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$\mathcal{Z}^*$-SEMILOCAL MODULES AND THE PROPER CLASS $\mathcal{RS}$

$\mathcal{Z}^*$-НАПІВЛОКАЛЬНІ МОДУЛІ ТА ВЛАСНИЙ КЛАС $\mathcal{RS}$

Over an arbitrary ring, a module M is said to be $\mathcal{Z}^*$-*semilocal* if every submodule U of M has a $\mathcal{Z}^*$-supplement V in M, i.e., $M = U + V$ and $U \cap V \subseteq \mathcal{Z}^*(V)$, where $\mathcal{Z}^*(V) = \{m \in V \mid Rm \text{ is a small module}\}$ is the Rad-small submodule. In this paper, we study basic properties of these modules as a proper generalization of semilocal modules. In particular, we show that the class of $\mathcal{Z}^*$-semilocal modules is closed under submodules, direct sums, and factor modules. Moreover, we prove that a ring R is $\mathcal{Z}^*$-semilocal if and only if every injective left R-module is semilocal. In addition, we show that the class $\mathcal{RS}$ of all short exact sequences $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}(\psi)$ has a $\mathcal{Z}^*$-supplement in N is a proper class over left hereditary rings. We also study some homological objects of the proper class $\mathcal{RS}$.

Над довільним кільцем модуль M називається $\mathcal{Z}^*$-*напівлокальним*, якщо кожний підмодуль U модуля M має $\mathcal{Z}^*$-доповнення V в M, тобто $M = U + V$ і $U \cap V \subseteq \mathcal{Z}^*(V)$, де $\mathcal{Z}^*(V) = \{m \in V \mid Rm - \text{малий модуль}\}$ — Rad-малий підмодуль. У цій роботі вивчаються базові властивості таких модулів, як відповідного узагальнення напівлокальних модулів. Зокрема, показано, що клас $\mathcal{Z}^*$-напівлокальних модулів є замкненим відносно підмодулів, прямих сум і фактор-модулів. Крім того, доведено, що кільце R є $\mathcal{Z}^*$-напівлокальним тоді і тільки тоді, коли кожен ін'єктивний лівий R-модуль є напівлокальним. Також встановлено, що клас $\mathcal{RS}$ усіх коротких послідовностей $\mathbb{E}:$ $0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ таких, що $\mathrm{Im}(\psi)$ має $\mathcal{Z}^*$-доповнення в N, є власним класом над лівими спадковими кільцями. Вивчено також деякі гомологічні об'єкти власного класу $\mathcal{RS}$.

1. Introduction. Throughout this study, all rings are associative with identity and all modules are unital left R-modules. Let R be a ring and M be a left R-module. The Jacobson radical of M will be denoted by $\mathrm{Rad}(M)$, and the injective hull of the module M will be denoted by $E(M)$. The notation $N \subseteq M$ ($N \subset M$) means that N is a (proper) submodule of M. A non-zero submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every non-zero submodule $N \subseteq M$. Dually, a proper submodule $N \subset M$ is said to be *small* in M, denoted by $N \ll M$, if $M \neq N + K$ for every proper submodule K of M (see [14], 19.1). A module M is said to be *small* if M is a small submodule of some R-module (see [7]). It is shown in [7] (Theorem 1) that a module M is *small* if and only if M is a small submodule of $E(M)$. It is clear that every small submodule of M is a small module. For a module M, we consider the following submodule of M:

$$\mathcal{Z}^*(M) = \{m \in M \mid Rm \textit{ is a small module}\}.$$

Since $\mathrm{Rad}(M)$ is the sum of all small submodules of M, we get $\mathrm{Rad}(M) \subseteq \mathcal{Z}^*(M)$. It is easy to see that $\mathcal{Z}^*(M) = M \cap \mathrm{Rad}(E(M))$. Clearly, $\mathcal{Z}^*(M) = M$ if and only if $M \subseteq \mathrm{Rad}(E(M))$. A module M is said to be Rad-*small* (according to [13], cosingular) if $\mathcal{Z}^*(M) = M$. Since $\mathcal{Z}^*(\mathcal{Z}^*(M)) = \mathcal{Z}^*(M)$, $\mathcal{Z}^*(M)$ is the largest Rad-small submodule of M. Small modules are Rad-small. Also, a finitely generated Rad-small module is small.

Let M be a module and U, $V \subseteq M$ be submodules. V is called a *supplement* (Rad-*supplement*, respectively) of U in M if $M = U + V$ and $U \cap V \ll V$ ($U \cap V \subseteq \mathrm{Rad}(V)$). M is called *supplemented* (Rad-*supplemented*, respectively) if every submodule of M has a (Rad-) supplement in M. Characterizations and structures of supplemented and Rad-supplemented modules are extensively studied by many authors. We specifically mention [4, 14, 15] among papers concerning supplemented and Rad-supplemented modules.

Since $\mathrm{Rad}(V) \subseteq \mathcal{Z}^*(V)$, it is natural to introduce another notion that we called a submodule V of M a *$\mathcal{Z}^*$-supplement* of U in M provided $M = U + V$ and $U \cap V \subseteq \mathcal{Z}^*(V)$. Following [13] (Lemma 2.6 and Proposition 3.10), we characterize modules whose submodules have a $\mathcal{Z}^*$-supplement.

Lemma 1.1. *Let R be a ring and M be an R-module. Then the following statements are equivalent:*

(1) *Every submodule U of M has a $\mathcal{Z}^*$-supplement V in M.*

(2) *For any submodule U of M, there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq \mathcal{Z}^*(M)$.*

(3) *If U is a submodule of M, then $M = U + V$ and $U \cap V$ is* Rad*-small for some submodule V of M.*

(4) $\dfrac{M}{\mathcal{Z}^*(M)}$ *is semisimple.*

We say that a module M *$\mathcal{Z}^*$-semilocal* if M has one of the equal conditions of Lemma 1.1 as a proper generalization of semilocal modules. In Section 2, we obtain the basic properties of these modules. We show that the class of $\mathcal{Z}^*$-semilocal modules is closed under submodules, direct sums and factor modules. We prove that a ring R is $\mathcal{Z}^*$-semilocal if and only if every left R-module is $\mathcal{Z}^*$-semilocal if and only if every injective left R-module is semilocal. Let $\mathcal{RS}$ be the class of all short exact sequences $\mathbb{E}\colon 0 \longrightarrow M \overset{\psi}{\longrightarrow} N \overset{\phi}{\longrightarrow} K \longrightarrow 0$ such that $\mathrm{Im}\,(\psi)$ has a $\mathcal{Z}^*$-supplement in N. In Section 3, we show that $\mathcal{RS}$ is a proper class over left hereditary rings. We study on some homological objects of the proper class $\mathcal{RS}$ in the same section. In particular, we show that over left hereditary rings the proper class $\mathcal{RS}$ is coinjectively generated by all Rad-small modules.

