UDC 517.9

T. Kenzizi (Univ. Tunis El Manar, Tunisia)

EXISTENCE OF NONNEGATIVE SOLUTIONS FOR A FRACTIONAL PARABOLIC EQUATION IN THE WHOLE SPACE

ІСНУВАННЯ НЕВІД'ЄМНИХ РОЗВ'ЯЗКІВ ДРОБОВОГО ПАРАБОЛІЧНОГО РІВНЯННЯ В УСЬОМУ ПРОСТОРІ

We study existence of nonnegative solutions for a parabolic problem $\frac{\partial u}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}} u + \frac{c}{|x|^\alpha} u$ in $\mathbb{R}^d \times (0,T)$. Here $0 < \alpha < \min(2,d)$, $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplacian on $\mathbb{R}^d$ and $u_0 \in L^2(\mathbb{R}^d)$.

Вивчається задача існування невід'ємних розв'язків параболічного рівняння $\frac{\partial u}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}} u + \frac{c}{|x|^\alpha} u$ на $\mathbb{R}^d \times$ $\times (0,T)$. Тут $0 < \alpha < \min(2,d)$, $(-\triangle)^{\frac{\alpha}{2}}$ — дробовий лапласіан на $\mathbb{R}^d$ й $u_0 \in L^2(\mathbb{R}^d)$.

1. Introduction. The purpose of the present paper is to verify that a similar critical behavior of the Cauchy problem holds when the classical Laplacian is replaced by the fractional Laplacian $-(-\triangle)^{\frac{\alpha}{2}}$ with $0 < \alpha < \min(2,d)$. So we give some existence results of positive solutions for negatively perturbed Dirichlet fractional Laplacian on $\mathbb{R}^d$.

For every $0 < \alpha < \min(2,d)$, we designate by $L_0 := (-\triangle)^{\frac{\alpha}{2}}$. Let us consider the parabolic perturbed problem

$$\begin{aligned} -\frac{\partial u}{\partial t} &= L_0 u - Vu \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\ u(x,0) &= u_0(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^d, \end{aligned} \tag{1}$$

where $u_0 \in L^2(\mathbb{R}^d)$, $u_0 \geq 0$ is a Borel measurable function and V is nonnegative potential in $L^1_{\text{loc}}(\Omega)$.

Although we shall focus on the very special case $V_c = \frac{c}{|x|^\alpha}$, $0 < c \leq c^* := \dfrac{2^\alpha \Gamma^2\left(\frac{d+\alpha}{4}\right)}{\Gamma^2\left(\frac{d-\alpha}{4}\right)}$.

The present paper addresses several important problems of the potential theory of fractional Laplacian. One of the results is the existence of nonnegative solution for a parabolic problem perturbed by potential. Its main results were motivated by the result of J. A. Goldstein and Q. S. Zhang [12] for the Laplacian perturbed by a singular potential.

By using the idea in [5, 10, 12], where the problem was addressed and solved for the Dirichlet Laplacian (i.e., $\alpha = 2$), Ali Ben Amor and T. Kenzizi [1] established conditions ensuring existence of nonnegative solutions for a nonlocal case, they proved that for bounded Ω and for $0 < c \leq$ $\leq c^* := \dfrac{2^\alpha \Gamma^2\left(\frac{d+\alpha}{4}\right)}{\Gamma^2\left(\frac{d-\alpha}{4}\right)}$ equation (1) has a nonnegative solution, whereas for $c > c^*$ and Ω a bounded Lipschitz domain then no nonnegative solutions occur.

The inspiring point for us was the papers of Baras, Goldstein [5, 6] and Goldstein, Zhang [12] where the problem was addressed and solved for the Lapacian operator (i.e., $\alpha = 2$). In the latter cited papers, the authors in [12] generalize the result of existence of nonnegative solutions in [5] to equations with variable coefficients in the principal part and to a degenerate equations, one of the most important degenerate equations is the heat equation on the Heisenberg group. However, there is a substantial difference between the Laplacian and the fractional Laplacian. Whereas it is known that the first one is local and therefore suitable for describing diffusions, the second one is nonlocal and commonly used for describing superdiffusions (Lévy flights). These differences are reflected in the way of computing for both operators (Green formula, integration by part, Leibniz formula, etc.). The fractional operator appears in numerous fields of mathematical physics, mathematical biology and mathematical finance and has attracted a lot of attention recently. Nonetheless, we shall show that the method used in [5, 12] still apply in our setting.

