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ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF DIFFERENCE EQUATION*

ПОРЯДОК РОСТУ МЕРОМОРФНИХ РОЗВ'ЯЗКІВ РІЗНИЦЕВОГО РІВНЯННЯ

We estimate the order of growth of meromorphic solutions of some linear difference equations and study the relationship between the exponent of convergence of zeros and the order of growth of the entire solutions of linear difference equations.

Оцінено порядок росту мероморфних розв'язків деяких лінійних різницевих рівнянь та вивчено співвідношення між показниками збіжності нулів та порядком зростання цілих розв'язків лінійних різницевих рівнянь.

1. Introduction and results. In this paper, we use the basic notions of Nevanlinna's theory (see [8, 12, 13]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Recently, many results of complex differences and difference equations are rapidly obtained (see [1–3, 5, 7, 9, 10]). Chiang and Feng [7] studied the growth of meromorphic solutions of homogeneous linear difference equation, when there exists only one coefficient having the maximal order, they obtained the following result.

Theorem A. *Let $A_0(z),\ldots,A_n(z)$ be entire functions such that there exists an integer l, $0 \leq l \leq n$, such that*

$$\sigma(A_l) > \max_{\substack{1\leq j\leq n \\ j\neq l}} \{\sigma(A_j)\}.$$

If $f(z)$ is a meromorphic solution to

$$A_n(z)y(z+n) + \ldots + A_1(z)y(z+1) + A_0(z)y(z) = 0,$$

then we have $\sigma(f) \geq \sigma(A_l) + 1$.

Laine and Yang [11] obtained that when the dominating coefficient depending on type but not order, Theorem A still holds. The result may be stated as follows.

Theorem B. *Let $A_0(z),\ldots,A_n(z)$ be entire functions of finite order such that among those coefficients having the maximal order $\sigma = \max\{\sigma(A_k), 0 \leq k \leq n\}$, exactly one has its type strictly greater than the others. If $f(z) \not\equiv 0$ is a meromorphic solution of equation*

$$A_n(z)f(z+\omega_n) + \ldots + A_1(z)f(z+\omega_1) + A_0(z)f(z) = 0, \tag{1.1}$$

then $\sigma(f) \geq \sigma + 1$.

Laine and Yang [11] raised the following question.

Question: Whether all meromorphic solutions $f(z)(\not\equiv 0)$ of equation (1.1) satisfy $\sigma(f) \geq 1 + \max_{0\leq j\leq n} \sigma(A_j)$, if there is no dominating coefficient.

Giving some restriction on the coefficients of difference equation, we answer this question and obtain the following results.

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Theorem 1.1. *Let $c_j,\ j = 1,\ldots,n,$ be distinct constants, $A_j(z) = P_j(z)e^{h_j(z)} + Q_j(z),\ j = 1,\ldots,n,$ where $h_j(z)$ are polynomials with degree $k \geq 1,\ P_j(z)(\not\equiv 0),\ Q_j(z)$ are entire functions of order less than k. Among leading coefficients of $h_j(z),\ j \in \{1,\ldots,n\}$, having the maximal modulus, there exists a term being unequal to the others. If $f(z)(\not\equiv 0)$ is a meromorphic solution of equation*

$$A_n(z)f(z+c_n) + \ldots + A_1(z)f(z+c_1) = 0, \tag{1.2}$$

then $\sigma(f) \geq k+1$.

***Corollary* 1.1.** *Let $k, A_j(z),\ j = 1,\ldots,n,$ be defined as in Theorem 1.1, $B_i(z),\ i = 1,\ldots,m,$ be entire functions of order less than k, and $c_j,\ j = 1,\ldots,n+m,$ be distinct constants. If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of equation*

$$B_m(z)f(z+c_{n+m}) + \ldots + B_1(z)f(z+c_{n+1}) + A_n(z)f(z+c_n) + \ldots + A_1(z)f(z+c_1) = 0, \tag{1.3}$$

then $\sigma(f) \geq k+1$.

***Example* 1.1.** The function $f(z) = e^{z^2}$ satisfies difference equation

$$e^{-2iz}f(z+i) + e^{2iz}f(z-i) - 2e^{-1}f(z) = 0.$$

Obviously, $\sigma(f) = 2 = \deg h_1 + 1 = \deg h_2 + 1$. This example shows that the equality in Corollary 1.1 can be arrived. So the estimation in Corollary 1.1 is precise.