The following lemma will be frequently used in this paper.

Lemma 1.2 (see [13], Lemma 2.6)**.** *The class of* Rad*-small left R-modules is closed under submodules, direct sums and factor modules.*

2. $\mathcal{Z}^*$-semilocal modules and rings.

Let M be a module. M is called *semilocal* if $\dfrac{M}{\mathrm{Rad}(M)}$ is semisimple, and a ring R is called *semilocal* if $\dfrac{R}{\mathrm{Rad}(R)}$ is a semisimple ring (see [8]).

It is clear that every semilocal module is $\mathcal{Z}^*$-semilocal, but the following example shows that the converse is not true, in general. Firstly, we need the following simple fact.

Lemma 2.1. *Every* Rad*-small module is $\mathcal{Z}^*$-semilocal.*

Proof. Let M be a Rad-small module. Then $\mathcal{Z}^*(M) = M$. Thus, it is $\mathcal{Z}^*$-semilocal.

Example **2.1.** Let $M =_{\mathbb{Z}} \mathbb{Z}$. Since M is a small submodule of the injective hull of $E(M)$, it is Rad-small. So, $Z^*(M) = M$. Applying Lemma 2.1, M is Z^*-semilocal. On the other hand, M is not semilocal.

Recall from [8] that a module M is *weakly supplemented* if every submodule U of M has a weak supplement V in M, that is, $M = U + V$ and $U \cap V \ll M$. Every supplemented module is weakly supplemented and weakly supplemented modules are semilocal.

Corollary **2.1.** *Let M be a module over an arbitrary ring. Suppose that $\mathcal{Z}^*(M)$ is a small submodule of M. Then the following statements are equivalent:*

(1) *M is weakly supplemented,*

(2) *M is semilocal,*

(3) *M is $\mathcal{Z}^*$-semilocal.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (1) Let $U \subseteq M$. By (3), there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq \mathcal{Z}^*(M)$. Since $\mathcal{Z}^*(M)$ is a small submodule of M, it follows from [14] (19.3.(4)) that $U \cap V \ll M$. Thus, V is a weak supplement of U in M. Hence, M is weakly supplemented.

Recall that a module M is *radical* if $M = \mathrm{Rad}(M)$, that is, M has no maximal submodules.

Lemma 2.2. *Every radical module is* Rad*-small.*

Proof. For a radical module M, let $m \in M$. Then $Rm \ll M$. So, Rm is small. Thus, $m \in \mathcal{Z}^*(M)$.

Let M be a module. By $P(M)$, we denote the sum of all radical submodules of M. $P(M)$ is the largest radical submodule of M. By using Lemmas 2.1 and 2.2, we obtain the following fact.

***Corollary* 2.2.** *$P(M)$ is $\mathcal{Z}^*$-semilocal for every module M.*

It is well known that any submodule of a semilocal module need not be semilocal. For example, ${}_{\mathbb{Z}}\mathbb{Z} \subseteq_{\mathbb{Z}} \mathbb{Q}$. But, we have the following proposition.

Proposition 2.1. *Every submodule of a $\mathcal{Z}^*$-semilocal module is $\mathcal{Z}^*$-semilocal.*

Proof. Let M be a $\mathcal{Z}^*$-semilocal module and $U \subseteq N \subseteq M$ be submodules. Since M is $\mathcal{Z}^*$-semilocal, we can write $M = U + V$ and $U \cap V$ is Rad-small for some submodule V of M. By using the modular law, $N = N \cap M = N \cap (U + V) = U + (N \cap V)$, and $U \cap (N \cap V) = = (U \cap N) \cap V = U \cap V$ is Rad-small. Hence, N is $\mathcal{Z}^*$-semilocal.

Proposition 2.2. *Every factor module of a $\mathcal{Z}^*$-semilocal module is $\mathcal{Z}^*$-semilocal.*

Proof. For a $\mathcal{Z}^*$-semilocal module M, let $N \subseteq U \subseteq M$ be submodules. Then there exists a submodule V of M such that $M = U + V$ and $U \cap V$ is Rad-small. Therefore, $\dfrac{M}{N} = \dfrac{U}{N} + \dfrac{V+N}{N}$. By using the canonical epimorphism $\pi : M \longrightarrow \dfrac{M}{N}$, we obtain that

$$\pi(U \cap V) = \frac{(U \cap V) + N}{N} = \frac{U \cap (V + N)}{N} = \frac{U}{N} \cap \frac{V+N}{N}$$

is Rad-small by Lemma 1.2. Hence, the factor module $\dfrac{M}{N}$ is $\mathcal{Z}^*$-semilocal.

Theorem 2.1. *Every direct sum of $\mathcal{Z}^*$-semilocal modules is $\mathcal{Z}^*$-semilocal.*

Proof. Let $\{M_i\}_{i \in I}$ be any collection of $\mathcal{Z}^*$-semilocal modules, where I is any index set. Put $M = \oplus_{i \in I} M_i$. It follows from [13] (Lemma 2.3) that

$$\frac{M}{\mathcal{Z}^*(M)} = \frac{\oplus_{i \in I} M_i}{\oplus_{i \in I} \mathcal{Z}^*(M_i)} \cong \oplus_{i \in I} \frac{M_i}{\mathcal{Z}^*(M_i)}$$

is semisimple as a direct sum of these semisimple modules $\dfrac{M_i}{\mathcal{Z}^*(M_i)}$. Therefore, M is $\mathcal{Z}^*$-semilocal.

***Corollary* 2.3.** *Any sum of $\mathcal{Z}^*$-semilocal submodules of a module M is $\mathcal{Z}^*$-semilocal.*

Proof. Let $\{N_i\}_{i \in I}$ be the family of $\mathcal{Z}^*$-semilocal submodules of the module M. Then, we can write the epimorphism $\Psi : \oplus_{i \in I} N_i \longrightarrow \sum_{i \in I} N_i$ via $\Psi((a_i)_{i \in I}) = \sum_{i \in I_0} a_i$, where I_0 is the finite set of the index set I. By Theorem 2.1, the external direct sum $\oplus_{i \in I} N_i$ is a $\mathcal{Z}^*$-semilocal module. It follows from Proposition 2.2 that the submodule $\sum_{i \in I} N_i$ is $\mathcal{Z}^*$-semilocal.

***Remark* 2.1.** Let R be a ring with identity. Suppose that ${}_RR$ is a $\mathcal{Z}^*$-semilocal R-module. Then, by Lemma 1.1, $\dfrac{R}{\mathcal{Z}^*(R)}$ is a semisimple left R-module. Therefore, $\dfrac{R}{\mathcal{Z}^*(R)}$ is a semisimple $\dfrac{R}{\mathcal{Z}^*(R)}$-module and so $\dfrac{R}{\mathcal{Z}^*(R)}$ is a semisimple ring. It follows that $\dfrac{R}{\mathcal{Z}^*(R)}$ is a semisimple right R-module. That is, R_R is a $\mathcal{Z}^*$-semilocal R-module. Similarly, if R_R is a $\mathcal{Z}^*$-semilocal R-module, it can be shown that ${}_RR$ is a $\mathcal{Z}^*$-semilocal R-module. By using this fact, we say that R is *a $\mathcal{Z}^*$-semilocal ring* if ${}_RR$ (or R_R) is a $\mathcal{Z}^*$-semilocal R-module.