2. Preliminaries and preparing results. To state our main results, it is convenient to introduce the following notations and definitions. In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \geq 1,\ dy$ is the Lebesgue measure on $\mathbb{R}^d$. We shall write $\int \ldots$ as a shorthand form $\int_{\mathbb{R}^d} \ldots$.

Throughout this paper, letter $k,\ C,\ c,\ C', \ldots$ will denote generic positive constants which may vary in value from line to line. $\big|A(x,r)\big|$ will denote the volume of the ball A centred at x and of radius $r,\ (a \wedge b) := \min(a,b)$ and $(a \vee b) := \max(a,b)$. Consider the bilinear symmetric form $\mathcal{E}^\alpha$ defined in L^2 by

$$\mathcal{E}^\alpha(f,g) = \frac{1}{2}\mathcal{A}(d,\alpha)\int\int \frac{\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)}{|x-y|^{d+\alpha}}dx\,dy,$$

$$D(\mathcal{E}^\alpha) = W^{\frac{\alpha}{2},2}(\mathbb{R}^d) := \big\{f \in L^2 : \mathcal{E}[f] : \mathcal{E}(f,f) < \infty\big\},$$

where

$$\mathcal{A}(d,\alpha) = \frac{\alpha\Gamma\left(\dfrac{d+\alpha}{2}\right)}{2^{1-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\dfrac{\alpha}{2}\right)},$$

and Γ is the Gamma function. It is well known that $\mathcal{E}^\alpha$ is a transient Dirichlet form and is related (via Kato representation theorem) to the self-adjoint operator commonly named the fractional Laplacian on $\mathbb{R}^d$, and which we shall denote by $(-\triangle)^{\frac{\alpha}{2}}$. We note that the domain of $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Sobolev space $W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$. For smooth compactly supported function $\phi \in C_c^\infty(\mathbb{R}^d)$, the fractional Laplacian is defined as the $L^2(\mathbb{R}^d)$-closure of the operator

$$\triangle^{\frac{\alpha}{2}}\phi(x) = \lim_{\epsilon \longrightarrow 0}\int\limits_{|y|>\epsilon}\big[\phi(x+y)-\phi(x)\big]\nu(y)\,dy, \quad x \in \mathbb{R}^d,$$

where ν is the Lévy measure given by the density function

$$\nu(y) = \frac{2^\alpha\Gamma\left(\dfrac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\left|\Gamma\left(\dfrac{-\alpha}{2}\right)\right|}|y|^{-d-\alpha}.$$

This definition is important for applications to probability. Its Fourier symbol is given by

$$\widehat{\triangle^{\frac{\alpha}{2}}\phi}(\xi) = -|\xi|^{\alpha}\widehat{\phi}(\xi).$$

If ϕ is regular enough and $\alpha \in (0,2)$, $(-\triangle)^{\frac{\alpha}{2}}\phi(x)$ can be computed by the formula

$$(-\triangle)^{\frac{\alpha}{2}}\phi(x) = c_{d,\alpha}\int \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}\,dy,$$

where $c_{d,\alpha}$ is a constant depending only on d and α. The inverse of the $\dfrac{\alpha}{2}$ power of the Laplacian is the $-\dfrac{\alpha}{2}$ power of the Laplacian $(-\triangle)^{-\frac{\alpha}{2}}$. For $0<\alpha<\min(2,d)$, there is an integral formula which says that $(-\triangle)^{-\frac{\alpha}{2}}u$ is the convolution of the function u with the Riesz potential

$$(-\triangle)^{-\frac{\alpha}{2}}\phi(x) = c_{d,\alpha}\int \frac{\phi(x-y)}{|y|^{d-\alpha}}\,dy,$$

which holds a long as ϕ is integrable enough for the right-hand side to make sense.