By Theorems A, B and 1.1, we deduce the following corollary.

***Corollary* 1.2.** *Let $c_j,\ j = 1,2,$ be distinct nonzero constants, $h_j(z),\ j = 1,2,$ be polynomials, and $A_j(z)\ (\not\equiv 0),\ j = 0,1,2,$ be entire functions such that*

$$\max\{\sigma(A_j), 0 \leq j \leq 2\} < \max\{\deg h_1, \deg h_2\}.$$

If $f(z)\ (\not\equiv 0)$ is a meromorphic solution of equation

$$A_2(z)e^{h_2(z)}f(z+c_2) + A_1(z)e^{h_1(z)}f(z+c_1) + A_0(z)f(z) = 0, \tag{1.4}$$

then $\sigma(f) \geq \max\{\deg h_1, \deg h_2\} + 1$.

Chen [6] studied complex oscillation problems of entire solutions $f(z)$ to homogeneous and nonhomogeneous linear difference equations respectively, and obtained some relations between $\lambda(f)$ and $\sigma(f)$. These results may be stated as follows.

Theorem C. *Let $A_j(z),\ j = 1,\ldots,n,$ be entire functions such that there is at least one A_j being transcendental, $c_j,\ j = 1,\ldots,n,$ be constants which are unequal to each other. Suppose that $f(z)$ is a finite order transcendental entire solution of the homogeneous linear difference equation* (1.2) *and satisfies $\sigma(f) > \max\{\sigma(A_j) : 1 \leq j \leq n\} + 1$.*

Then $\lambda(f) \geq \sigma(f) - 1$. Moreover, if assume $n = 2$, then $\lambda(f) = \sigma(f)$.

Theorem D. *Let $F(z),\ A_j(z),\ j = 1,\ldots,n,$ be entire functions such that $F(z)A_n(z) \not\equiv 0,\ c_k,$ $k = 1,\ldots,n,$ be constants which are unequal to each other. Suppose that $f(z)$ is a finite order entire solution of the nonhomogeneous linear difference equation*

$$A_n(z)f(z+c_n) + \ldots + A_1(z)f(z+c_1) = F(z).$$

If $\sigma(f) > \max\{\sigma(F), \sigma(A_j) : 1 \leq j \leq n\}$, then $\lambda(f) = \sigma(f)$.

In the following, we continue to study the complex oscillation problems of entire solutions to linear difference equations (1.2) and (1.4), and obtain the following results, which extend Theorems C and D.

Theorem 1.2. *Let $c_j,\ j = 1, \ldots, n,$ be distinct constants, and $A_j(z)\ (\not\equiv 0),\ j = 1, \ldots, n,$ be entire functions with finite order. Suppose that $f(z)$ is a finite order entire solution of equation* (1.2) *and satisfies $\sigma(f) > \max\{\sigma(A_j) : 1 \le j \le n\} + 1$. Then $f(z)$ assumes every finite value d infinitely often and $\lambda(f - d) = \sigma(f)$.*

***Example* 1.2.** The entire function $f(z) = e^{z^2}$ satisfies linear difference equation

$$f(z + 1) - e^{2z+1} f(z) = 0.$$

Obviously, $A_2(z) \equiv 1, A_1(z) = -e^{2z+1}$. We see $\sigma(f) = 2 = \max\{\sigma(A_1), \sigma(A_2)\} + 1$, but $\lambda(f) = 0 < \sigma(f)$. This example shows that the condition in Theorem 1.2, $\sigma(f) > \max\{\sigma(A_j) : 1 \le j \le n\} + 1$, can not be weaken.

By Theorems D and 1.2, we have the following corollary.

***Corollary* 1.3.** *Under conditions of Theorem* 1.2, *for any small entire function $\varphi(z)\ (\not\equiv 0)$ satisfying $\sigma(\varphi) < \sigma(f)$, we have $\lambda(f - \varphi) = \sigma(f)$.*

***Corollary* 1.4.** *Let $h_1(z), h_2(z)$ be polynomials such that*

$$h_1(z) = a_n z^n + \ldots + a_0, \qquad h_2(z) = b_m z^m + \ldots + b_0,$$

where $a_n b_m \ne 0,\ A_j(z)\ (\not\equiv 0),\ j = 0, 1, 2,$ be entire functions of order less than $\max\{n, m\}$ and $c_k,\ k = 1, 2,$ be distinct nonzero constants such that $c_2 a_n - c_1 b_m \ne 0$ while $n = m$. If $f(z)\ (\not\equiv 0)$ is a finite order entire solution of (1.4), *then $\lambda(f) = \sigma(f) \ge \max\{n, m\} + 1$.*

Example 1.1 shows that the condition, $c_2 a_n - c_1 b_m \ne 0$ while $n = m$, in Corollary 1.4 can not be weaken.