It is shown in [8] (Theorem 3.5) that a ring R is semilocal if and only if every left R-module is semilocal. Now, we give an analogue of this fact for $\mathcal{Z}^*$-semilocal rings.

Lemma 2.3. *Let E be an injective module. Then E is $\mathcal{Z}^*$-semilocal if and only if it is semilocal.*

Proof. Let E be a $\mathcal{Z}^*$-semilocal module and $U \subseteq E$. Then there exists a submodule V of E such that $E = U + V$ and $U \cap V$ is Rad-small. Since E is injective, $\mathcal{Z}^*(E) = E \cap \mathrm{Rad}(E) = \mathrm{Rad}(E)$. So $U \cap V \subseteq \mathrm{Rad}(E)$. Hence, E is semilocal.

Theorem 2.2. *The following statements are equivalent for a ring R:*

(1) *R is $\mathcal{Z}^*$-semilocal,*

(2) *every left R-module is $\mathcal{Z}^*$-semilocal,*

(3) *every injective left R-module is semilocal.*

Proof. (1) $\Longrightarrow$ (2) Let M be any left R-module. Then, for an index set I, there exists an epimorphism $\Psi\colon R^{(I)} \longrightarrow M$. Since R is $\mathcal{Z}^*$-semilocal, it follows from Theorem 2.1 that the left free R-module $R^{(I)}$ is $\mathcal{Z}^*$-semilocal. Therefore, M is $\mathcal{Z}^*$-semilocal by Proposition 2.2.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (2) For any module M, the injective hull $E(M)$ is semilocal. Therefore, $E(M)$ is $\mathcal{Z}^*$-semilocal. Applying Proposition 2.1, we deduce that M is $\mathcal{Z}^*$-semilocal.

(2) $\Longrightarrow$ (1) It follows from (2) that ${}_RR$ is $\mathcal{Z}^*$-semilocal. Thus, R is a $\mathcal{Z}^*$-semilocal ring.

In [13], a ring R is called *left cosingular* if ${}_RR$ is Rad-small. Every commutative domain (which is not field) is left (right) cosingular. It is proven in [13] (Lemma 2.8) that R is a left cosingular ring if and only if every injective left R-module is radical. By using this fact, Theorem 2.2 and Lemma 2.1, we obtain that every left cosingular ring is $\mathcal{Z}^*$-semilocal. Now, we shall show that a $\mathcal{Z}^*$-semilocal ring need not be left cosingular in the following example.

***Example* 2.2.** Let $n > 1$ be a non-prime positive element of $\mathbb{Z}$. Then the ring $\mathbb{Z}_n$ is $\mathcal{Z}^*$-semilocal but not cosingular.

A ring R is called *left hereditary* if every factor module of an injective left R-module is injective (see [6]).

Lemma 2.4 (see [7], Theorem 3)**.** *Let R be a left hereditary ring and $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi}$ $\xrightarrow{\phi} K \longrightarrow 0$ be a short exact sequence of left R-modules. Then M and K are small modules if and only if N is a small module.*

We give an analogous characterization of this fact for Rad-small modules.

Lemma 2.5. *Let R be a left hereditary ring and $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0$ be a short exact sequence of left R-modules. Then M and K are* Rad*-small modules if and only if N is a* Rad*-small module.*

Proof. ($\Longrightarrow$) To simplify the notation, we think of M as a submodule of N. Since M is Rad-small, we get $M \subseteq \mathrm{Rad}(E(M))$. Therefore, $M \subseteq \mathrm{Rad}(E(N))$. Moreover, $\dfrac{N}{M}$ is Rad-small in

$\dfrac{E(N)}{M}$ is injective over a left hereditary ring R. Thus, $N \subseteq \mathrm{Rad}(E(N))$. This means that N is Rad-small.

($\Longleftarrow$) It follows from Lemma 1.2.

Lemma 2.6. *Let R be a left hereditary ring and M be a left R-module. Suppose that a submodule N of M is Rad-small. Then $\mathcal{Z}^*\left(\dfrac{M}{N}\right) = \dfrac{\mathcal{Z}^*(M)}{N}$.*

Proof. By the hypothesis, we have $N \subseteq \mathcal{Z}^*(M)$. It follows that

$$\frac{\mathcal{Z}^*(M)+N}{N} = \frac{\mathcal{Z}^*(M)}{N} \subseteq \mathcal{Z}^*\left(\frac{M}{N}\right).$$

Let $m+N \in \mathcal{Z}^*\left(\dfrac{M}{N}\right)$. Then $R(m+N) = \dfrac{Rm+N}{N}$ is a Rad-small module. Now, consider the following exact sequence:

$$0 \longrightarrow Rm \cap N \stackrel{i}{\longrightarrow} Rm \stackrel{\pi}{\longrightarrow} \frac{Rm+N}{N} \longrightarrow 0,$$

where i is the canonical injection and π is the canonical projection. Applying Lemma 2.5, since R is left hereditary, Rm is Rad-small and so $m \in \mathcal{Z}^*(M)$. This means that $\mathcal{Z}^*\left(\dfrac{M}{N}\right) \subseteq \dfrac{\mathcal{Z}^*(M)}{N}$. Hence, $\mathcal{Z}^*\left(\dfrac{M}{N}\right) = \dfrac{\mathcal{Z}^*(M)}{N}$.

Proposition 2.3. *Let R be a left hereditary ring and M be a left R-module. If a submodule N of M is Rad-small, M is $\mathcal{Z}^*$-semilocal if and only if $\dfrac{M}{N}$ is $\mathcal{Z}^*$-semilocal.*

Proof. ($\Longrightarrow$) By Proposition 2.2.

($\Longleftarrow$) Let $U \subseteq M$ be a submodule. By the hypothesis, we can write

$$\frac{M}{N} = \frac{U+N}{N} + \frac{V}{N} \qquad \text{and} \qquad \frac{U+N}{N} \cap \frac{V}{N}$$

is Rad-small for some submodule $\dfrac{V}{N}$ of $\dfrac{M}{N}$. Then $M = U + V$. Now,

$$\frac{U+N}{N} \cap \frac{V}{N} = \frac{(U+N)\cap V}{N} = \frac{U\cap V+N}{N} \subseteq \mathcal{Z}^*\left(\frac{M}{N}\right) = \frac{\mathcal{Z}^*(M)}{N}$$

according to Lemma 2.6. So, $U \cap V \subseteq \mathcal{Z}^*(M)$. Thus, M is $\mathcal{Z}^*$-semilocal.

