ be a local orthonormal basis of D and D is Riemannian therefore using (14) we have

$$g_1(\csc\theta\omega e_i, \csc\theta\omega e_j) = \csc^2\theta \sin^2\theta g_1(e_i, e_j) = \delta_{ij},$$

this proves the lemma.

Since for any $Q_2X \in \Gamma(D)$, $\phi Q_2 X \in \Gamma(D)$ therefore the distribution D is even dimensional. Hence we have the following result similar to the above lemma.

Lemma 3.4. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1^m, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2^n, g_2). If $\{e_1, \ldots, e_{\frac{m-n-3r}{2}}\}$ are unit vector fields in D, then $\{e_1, \sec\theta\phi e_1, e_2, \sec\theta\phi e_2, \ldots \ldots, e_{\frac{m-n-3r}{2}}, \sec\theta\phi e_{\frac{m-n-3r}{2}}\}$ is a local orthonormal basis of D.*

Theorem 3.5. *Let f be a lightlike submersion from an indefinite almost Hermitian manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). If ω is parallel with respect to ∇, then we have*

$$\mathcal{T}_{\phi X}\phi X = -cos^2\theta \mathcal{T}_X X, \qquad \mathcal{T}_{\phi X}\phi X = -\mathcal{T}_X X, \quad \text{and} \quad \mathcal{T}_{\phi X}\phi X = 0, \tag{17}$$

for any $X \in \Gamma(D)$, $X \in \Gamma(\triangle \perp J\triangle)$ and $X \in \Gamma(J(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, respectively.

Proof. Let ω be parallel, then from (9) we have $C\mathcal{T}_X Y = \mathcal{T}_X \phi Y$ for $X, Y \in \Gamma(TM_1)$. Interchanging the role of X and Y, we get $C\mathcal{T}_Y X = \mathcal{T}_Y \phi X$. Thus we obtain

$$C\mathcal{T}_X Y - C\mathcal{T}_Y X = \mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X.$$

Using (8), we derive

$$\mathcal{T}_X \phi Y = \mathcal{T}_Y \phi X.$$

Then substituting Y by ϕX we get $\mathcal{T}_X \phi^2 X = \mathcal{T}_{\phi X}\phi X$. Thus using Theorem 3.1 with the fact that $\phi^2 X = -X$, for any $X \in \Gamma(\triangle \perp J\triangle)$ and $\phi X = 0$ for any $X \in \Gamma(J(\mathrm{ltr}(\mathrm{Ker}\, f_*))$, (17) follows.

Theorem 3.6. *Let f be a lightlike submersion from an indefinite Kaehler manifold (M_1, g_1), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then the distribution $\mathcal{V}$ defines a totally geodesic foliation on M_1 if and only if*

$$\omega(\nu\nabla_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + h\nabla_X \omega Y) = 0$$

for any $X, Y \in \mathcal{V}$.

Proof. Let $X, Y \in \Gamma(\mathcal{V})$ then using Lemma 3.1 with (2) and (3) we obtain

$$\nabla_X Y = -J\nabla_X JY = -J(\mathcal{T}_X \phi Y + \nu\nabla_X \phi Y + \mathcal{T}_X \omega Y + h\nabla_X \omega Y) =$$

$$= -(B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi\nu\nabla_X \phi Y + \omega\nu\nabla_X \phi Y + \phi\mathcal{T}_X \omega Y + \omega\mathcal{T}_X \omega Y +$$

$$+ Bh\nabla_X \omega Y + Ch\nabla_X \omega Y).$$

Hence $\nabla_X Y \in \Gamma(\mathcal{V})$ if and only if $\omega(\nu\nabla_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + h\nabla_X \omega Y) = 0$.

Similarly, we can prove the following theorem.

Theorem 3.7. *Let f be a lightlike submersion from an indefinite Kaehler manifold (M_1, g_1), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then the distribution $\mathcal{H}$ defines a totally geodesic foliation on M_1 if and only if*

$$\phi(\nu\nabla_U BV + \mathcal{A}_U CV) + B(\mathcal{A}_U BV + h\nabla_U CV) = 0$$

for any $U, V \in \mathcal{H}$.

***Corollary* 3.2.** *Let f be a lightlike submersion from an indefinite Kaehler manifold (M_1, g_1), where g_1 is a semi-Riemannian metric of index $2r$, onto an r-lightlike manifold (M_2, g_2). Then M_1 is a locally product Riemannian manifold if and only if*

$$\omega(\nu\nabla_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + h\nabla_X \omega Y) = 0,$$

$$\phi(\nu\nabla_U BV + \mathcal{A}_U CV) + B(\mathcal{A}_U BV + h\nabla_U CV) = 0$$

for $X, Y \in \Gamma(\mathcal{V})$ and $U, V \in \Gamma(\mathcal{H})$.

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