2. Proofs of theorems and corollaries. We need the following lemmas for the proof of theorems and corollaries.

Lemma 2.1 [4]. *Suppose that $f(z)$ is a meromorphic function with $\sigma(f) = \sigma < \infty$, then for any given $\varepsilon > 0$, there is a set $E \subset (1, \infty)$ that has finite linear measure or finite logarithmic measure, such that*

$$|f(z)| \le \exp\{r^{\sigma+\varepsilon}\}$$

for all z satisfying $|z| = r \notin [0, 1] \cup E,\ r \to \infty$.

Lemma 2.2 [7]. *Let $\eta_1,\ \eta_2$ be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (0, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \le \left|\frac{f(z + \eta_1)}{f(z + \eta_2)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}.$$

***Proof of Theorem* 1.1.** Contrary to our assertion, we assume $\sigma(f) < k + 1$. Let

$$h_j(z) = a_{jk} z^k + h_j^*(z), \tag{2.1}$$

where $a_{jk} \ne 0$ are constants, $h_j^*(z)$ are polynomials with $\deg h_j^* \le k - 1,\ j = 1, \ldots, n.$

Set

$$I = \left\{ i \colon |a_{ik}| = \max_{1 \le j \le n} |a_{jk}| \right\}, \qquad \theta_j = \arg a_{jk} \in [0, 2\pi), \quad j \in I.$$

There exists $l \in I$ such that $a_{lk} \neq a_{jk}, j \in I \setminus \{l\}$. By this and the definitions of I and θ_j, we see that

$$|a_{jk}| = |a_{lk}|, \qquad \theta_j \neq \theta_l, \quad j \in I \setminus \{l\}.$$

Choosing θ such that

$$\cos(k\theta + \theta_l) = 1. \tag{2.2}$$

By $\theta_j \neq \theta_l,\ j \in I \setminus \{l\}$, we have

$$\cos(k\theta + \theta_j) < 1, \quad j \in I \setminus \{l\}. \tag{2.3}$$

Denote

$$a = \max_{1 \le j \le n} \{|a_{jk}|\}, \qquad b = \max_{j \notin I}\{|a_{jk}|\}, \qquad c = \max\left\{b, a\cos(k\theta + \theta_j),\ j \in I \setminus \{l\}\right\} < a, \tag{2.4}$$

and

$$\sigma = \sigma(f) < k + 1, \qquad \beta = \max_{1 \le j \le n} \{\sigma(P_j), \sigma(Q_j)\} < k. \tag{2.5}$$

Obviously,

$$\sigma\left(\frac{P_j}{P_l}\right) \le \max\{\sigma(P_j), \sigma(P_l)\} \le \beta, \qquad 1 \le j \le n, \quad j \neq l,$$

$$\sigma\left(\frac{Q_j}{P_l}\right) \le \max\{\sigma(Q_j), \sigma(P_l)\} \le \beta, \quad 1 \le j \le n.$$

By Lemma 2.1, for any given $\varepsilon,\ 0 < 2\varepsilon < \min\{1, k+1-\sigma, k-\beta, a-c\}$, there is a set $E_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfies $|z| = r \notin E_1 \cup [0,1]$, we obtain

$$\left|\frac{P_j(z)}{P_l(z)}\right| \le \exp\{r^{\beta+\varepsilon}\}, \quad 1 \le j \le n, j \neq l; \qquad \left|\frac{Q_j(z)}{P_l(z)}\right| \le \exp\{r^{\beta+\varepsilon}\}, \quad 1 \le j \le n. \tag{2.6}$$

It is clear that $\exp\{-h_l^*(z)\}$ is of regular order $\deg h_l^*$, $\exp\{h_j^*(z)\}$, $1 \le j \le n$, $j \neq l$, is of regular order $\deg h_j^*$. By $\deg h_j^* \le k-1$, $1 \le j \le n$, then for all large z, $|z| = r$, we get

$$|\exp\{-h_l^*(z)\}| \le \exp\{r^{k-1+\varepsilon}\}, \qquad \left|\exp\{h_j^*(z)\}\right| \le \exp\{r^{k-1+\varepsilon}\}, \quad 1 \le j \le n, j \neq l. \tag{2.7}$$