In [14], over an arbitrary ring a module P is said to be a *small cover* of a module M if there exists an epimorphism $f\colon P \longrightarrow M$ with $\mathrm{Ker}\,(f) \ll P$. A submodule K of M is small in M if and only if M is a small cover of $\dfrac{M}{K}$. By using Proposition 2.3, we obtain the following result.

***Corollary* 2.4.** *Let R be a left hereditary ring and M be a $\mathcal{Z}^*$-semilocal R-module. Then every small cover of M is $\mathcal{Z}^*$-semilocal.*

Proof. Let $f\colon P \longrightarrow M$ be a small cover. Then $\mathrm{Ker}\,(f)$ is a small submodule of P and so $\mathrm{Ker}\,(f)$ is Rad-small. Since M is $\mathcal{Z}^*$-semilocal, we get $\dfrac{P}{\mathrm{Ker}\,(f)}$ is $\mathcal{Z}^*$-semilocal. Applying Proposition 2.3, we deduce that P is $\mathcal{Z}^*$-semilocal.

3. The proper class $\mathcal{RS}$.

Definition 3.1. *Let $\mathcal{P}$ be a class of short exact sequences of left R-modules and R-module homomorphisms. If a short exact sequence $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ belongs to $\mathcal{P}$, then ψ is said to be a $\mathcal{P}$-monomorphism and ϕ is said to be an $\mathcal{P}$-epimorphism.*

The class $\mathcal{P}$ is said to be a *proper class* (in the sense of Buchsbaum) if it has the following properties:

(P_1) If the short exact sequence $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

(P_2) $\mathcal{P}$ contains all splitting short exact sequences.

(P_3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

(P_3') The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

(P_4) If $\psi_1,\ \psi_2$ are monomorphisms and $\psi_2\psi_1$ is a $\mathcal{P}$-monomorphism, then ψ_1 is a $\mathcal{P}$-monomorphism.

(P_4') If $\phi_1,\ \phi_2$ are epimorphisms and $\phi_2\phi_1$ is an $\mathcal{P}$-epimorphism, then ϕ_2 is an $\mathcal{P}$-epimorphism.

***Example* 3.1.** We list some examples of proper classes:

(1) The smallest proper class $\mathcal{S}plit$ of all splitting short exact sequences of left R-modules.

(2) The largest proper class $\mathcal{A}bs$ of all short exact sequences of left R-modules.

(3) The proper class $\mathcal{S}upp$ of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}(\psi)$ is a supplement of some submodule of N (see [5]).

(4) The proper class $\mathcal{C}o\text{-}\mathcal{N}eat$ of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}(\psi)$ is a Rad-supplement of some submodule of N (see [10]).

(5) Over left hereditary rings the proper class $\mathcal{SS}$ of all short exact sequences $0 \longrightarrow M \xrightarrow{\psi}$ $\xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}\,(\psi)$ has a small supplement in N, that is, $N = \mathrm{Im}(\psi) + V$ and $\mathrm{Im}(\psi) \cap V$ is a small module (see [1]).

Now, we have the following implications on the the above classes of left R-modules:

$$\mathcal{S}plit \subseteq \mathcal{S}upp \subseteq \mathcal{C}o\text{-}\mathcal{N}eat \subseteq \mathcal{A}bs \qquad \text{and} \qquad \mathcal{S}plit \subseteq \mathcal{S}upp \subseteq \mathcal{SS} \subseteq \mathcal{A}bs.$$

Let $\mathcal{RS}$ be the class of all short exact sequences $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}(\psi)$ has a $\mathcal{Z}^*$-supplement in N, that is, $\mathrm{Im}(\psi) + V = N$ and $\mathrm{Im}(\psi) \cap V$ is Rad-small for some submodule V of N. It is obvious that $\mathcal{C}o\text{-}\mathcal{N}eat \subseteq \mathcal{RS}$ and $\mathcal{SS} \subseteq \mathcal{RS}$. The following example shows that $\mathcal{RS}$ contains properly the class $\mathcal{SS}$ and the class $\mathcal{C}o\text{-}Neat$.

***Example* 3.2.** (1) Let R be a local Dedekind domain (i.e., DVR) with quotient $K \neq R$ $\Big($e.g., the ring $\mathbb{Z}_{(p)}$ containing all rational numbers of the form $\dfrac{a}{b}$ with $p \nmid b$ for any prime p in $\mathbb{Z}\Big)$. Put $N = R^{(\mathbb{N})}$ and $M = \mathrm{Rad}(N)$. Consider the extension $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} K \longrightarrow 0$, where $K = \dfrac{N}{M}$. Then $\mathbb{E}$ is an element of $\mathcal{RS}$. However, it is not in $\mathcal{SS}$ because M has no (weak) supplements in the projective module N.

(2) Let $N =_{\mathbb{Z}} \mathbb{Z}$ and $M =_{\mathbb{Z}} 2\mathbb{Z}$. Put $K =_{\mathbb{Z}} \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)$. Then the extension $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\iota}$ $\xrightarrow{\iota} N \xrightarrow{\pi} K \longrightarrow 0$ is in the class $\mathcal{RS}$ by Theorem 2.2. On the other hand, $\mathbb{E}$ is not an element of $\mathcal{C}o\text{-}Neat$ since M is not Rad-supplement in N.

Proposition 3.1. *Let R be an arbitrary ring. If every injective left R-module has a small radical, then $\mathcal{RS} = \mathcal{SS}$.*

Proof. Let $\mathbb{E}\colon 0 \longrightarrow M \stackrel{\psi}{\longrightarrow} N \stackrel{\phi}{\longrightarrow} K \longrightarrow 0$ be any element of the class $\mathcal{RS}$. Then there exists a submodule V of N such that $N = M + V$ and $M \cap V$ is Rad-small. Therefore, $M \cap V \subseteq$ $\subseteq \mathrm{Rad}(E(M \cap V))$. By the assumption and [6] (11.5.5, § 11.6.3), we obtain that $M \cap V$ is a small submodule of the injective hull $E(M \cap V)$. It means that M has a small supplement in N. Hence, $\mathcal{RS} = \mathcal{SS}$.

A ring R is said to be *a left max ring* if every non-zero left R-module has a maximal submodule. Now we have the following:

***Corollary* 3.1.** *Let R be a left max ring. Then $\mathcal{RS} = \mathcal{SS}$.*

Proof. Since R is left max, every left R-module has a small radical. Hence, the proof follows from Proposition 3.1.