Applying Lemma 2.2 to $f(z)$, there is a set $E_2 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfies $|z| = r \notin E_2 \cup [0,1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right| \le \exp\{r^{\sigma-1+\varepsilon}\}, \qquad 1 \le j \le n, \quad j \neq l. \tag{2.8}$$

By (1.2) and (2.1), we obtain

$$-\exp\{a_{lk}z^k\} = \sum_{j \in I \setminus \{l\}} \exp\{-h_l^*(z)\} \frac{f(z+c_j)}{f(z+c_l)} \left(\frac{P_j(z)}{P_l(z)} \exp\{a_{jk}z^k\} \exp\{h_j^*(z)\} + \frac{Q_j(z)}{P_l(z)}\right) +$$

$$+\sum_{j\notin I}\exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\}+\frac{Q_j(z)}{P_l(z)}\right)+$$

$$+\exp\{-h_l^*(z)\}\frac{Q_l(z)}{P_l(z)}. \tag{2.9}$$

Let $z=re^{i\theta}$, where $r\notin E_1\cup E_2\cup[0,1]$. Substituting (2.2) – (2.4), (2.6) – (2.8) into (2.9), we get

$$\exp\{ar^k\}\le\sum_{j\in I\setminus\{l\}}\exp\{r^{k-1+\varepsilon}+r^{\sigma-1+\varepsilon}+r^{\beta+\varepsilon}\}\Big(\exp\{a\cos(k\theta+\theta_j)r^k+r^{k-1+\varepsilon}\}+1\Big)+$$

$$+\sum_{j\notin I}\exp\{r^{k-1+\varepsilon}+r^{\sigma-1+\varepsilon}+r^{\beta+\varepsilon}\}\Big(\exp\{(b+\varepsilon)r^k+r^{k-1+\varepsilon}\}+1\Big)+$$

$$+\exp\{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}\}\le$$

$$\le n\exp\{(c+\varepsilon)r^k+2r^{k-1+\varepsilon}+r^{\sigma-1+\varepsilon}+r^{\beta+\varepsilon}\}\le$$

$$\le n\exp\{(c+2\varepsilon)r^k\}. \tag{2.10}$$

Dividing by $\exp\{ar^k\}$ both sides of (2.10) and letting $r\to\infty$, we have $1\le 0$. A contradiction. Hence, $\sigma(f)\ge k+1$.

***Proof of Corollary* 1.1.** We assume $\sigma(f)<k+1$. Using a same method as the proof of Theorem 1.1, we also obtain (2.1) – (2.7).

By Lemma 2.1, there is a set $E_3\subset(1,\infty)$ with finite logarithmic measure such that for all z satisfies $|z|=r\notin E_3\cup[0,1]$, we obtain

$$|B_j(z)|\le\exp\{r^{\beta_1+\varepsilon}\},\quad 1\le j\le m, \tag{2.11}$$

where $\beta_1=\max\{\sigma(B_j),\ 1\le j\le m\}<k$.

Applying Lemma 2.2 to $f(z)$, there is a set $E_4\subset(1,\infty)$ with finite logarithmic measure such that for all z satisfies $|z|=r\notin E_4\cup[0,1]$, we get

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right|\le\exp\{r^{\sigma-1+\varepsilon}\},\qquad 1\le j\le n+m,\quad j\ne l. \tag{2.12}$$

By (1.3) and (2.1), we have

$$-\exp\{a_{lk}z^k\}=\sum_{j\in I\setminus\{l\}}\exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\}+\frac{Q_j(z)}{P_l(z)}\right)+$$

$$+\sum_{j\notin I}\exp\{-h_l^*(z)\}\frac{f(z+c_j)}{f(z+c_l)}\left(\frac{P_j(z)}{P_l(z)}\exp\{a_{jk}z^k\}\exp\{h_j^*(z)\}+\frac{Q_j(z)}{P_l(z)}\right)+$$

$$+\sum_{j=n+1}^{n+m}B_j(z)\frac{f(z+c_j)}{f(z+c_l)}+\exp\{-h_l^*(z)\}\frac{Q_l(z)}{P_l(z)}. \tag{2.13}$$

Let $z=re^{i\theta}$, where $r\notin E_1\cup E_2\cup E_3\cup E_4\cup[0,1]$. Substituting (2.2) – (2.7), (2.11) and (2.12) into (2.13), we obtain

$$\exp\{ar^k\} \leq \sum_{j\in I\setminus\{l\}} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\} \Big(\exp\{a\cos(k\theta+\theta_j)r^k + r^{k-1+\varepsilon}\} + 1\Big) +$$
$$+ \sum_{j\notin I} \exp\{r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\} \Big(\exp\{(b+\varepsilon)r^k + r^{k-1+\varepsilon}\} + 1\Big) +$$
$$+m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\} + \exp\{r^{k-1+\varepsilon} + r^{\beta+\varepsilon}\} \leq$$
$$\leq n\exp\{(c+\varepsilon)r^k + 2r^{k-1+\varepsilon} + r^{\sigma-1+\varepsilon} + r^{\beta+\varepsilon}\} + m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\} \leq$$
$$\leq n\exp\{(c+2\varepsilon)r^k\} + m\exp\{r^{\beta_1+\varepsilon} + r^{\sigma-1+\varepsilon}\}. \tag{2.14}$$

Dividing by $\exp\{ar^k\}$ both sides of (2.14) and letting $r\to\infty$, we have $1 \leq 0$. It is a contradiction. So, $\sigma(f) \geq k+1$ holds.

***Proof of Theorem* 1.2.** Consider the following two cases.

Case 1: $d = 0$.

Contrary to our assertion, suppose that $\lambda(f) < \sigma(f)$. Then $f(z)$ can be written as

$$f(z) = H(z)e^{h(z)}, \tag{2.15}$$

where $H(z)(\not\equiv 0)$ is canonical product (or polynomial) formed by zeros of $f(z)$ such that

$$\lambda(H) = \sigma(H) = \lambda(f) < \sigma(f)$$

and

$$h(z) = a_k z^k + a_{k-1}z^{k-1} + \ldots + a_0, \tag{2.16}$$

where $k \in \mathbb{N}^+$ satisfying $k = \sigma(f) > \lambda(f)$, and $a_k(\neq 0), a_{k-1}, \ldots, a_0$ are constants.

Substituting (2.15) into (1.2), we obtain

$$A_n(z)H(z+c_n)\exp\{h(z+c_n)\} + \ldots + A_1(z)H(z+c_1)\exp\{h(z+c_1)\} = 0,$$

or

$$A_n(z)\exp\{h(z+c_n) - h(z+c_1)\}H(z+c_n) + \ldots$$
$$\ldots + A_2(z)\exp\{h(z+c_2) - h(z+c_1)\}H(z+c_2) + A_1(z)H(z+c_1) = 0. \tag{2.17}$$

Since $\sigma(f) > \max\{\sigma(A_j)\colon 1 \leq j \leq n\} + 1$, then $\deg h(z) = k \geq 2$. By (2.16), we get

$$h(z+c_j) - h(z+c_1) = ka_k(c_j - c_1)z^{k-1} + h_j^*(z), \tag{2.18}$$

where $h_j^*(z)$ are polynomials with $\deg h_j^* \leq k-2,\ j = 2,\ldots,n$.

Set

$$I = \{i\colon |c_i - c_1| = \max_{2\leq j\leq n} |c_j - c_1|\}.$$

We consider two cases in the following.

Case 1.1. I contains exactly one term.

Without loss of generality, assume $I = \{n\}$. By $\sigma(A_j) < \sigma(f) - 1 = k - 1,\ j = 1, \ldots, n$, and (2.18), we have

$$\sigma\left(A_j \exp\{h(z + c_j) - h(z + c_1)\}\right) = \deg(h(z + c_j) - h(z + c_1)) = k - 1, \quad j = 2, \ldots, n.$$

By the definition of I and $I = \{n\}$, we see in the equation (2.17), the type $k|a_k(c_n - c_1)|$ of coefficient $A_n \exp\{h(z+c_n) - h(z+c_1)\}$ is strictly greater than types $k|a_k(c_j - c_1)|$ of coefficients $A_j \exp\{h(z+c_j) - h(z+c_1)\}$, $j = 2, \ldots, n-1$. By this and applying Theorem B to equation (2.17), we get $\sigma(H) \geq (k - 1) + 1 = k = \sigma(f)$. A contradiction. So, $\lambda(f) = \sigma(f)$.

Case 1.2**.** I contains more than one term.