***Example* 3.3.** Consider the non-Noetherian commutative ring which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = F$ is any field. Suppose that R is the subring of the ring consisting of all sequences $(r_n)_{n\in\mathbb{N}}$ such that there exist $r \in F$, $m \in \mathbb{N}$ with $r_n = r$ for all $n \geq m$. Let $N = {}_R R$. Then N is a regular module which is not semisimple. Put $M = \mathrm{Soc}\,(N)$ and $K = \dfrac{N}{M}$. Then the extension $\mathbb{E}\colon 0 \longrightarrow M \stackrel{\iota}{\longrightarrow} N \stackrel{\pi}{\longrightarrow} K \longrightarrow 0$ is not in $\mathcal{RS}$.

Theorem 3.1. *A ring R is a Z^*-semilocal ring if and only if $\mathcal{RS} = \mathcal{A}bs$.*

Proof. By Theorem 2.2.

Observe from Theorem 3.1 that over $\mathcal{Z}^*$-semilocal rings (in particular, semilocal rings or commutative domains), $\mathcal{RS}$ is a proper class.

The following the structure of the Abelian group $\mathrm{Ext}_R(K, M)$ is given in the book [9, p. 63 – 71], and we recall them for the convenience of the reader:

Let R be an arbitrary ring with identity and $\mathbb{E}\colon 0 \longrightarrow M \stackrel{\psi}{\longrightarrow} N \stackrel{\phi}{\longrightarrow} K \longrightarrow 0$ be a short exact sequence of left R-modules and module homomorphisms. Then $\mathbb{E}$ is called *an extension* of M by K. By $\mathrm{Ext}_R(K, M)$ we will denote the set of all equivalence classes of extensions of M by K. Let $\mathbb{E}_1$: $0 \longrightarrow M \stackrel{\psi_1}{\longrightarrow} N_1 \stackrel{\phi_1}{\longrightarrow} K \longrightarrow 0$ and $\mathbb{E}_2 : 0 \longrightarrow M \stackrel{\psi_2}{\longrightarrow} N_2 \stackrel{\phi_2}{\longrightarrow} K \longrightarrow 0$ be any elements of $\mathrm{Ext}_R(K, M)$. We define the direct sum of $\mathbb{E}_1$ and $\mathbb{E}_2$ as follows:

$$\mathbb{E}_1 \oplus \mathbb{E}_2 : 0 \longrightarrow M \oplus M \stackrel{\psi}{\longrightarrow} N_1 \oplus N_2 \stackrel{\phi}{\longrightarrow} K \oplus K \longrightarrow 0,$$

where $\psi(m_1, m_2) = (\psi_1 \oplus \psi_2)(m_1, m_2) = (\psi_1(m_1), \psi_2(m_2))$ for all $(m_1, m_2) \in M \oplus M$ and $\phi(n_1, n_2) = (\phi_1 \oplus \phi_2)(n_1, n_2) = (\phi_1(n_1), \phi_2(n_2))$ for all $(n_1, n_2) \in N_1 \oplus N_2$. Then $\mathbb{E}_1 \oplus \mathbb{E}_2$ is a short exact sequence. The *Baer sum* of $\mathbb{E}_1$ and $\mathbb{E}_2$, $\mathbb{E}_1 + \mathbb{E}_2 = \bigtriangledown_M(\mathbb{E}_1 \oplus \mathbb{E}_2)\triangle_K$, where the *diagonal* map $\triangle_K(k) = (k, k)$ for all $k \in K$ and the *codiagonal* map $\bigtriangledown_M(m_1, m_2) = m_1 + m_2$ for all $(m_1, m_2) \in M \oplus M$. Therefore, $\mathrm{Ext}_R(K, M)$ is an Abelian group under Baer sum of extensions. Note that the split extension $0 \longrightarrow M \longrightarrow M \oplus K \longrightarrow K \longrightarrow 0$ is the zero element of this group and the inverse of an extension $\mathbb{E}\colon 0 \longrightarrow M \stackrel{\psi}{\longrightarrow} N \stackrel{\phi}{\longrightarrow} K \longrightarrow 0$ is the extension $(-I_M)\mathbb{E}$.

The set $\mathrm{Ext}_{\mathcal{P}}(K, M)$ of all short exact sequences of $\mathrm{Ext}_R(K, M)$ that belongs to a proper class $\mathcal{P}$ is a subgroup of the group of $\mathrm{Ext}_R(K, M)$.

Theorem 3.2 (see [12], Theorem 1.1)**.** *Let $\mathcal{P}$ be a class of short exact sequences for left R-modules. If $\mathrm{Ext}_{\mathcal{P}}(K, M)$ is a subfunctor of $\mathrm{Ext}_R(K, M)$, $\mathrm{Ext}_{\mathcal{P}}(K, M)$ is a subgroup of $\mathrm{Ext}_R(K, M)$ for every R-modules M, K and the composition of two $\mathcal{P}$-monomorphism* (*or* $\mathcal{P}$*-epimorphisms*) *is a* $\mathcal{P}$*-monomorphism* (*an* $\mathcal{P}$*-epimorphism, respectively*), *then* $\mathcal{P}$ *is a proper class.*

Using Theorem 3.2, we shall prove that $\mathcal{RS}$ is a proper class over left hereditary rings.

Lemma 3.1. *Let* $f\colon M \longrightarrow M'$ *be any homomorphism of left* R*-modules. Then*

$$f_*\colon \mathrm{Ext}_R(K,M) \longrightarrow \mathrm{Ext}_R\left(K,M'\right)$$

preserves the elements of the class $\mathcal{RS}$.

Proof. Let $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be any element of $\mathcal{RS}$. Take the left R-module $N' = \dfrac{M' \oplus N}{H}$, where $H = \{(-f(m), \psi(m)) \in M' \oplus N \mid m \in M\}$ is a submodule of $M' \oplus N$. Define these homomorphisms of left R-modules $\psi'\colon M' \longrightarrow N'$ via $\psi'(m') = (m',0) + H$, $\phi'\colon N' \longrightarrow K$ via $\phi'((m',n)) = \phi(n)$ and $h\colon N \longrightarrow N'$ via $h(n) = (0,n) + H$. Then $f_*(\mathbb{E}) = f\mathbb{E}\colon 0 \longrightarrow M' \xrightarrow{\psi'} N' \xrightarrow{\phi'} K \longrightarrow 0 \in \mathrm{Ext}_R(K,M')$ and we obtain the following commutative diagram with exact rows:

$$\begin{array}{lccccccccc}
\mathbb{E}: & 0 & \longrightarrow & M & \xrightarrow{\psi} & N & \xrightarrow{\phi} & K & \longrightarrow & 0 \\
 & & & \downarrow f & & \downarrow h & & \| & & \\
f\mathbb{E}: & 0 & \longrightarrow & M' & \xrightarrow{\psi'} & N' & \xrightarrow{\phi'} & K & \longrightarrow & 0
\end{array}$$

that is, $\psi' f = h\psi$ and $\phi' h = \phi$. Since the extension $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ is in the class $\mathcal{RS}$ of left R-modules, there exists a submodule V of N such that $N = \mathrm{Im}\,(\psi) + V$ and $\mathrm{Im}\,(\psi) \cap V$ is Rad-small. By using the above commutative diagram, we obtain that $N' = \mathrm{Im}\,(\psi') + \mathrm{Im}\,(h)$ and $\mathrm{Im}\,(h) \cap \mathrm{Im}\,(\psi') = h(\mathrm{Im}\,(\psi) \cap V)$. It follows from Lemma 1.2 that $\mathrm{Im}\,(h) \cap \mathrm{Im}\,(\psi')$ is Rad-small as a homomorphic image of the Rad-small module $\mathrm{Im}\,(\psi) \cap V$. So $\mathrm{Im}\,(h)$ is a $\mathcal{Z}^*$-supplement of $\mathrm{Im}\,(\psi')$ in N'. Thus, $f\mathbb{E} = f_*(\mathbb{E}) \in \mathcal{RS}$.