Without loss of generality, assume $I = \{s, s + 1, \ldots, n\},\ 2 \leq s < n$. Set

$$a_k = |a_k|e^{i\theta_0}, \qquad \theta_j = \arg(c_j - c_1), \quad j = s, \ldots, n.$$

From the definition of I, we deduce

$$|c_j - c_1| < |c_n - c_1|, \quad j = 1, \ldots, s - 1,$$
$$|c_j - c_1| = |c_n - c_1|, \quad j = s, \ldots, n.$$

Since c_j are distinct constants, θ_j are distinct constants, too. So we may choose $\theta \in [0, 2\pi)$ such that

$$\cos((k - 1)\theta + \theta_0 + \theta_n) = 1. \tag{2.19}$$

By $\theta_j \neq \theta_n,\ j = s, \ldots, n - 1$, and (2.19), we see

$$\cos((k - 1)\theta + \theta_0 + \theta_j) < 1, \quad j = s, \ldots, n - 1. \tag{2.20}$$

Denote

$$\begin{gathered} a = |a_k(c_n - c_1)|, \qquad \beta = \max_{1\leq j<s}\{|a_k(c_j - c_1)|\}, \\ b = \max_{s\leq j\leq n-1}\{a\cos((k - 1)\theta + \theta_0 + \theta_j), \beta\}, \qquad \alpha = \max_{1\leq j\leq n}\{\sigma(A_j), \lambda(f) - 1, k - 2\}. \end{gathered} \tag{2.21}$$

Obviously,

$$\beta < a, \qquad b < a, \qquad \alpha < k - 1. \tag{2.22}$$

By Lemma 2.1, for any given $\varepsilon,\ 0 < \varepsilon < \min\{a - b, 1\}$, there exists a set $E_1 \subset (1, \infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{A_j(z)}{A_n(z)}\right| \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j = 1, \ldots, n - 1. \tag{2.23}$$

We know both $\exp\{-h_n^*\}$ and $\exp\{h_j^* - h_n^*\}$ are of regular order $\leq k - 2 \leq \alpha$. Then for large $z, |z| = r$, we obtain

$$|\exp\{-h_n^*\}| \leq \exp\{r^{\alpha+\varepsilon}\}, \qquad \left|\exp\{h_j^* - h_n^*\}\right| \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j = 2, \ldots, n - 1. \tag{2.24}$$

Applying Lemma 2.2 to $H(z)$, there exists a set $E_2 \subset (1, \infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, we get

$$\left|\frac{H(z+c_j)}{H(z+c_n)}\right| \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j=1,\ldots,n-1. \tag{2.25}$$

By (2.17), we have

$$-\exp\{ka_k(c_n-c_1)z^{k-1}\} = \sum_{j=s}^{n-1}\frac{A_j}{A_n}\frac{H(z+c_j)}{H(z+c_n)}\exp\{h_j^*-h_n^*\}\exp\{ka_k(c_j-c_1)z^{k-1}\}+$$

$$+\sum_{j=2}^{s-1}\frac{A_j}{A_n}\frac{H(z+c_j)}{H(z+c_n)}\exp\{h_j^*-h_n^*\}\exp\{ka_k(c_j-c_1)z^{k-1}\}+$$

$$+\frac{A_1}{A_n}\frac{H(z+c_1)}{H(z+c_n)}\exp\{-h_n^*\}. \tag{2.26}$$

Take $z=re^{i\theta}$, where $r\notin[0,1]\cup E_1\cup E_2$. Substituting (2.19)–(2.25) into (2.26), we obtain

$$\exp\{kar^{k-1}\} \leq (n-2)\exp\{3r^{\alpha+\varepsilon}\}\exp\{kbr^{k-1}\}+\exp\{3r^{\alpha+\varepsilon}\} \leq$$

$$\leq (n-1)\exp\{kbr^{k-1}+3r^{\alpha+\varepsilon}\},$$

thus,

$$1 \leq (n-1)\exp\{3r^{\alpha+\varepsilon}+kbr^{k-1}-kar^{k-1}\}.$$

Letting $r\to\infty$, by (2.22), we get $1\leq 0$. It is impossible. Hence, $\lambda(f)=\sigma(f)$.

Case 2: $d\neq 0$.