Observe from Lemma 3.1 that if, for all modules M and K, $\mathbb{E} \in \mathrm{Ext}_{\mathcal{RS}}(K,M)$, then the inverse extension $(-I_M)\mathbb{E} \in \mathrm{Ext}_{\mathcal{RS}}(K,M)$.

Lemma 3.2. *Let* $g\colon K' \longrightarrow K$ *be any homomorphism of left* R*-modules. Then*

$$g^*\colon \mathrm{Ext}_R(K,M) \longrightarrow \mathrm{Ext}_R\left(K',M\right)$$

preserves the elements of the class $\mathcal{RS}$.

Proof. Let $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be a short exact sequence in $\mathcal{RS}$. Consider the left R-submodule $N' = \{(n,k') \in N \oplus K' \mid \phi(n) = g(k')\}$ of the left R-module $N \oplus K'$. Define these homomorphisms $\phi'\colon N' \longrightarrow K'$ via $\phi'(n,k') = k'$, $h\colon N' \longrightarrow N$ via $h(n,k') = n$ and $\psi'\colon M \longrightarrow N'$ via $\psi'(m) = (\psi(m),0)$. Then we can write the following commutative diagram with rows:

$$\begin{array}{lccccccccc}
\mathbb{E}g: & 0 & \longrightarrow & M & \xrightarrow{\psi'} & N' & \xrightarrow{\phi'} & K' & \longrightarrow & 0 \\
 & & & \| & & \downarrow h & & \downarrow g & & \\
\mathbb{E}: & 0 & \longrightarrow & M & \xrightarrow{\psi} & N & \xrightarrow{\phi} & K & \longrightarrow & 0
\end{array}$$

where $g^*(\mathbb{E}) = \mathbb{E}g$. Since $\mathbb{E}$ is an element of $\mathcal{RS}$, there exists a submodule V of N such that $N = \mathrm{Im}\,(\psi) + V =$ and $\mathrm{Im}\,(\psi) \cap V$ is Rad-small. To show $N' = \mathrm{Im}\,(\psi') + h^{-1}(V)$, let a

be any element of N'. Then we can write $h(a) = \psi(m) + v$ where $m \in M$ and $v \in V$. Since $\psi(m) = (h\psi')\,(m)$, we have $a - \psi'(m) \in h^{-1}(V)$ and this implies that $N' = \mathrm{Im}\,(\psi') + h^{-1}(V)$.

Let (n, k') be any element of $\mathrm{Im}\,(\psi') \cap h^{-1}(V)$. Since $\mathrm{Im}\,(\psi') = \mathrm{Ker}\,(\phi')$, we obtain that $\phi'\,(n, k') = k' = 0$. Then $g\,(k') = \phi(n) = 0$, that is, $n \in \mathrm{Ker}\,(\phi)$. Therefore, $n \in \mathrm{Im}\,(\psi)$ because $\mathrm{Im}\,(\psi) = \mathrm{Ker}\,(\phi)$. It follows that $n \in \mathrm{Im}\,(\psi) \cap V$. Since $\mathrm{Im}\,(\psi) \cap V$ is Rad-small, the module Rn is small, and so $R\,(n, k')$ is small. Thus, $\mathrm{Im}\,(\psi') \cap h^{-1}(V)$ is Rad-small. Hence, $h^{-1}(V)$ is a $\mathcal{Z}^*$-supplement of $\mathrm{Im}\,(\psi')$ in N'.

Lemma 3.3. *If* $\mathbb{E}_1, \mathbb{E}_2 \in \mathrm{Ext}_{RS}(K, M)$, *then* $\mathbb{E}_1 \oplus \mathbb{E}_2 \in \mathrm{Ext}_{RS}(K \oplus K, M \oplus M)$.

Proof. Let $\mathbb{E}_1 : 0 \longrightarrow M \xrightarrow{\psi_1} N_1 \xrightarrow{\phi_1} K \longrightarrow 0$ and $\mathbb{E}_2 : 0 \longrightarrow M \xrightarrow{\psi_2} N_2 \xrightarrow{\phi_2} K \longrightarrow 0$ be two elements of $\mathrm{Ext}_{\mathcal{RS}}(K, M)$. Then, for $i = 1, 2$, $N_i = M + V_i$ and $M \cap V_i$ is Rad-small for some submodules V_i of N_i. Since $(M \oplus M) + (V_1 \oplus V_2) = N_1 \oplus N_2$ and $(M \oplus M) \cap (V_1 \oplus \oplus V_2) = (M \cap V_1) \oplus (M \cap V_2)$, it follows from Lemma 1.2 that the short exact sequence $\mathbb{E}_1 \oplus \mathbb{E}_2 :$ $0 \longrightarrow M \oplus M \xrightarrow{\psi} N_1 \oplus N_2 \xrightarrow{\phi} K \oplus K \longrightarrow 0$ is in $\mathrm{Ext}_{RS}(K \oplus K, M \oplus M)$, where $\psi = \psi_1 \oplus \psi_2$ and $\phi = \phi_1 \oplus \phi_2$.

***Corollary* 3.2.** $\mathrm{Ext}_{\mathcal{RS}}(K, M)$ *is a subgroup of the extension* $\mathrm{Ext}_R(K, M)$ *for every module* K *and* M. *Moreover,* $\mathrm{Ext}_{\mathcal{RS}}(K, M)$ *is a subfunctor of the functor* $\mathrm{Ext}_R(K, M)$.

Proof. Let $\mathbb{E}_1 : 0 \longrightarrow M \xrightarrow{\psi_1} N_1 \xrightarrow{\phi_1} K \longrightarrow 0$ and $\mathbb{E}_2 : 0 \longrightarrow M \xrightarrow{\psi_2} N_2 \xrightarrow{\phi_2} K \longrightarrow 0$ be any elements of $\mathrm{Ext}_{\mathcal{RS}}(K, M)$. It follows from Lemmas 3.1, 3.2 and 3.3 that the Baer sum $\mathbb{E}_1 + \mathbb{E}_2$ of these extensions $\mathbb{E}_1$ and $\mathbb{E}_2$ is in $\mathrm{Ext}_{\mathcal{RS}}(K, M)$. Hence, $\mathrm{Ext}_{\mathcal{RS}}(K, M)$ is a subgroup of $\mathrm{Ext}_R(K, M)$.

Theorem 3.3. *Let* R *be a left hereditary ring. Then* $\mathcal{RS}$ *is a proper class.*

Proof. By Theorem 3.2 and Corollary 3.2, it suffices to show that the composition of two $\mathcal{RS}$-epimorphisms is an $\mathcal{RS}$-epimorphism. Let $f : N \longrightarrow N'$ and $g : N' \longrightarrow K$ be $\mathcal{RS}$-epimorphisms. Now we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \xrightarrow{i_{\mathrm{Ker}(f)}} & M & \longrightarrow & \mathrm{Ker}(g) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle i_{\mathrm{Ker}(g)}} & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & N & \xrightarrow{f} & N' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle g} & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $i_{\mathrm{Ker}\,(f)}$ and $i_{\mathrm{Ker}\,(g)}$ are the canonical inclusions. By the hypothesis, we can write $N = \mathrm{Ker}\,(f) + V$ and $\mathrm{Ker}\,(f) \cap V$ is Rad-small for some submodule V of N, and $\dfrac{N}{\mathrm{Ker}\,(f)} = \dfrac{M}{\mathrm{Ker}\,(f)} + \dfrac{L}{\mathrm{Ker}\,(f)}$ and $\dfrac{M \cap L}{\mathrm{Ker}\,(f)}$ is Rad-small for some submodule $\dfrac{L}{\mathrm{Ker}\,(f)}$ of $\dfrac{N}{\mathrm{Ker}\,(f)}$. Therefore, $M = M \cap N = M \cap (\mathrm{Ker}\,(f) + V) = \mathrm{Ker}\,(f) + M \cap V$, $M \cap L = \mathrm{Ker}\,(f) + M \cap V \cap L$ and

$L = \mathrm{Ker}\,(f) + L \cap V$. It follows that $N = M + (V \cap L)$. Applying Lemma 2.5, we deduce that $M \cap V \cap L$ is Rad-small. This means that the composition gf is an $\mathcal{RS}$-epimorphism.

Let $\mathcal{M}$ be a class of short exact sequences. The smallest proper class containing $\mathcal{M}$ is said to be *generated* by $\mathcal{M}$ and denoted by $\langle\mathcal{M}\rangle$. Since the intersection of any proper classes is proper, we have $\langle\mathcal{M}\rangle = \cap\{\mathcal{P} \mid \mathcal{M} \subseteq \mathcal{P} \ \textit{and}\ \mathcal{P}\ \textit{is proper class}\}$.

By $\mathcal{RS}mall$, we will denote the class of all short exact sequences $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}(\psi) \subseteq \mathrm{Rad}(N)$, and by $\mathcal{WRS}$ we will denote the class of all short exact sequences $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ such that $\mathrm{Im}\,(\psi)$ has a weak Rad-supplement in N, that is, $\mathrm{Im}\,(\psi) + V = N$ and $\mathrm{Im}\,(\psi) \cap V \subseteq \mathrm{Rad}(N)$ for some submodule V of N.

Clearly, $\mathcal{RS}mall \subseteq \mathcal{WRS} \subseteq \mathcal{RS}$, and so $\langle\mathcal{RS}mall\rangle \subseteq \langle\mathcal{WRS}\rangle \subseteq \mathcal{RS}$ whenever $\mathcal{RS}$ is a proper class. Motivated by [1] (Corollary 3.13), we shall prove that $\langle\mathcal{RS}mall\rangle = \langle\mathcal{WRS}\rangle = \mathcal{RS}$ in the following theorem.

Theorem 3.4. *For the proper class* $\mathcal{RS}$, $\langle\mathcal{RS}mall\rangle = \langle\mathcal{WRS}\rangle = \mathcal{RS}$.

Proof. Let $\mathbb{E}\colon 0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} K \longrightarrow 0$ be any element of the proper class $\mathcal{RS}$. Then, for some submodule V of N, we can write $N = M + V$ and $M \cap V$ is Rad-small. Put $L = M \cap V$. Therefore, the extension $\mathbb{E}\colon 0 \longrightarrow \dfrac{M}{L} \xrightarrow{\iota} \dfrac{N}{L} \xrightarrow{\Phi} K \longrightarrow 0$ is in the class $\langle\mathcal{RS}mall\rangle$, where ι is the canonical injection, $\pi\colon N \longrightarrow \dfrac{N}{L}$ is the canonical projection and $\phi = \Phi\pi$. Since π and Φ are $\langle\mathcal{RS}mall\rangle$-epimorphisms, we get that ϕ is $\langle\mathcal{RS}mall\rangle$-epimorphism. It means that $\mathcal{RS} \subseteq \langle\mathcal{RS}mall\rangle$.

Let $\mathcal{P}$ be a proper class. A module M is said to be $\mathcal{P}$*-injective* (respectively, $\mathcal{P}$*-coinjective*) if the subgroup $\mathrm{Ext}_{\mathcal{P}}(K, M) = 0$ (respectively, $\mathrm{Ext}_{\mathcal{P}}(K, M) = \mathrm{Ext}_R(K, M)$) for all left R-modules K.

Now we prove that weak Rad-supplement submodules of $\mathcal{RS}$-coinjective modules are $\mathcal{RS}$-coinjective.

Proposition 3.2. *Let R be a left hereditary ring and M be a $\mathcal{RS}$-coinjective R-module. Then every weak* Rad*-supplement submodule of M is $\mathcal{RS}$-coinjective.*

Proof. Let A be a weak Rad-supplement submodule of M. Then the extension $\mathbb{E}\colon 0 \longrightarrow A \xrightarrow{\iota} M \xrightarrow{\pi} \dfrac{M}{A} \longrightarrow 0$ is an element of the class $\mathcal{WRS}$ and so it is in $\mathcal{RS}$. Hence, by [11] (Proposition 1.8), A is $\mathcal{RS}$-coinjective.

Now we characterize $\mathcal{RS}$-coinjective modules via weak Rad-supplements in the following theorem which is adapted of [3] (Theorem 4.1).

Theorem 3.5. *For a module M over a left hereditary ring R, the following statements are equivalent:*

(1) *M is $\mathcal{RS}$-coinjective,*

(2) *M has a weak* Rad*-supplement in $E(M)$.*

Proof. (1) $\Longrightarrow$ (2) Let $\delta\colon M \longrightarrow E(M)$ be the essential monomorphism. Without loss of generality, we take $M \subseteq E(M)$. By (1), there exists a submodule V of $E(M)$ such that $M + V = E(M)$ and $M \cap V$ is Rad-small. Since $E(M)$ is injective, $\mathcal{Z}^*(E(M)) = \mathrm{Rad}(E(M))$, and so $M \cap V \subseteq \mathrm{Rad}(E(M))$. Thus, V is a weak Rad-supplement of M in $E(M)$.