Set $g(z)=f(z)-d$, then

$$f(z)=g(z)+d \tag{2.27}$$

and

$$\sigma(g)=\sigma(f)>\max\{\sigma(A_j)\colon 1\leq j\leq n\}+1. \tag{2.28}$$

Substituting (2.27) into (1.2), we obtain

$$A_n(z)g(z+c_n)+\ldots+A_1(z)g(z+c_1)=-d(A_n(z)+\ldots+A_1(z)). \tag{2.29}$$

If $A_n(z)+\ldots+A_1(z)\not\equiv 0$, by (2.28), (2.29) and Theorem D, we have $\lambda(g)=\sigma(g)$, that is, $\lambda(f-d)=\sigma(f)$.

If $A_n(z)+\ldots+A_1(z)\equiv 0$, then $g(z)$ is an entire solution of difference equation

$$A_n(z)g(z+c_n)+\ldots+A_1(z)g(z+c_1)=0.$$

By (2.28) and the above Case 1, we have $\lambda(g)=\sigma(g)$, that is, $\lambda(f-d)=\sigma(f)$.

From the above Cases 1 and 2, we see $f(z)$ assumes every finite value d infinitely often and $\lambda(f-d)=\sigma(f)$.

***Proof of Corollary* 1.4.** Without loss of generality, assume that $n\geq m$. By Corollary 1.2, we know $\sigma(f)\geq n+1$. If $\sigma(f)>n+1$, by Theorem 1.2, $\lambda(f)=\sigma(f)$ holds. So, we assume $\sigma(f)=n+1$.

Suppose that $\lambda(f)<\sigma(f)$, then $f(z)$ can be written as

$$f(z)=g(z)e^{h(z)}, \tag{2.30}$$

where $g(z)(\not\equiv 0)$ is canonical product (or polynomial) formed by zeros of $f(z)$ such that

$$\sigma(g) = \lambda(g) = \lambda(f) < \sigma(f) = n + 1,$$

and

$$h(z) = d_{n+1}z^{n+1} + d_n z^n + \ldots + d_0 \tag{2.31}$$

is a polynomial, where $d_{n+1} \neq 0, d_n \ldots, d_0$ are constants.

Substituting (2.30), (2.31) into (1.4) and dividing by $e^{h(z)}$, we obtain

$$A_2(z)e^{h(z+c_2)-h(z)+h_2(z)}g(z+c_2) + A_1(z)e^{h(z+c_1)-h(z)+h_1(z)}g(z+c_1) + A_0(z)g(z) = 0. \tag{2.32}$$

By (2.31), we see

$$\begin{aligned} h(z+c_1) - h(z) + h_1(z) &= ((n+1)c_1 d_{n+1} + a_n)z^n + h_1^*(z), \\ h(z+c_2) - h(z) + h_2(z) &= (n+1)c_2 d_{n+1} z^n + b_m z^m + h_2^*(z), \end{aligned} \tag{2.33}$$

where $h_1^*(z), h_2^*(z)$ are polynomials with degree no more than $n-1$.

Consider the following two cases.

Case 1: $n > m$.

By $(n+1)c_2 d_{n+1} \neq 0$, we see

$$\deg(h(z+c_2) - h(z) + h_2(z)) = n \geq \deg(h(z+c_1) - h(z) + h_1(z)).$$

Combining this with (2.32) and Corollary 1.2, we have $\sigma(g) \geq n+1$. A contradiction. So, $\lambda(f) = \sigma(f) = n+1$.

Case 2: $n = m$.

If $(n+1)c_1 d_{n+1} + a_n \neq 0$, it follows from (2.33) that

$$\deg(h(z+c_1) - h(z) + h_1(z)) = n \geq \deg(h(z+c_2) - h(z) + h_2(z)).$$

Combining this with (2.32) and Corollary 1.2, we have $\sigma(g) \geq n+1 = \sigma(f)$. A contradiction. So, $\lambda(f) = \sigma(f) = n+1$.

If $(n+1)c_1 d_{n+1} + a_n = 0$, since $c_1 \neq 0$, we get

$$(n+1)c_2 d_{n+1} + b_m = -\frac{a_n}{c_1}c_2 + b_m = \frac{c_1 b_m - c_2 a_n}{c_1} \neq 0,$$

then

$$\deg(h(z+c_2) - h(z) + h_2(z)) = n > \deg(h(z+c_1) - h(z) + h_1(z)).$$

Together with (2.32) and Corollary 1.2, we have $\sigma(g) \geq n+1$. A contradiction. So, $\lambda(f) = \sigma(f) =$ $= n+1$.

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