(2) $\Longrightarrow$ (1) is clear by [11] (Proposition 1.7).

***Corollary* 3.3.** *Let R be a left hereditary ring. Then ${}_RR$ is $\mathcal{RS}$-coinjective if and only if there exists a submodule S of $E({}_RR)$ such that $E({}_RR) = R + S$ and $R \cap S \subseteq \mathrm{Rad}(E({}_RR))$.*

The following fact is a direct consequence of Theorem 3.5.

***Corollary* 3.4.** *Every* Rad*-small module over a left hereditary ring is* $\mathcal{RS}$*-coinjective.*

Proof. Let M be a Rad-small module. Then, $M \subseteq \mathrm{Rad}(E(M))$. Therefore, $E(M) = E(M) + M$ and $M \cap E(M) \subseteq \mathrm{Rad}(E(M))$. So $E(M)$ is a weak Rad-supplement of M in the injective hull $E(M)$. Hence, M is $\mathcal{RS}$-coinjective by Theorem 3.5.

The smallest proper class for which every module from the class of modules $\mathcal{M}$ is coinjective is denoted by $\underline{k}(\mathcal{M})$. Such classes are said to be *coinjectively generated* by $\mathcal{M}$.

Proposition 3.3. *Let R be a left hereditary ring. The proper class $\mathcal{RS}$ is coinjectively generated by all* Rad*-small left R-modules.*

Proof. We shall show that $\mathcal{RS} = \underline{k}(\mathcal{RS}mall)$. It follows from Corollary 3.4 that every Rad-small R-module is $\mathcal{RS}$-coinjective, and so $\underline{k}(\mathcal{RS}mall) \subseteq \mathcal{RS}$. By Proposition 3.2, we get $\mathcal{RS} = \langle \mathcal{RS}mall \rangle \subseteq \underline{k}(\mathcal{RS}mall)$. Hence, $\mathcal{RS} = \underline{k}(\mathcal{RS}mall)$.

Let $\mathcal{P}$ be a proper class. The *global dimension* of $\mathcal{P}$ is defined as

$$\text{gl.dim}\ \mathcal{P} = \big\{ n \mid \mathrm{Ext}^{n+1}_{\mathcal{P}}(K, M) = 0 \text{ for all } M \text{ and } K \text{ left } R\text{-modules} \big\}.$$

If there no such n, then gl.dim $\mathcal{P} = \infty$.

Theorem 3.6. $\text{gl.dim}\ \mathcal{RS} \leq 1$.

Proof. It follows from Theorem 3.3 and [2].

Recall that a ring R is said to be *a left V-ring* if every simple left R-module is injective. The following next theorem characterizes the left hereditary rings in which $\mathcal{RS}$-coinjective modules are injective.

Theorem 3.7. *The following statements are equivalent for a left hereditary ring R:*

(1) *every $\mathcal{RS}$-coinjective module is injective,*

(2) *every* Rad*-small module is injective,*

(3) *every small module is injective,*

(4) *R is a left V-ring.*

Proof. (1) $\Longrightarrow$ (2) If follows from Corollary 3.4.

(2) $\Longrightarrow$ (3) Since small modules are Rad-small.

(3) $\Longrightarrow$ (4) By [14] (23.1), it suffices to prove that, for any left R-module M, $\mathrm{Rad}(M) = 0$. Let $m \in \mathrm{Rad}(M)$. Then Rm is a small submodule of M. By (3), we can write the decomposition $M = Rm \oplus K$ for some submodule K of M. It follows that $m = 0$. Hence, we obtain that $\mathrm{Rad}(M) = 0$.

(4) $\Longrightarrow$ (1) Let M be a $\mathcal{RS}$-coinjective module and N be any extension of M. Then $N = M + V$ and $M \cap V \subseteq \mathrm{Rad}(E(M \cap V))$ for some submodule V of N. Since R is a left V-ring, by [14] (23.1), we get $M \cap V \subseteq \mathrm{Rad}(E(M \cap V)) = 0$. Thus, M is a direct summand of N. It means that M is injective.

Let $\mathcal{P}$ be a proper class. A module M is said to be $\mathcal{P}$*-projective* (respectively, $\mathcal{P}$*-coprojective*) if the subgroup $\mathrm{Ext}_{\mathcal{P}}(M, K) = 0$ (respectively, $\mathrm{Ext}_{\mathcal{P}}(M, K) = \mathrm{Ext}_R(M, K)$) for all left R-module K.

Theorem 3.8. *Let M be a module over a left hereditary ring. Then, the following statements are equivalent:*

(1) *M is $\mathcal{RS}$-projective.*

(2) $\mathrm{Ext}_R(M, K) = 0$ *for every* Rad*-small module* K.

Proof. (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (1) Let $0 \longrightarrow A \stackrel{\psi}{\longrightarrow} B \stackrel{\phi}{\longrightarrow} C \longrightarrow 0$ be any element of $\mathcal{RS}$. So, $B = A + D$ and $A \cap D$ is Rad-small for some submodule D of B. Put $F = A \cap D$. Then the short exact sequence $0 \longrightarrow \dfrac{A}{F} \stackrel{i_1}{\longrightarrow} \dfrac{B}{F} \stackrel{\pi_1}{\longrightarrow} \dfrac{D}{F} \longrightarrow 0$ splits, where i_1 is the canonical injection and π_1 is the canonical projection. Now we can write the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \stackrel{\psi}{\longrightarrow} & B & \stackrel{\phi}{\longrightarrow} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle I_C} & & \\ 0 & \longrightarrow & \dfrac{A}{F} & \stackrel{i_1}{\longrightarrow} & \dfrac{B}{F} & \stackrel{f\pi_1}{\longrightarrow} & C & \longrightarrow & 0 \end{array}$$

where π_F and π are canonical projections. Applying the functor $\mathrm{Hom}\,(M, .)$, we get

$$\begin{array}{ccccc} \mathrm{Hom}\,(M, B) & \stackrel{\phi_*}{\longrightarrow} & \mathrm{Hom}\,(M, C) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \pi_*} & & \Vert & & \\ \mathrm{Hom}\left(M, \dfrac{B}{F}\right) & \stackrel{(f\pi_1)_*}{\longrightarrow} & \mathrm{Hom}\,(M, C) & \longrightarrow & 0 \\ \downarrow & & & & \\ \mathrm{Ext}_R(M, F) & & & & \end{array}$$

Then $(f\pi_1)_*$ is an epimorphism. It follows from (2) that $\mathrm{Ext}_R(M, F) = 0$. So π_* is an epimorphism. This means that ϕ_* is an epimorphism. Consequently, M is $\mathcal{RS}$-projective.

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