Let $r>0$ and $\phi_r(x)=\phi(rx)$, then we obtain

$$\triangle^{\frac{\alpha}{2}}\phi_r(x) = r^{\alpha}\triangle^{\frac{\alpha}{2}}\phi(rx), \quad x\in\mathbb{R}^d.$$

We let p_t the fractional heat kernel which is the fundamental solution to the heat equation

$$\frac{\partial p_t(x)}{\partial t} + (-\triangle)^{\frac{\alpha}{2}}p = 0,$$

$$p_0(x) = \delta_0(x),$$

with Fourier transform

$$\hat{p}_t(\xi) = \int p_t(x)e^{ix\xi}dx = e^{-t|\xi|^{\alpha}}, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{2}$$

yield the following identity:

$$p_t(x) = (2\pi)^{-d}\int e^{-t|\xi|^{\alpha}}e^{-ix\xi}d\xi, \quad x\in\mathbb{R}^d.$$

Consequently, we get the scaling property

$$p_t(x) = t^{-\frac{d}{\alpha}}p_1(t^{-\frac{1}{\alpha}}x), \qquad t>0, \quad x\in\mathbb{R}^d.$$

It is well-known (see [4]) that $p_1(x)\approx 1\wedge|x|^{-d-\alpha}$, hence the following inequalities holds for some constant C:

$$C^{-1}\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right) \le p_t(x) \le C\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right), \qquad t>0, \quad x\in\mathbb{R}^d.$$

In particular, the maximum of p_t is $p_t(0) = 2^{1-\alpha}\pi^{-\frac{d}{2}}\alpha^{-1}\dfrac{\Gamma\left(\dfrac{d}{\alpha}\right)}{\Gamma\left(\dfrac{d}{2}\right)}t^{-\frac{d}{\alpha}}$.

The semigroup $P_t\phi(x) = \int p_t(x,y)\phi(y)\,dy$ has the fractional Laplacian as generator (see [2, 8, 13]). In particular, for $\phi \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have

$$(-\triangle)^{-\frac{\alpha}{2}}\phi(x) = \lim_{t\longrightarrow 0^+}\frac{1}{t}(P_t\phi(x)-\phi(x)) = \lim_{\varepsilon\longrightarrow 0^+}\int_{|y|>\varepsilon}\frac{\phi(x+y)-\phi(x)}{|y|^{d+\alpha}}\,dy.$$

Using (2) one proves that p is the heat kernel of the fractional Laplacian

$$\int_s^\infty\int p(u-s,x,z)\big[\partial_u\phi(u,z)+\triangle_z^{\frac{\alpha}{2}}\phi(u,z)\big]dzdu = -\phi(s,x),$$

where $p(t,x,y) = p_t(y-x),\ s\in\mathbb{R},\ x\in\mathbb{R}^d$ and $\phi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$.

Let $D\subseteq\mathbb{R}^d$ be an open set. We denote by p_D the heat kernel of the Dirichlet fractional Laplacian on D. Also p_D is jointly continuous when $t\neq 0$, and we have

$$0\le p_D(t,x,y) = p_D(t,y,x)\le p(t,x,y),\qquad t>0,\quad x,y\in\mathbb{R}^d.$$

In particular,

$$\int p_D(t,x,y)\le 1.$$

We define the Green function for $\triangle^{\frac{\alpha}{2}}$ on D by

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt,$$

and the scaling property of p_D yields the following scaling of G_D:

$$G_{rD}(rx,ry) = r^{\alpha-d}G_D(x,y),\qquad r>0,\quad x,y\in\mathbb{R}^d.$$

Definition 2.1. *Let* $0<T\le\infty$. *A Borel measurable function* $u\colon[0,T)\times\mathbb{R}^d\longrightarrow\mathbb{R}$ *is a solution of problem* (1) *if*:

1) $u\in L^1_{\rm loc}\big(\mathbb{R}^d\times(0,T)\big)$,

2) $u\in\mathcal{L}^2_{\rm loc}\big([0,T],L^2_{\rm loc}(\mathbb{R}^d)\big)$,

3) *for every* $0\le t<T$, *every* $\Omega\subset\mathbb{R}^d$ *and every* $\phi\in C_c^\infty\big([0,T]\times\Omega\big)$, *the following identity holds true*:

$$\int_\Omega\big((u\phi)(t,x)-u_0(x)\phi(0,x)\big)\,dx + \int_0^t\int_\Omega u(s,x)\Big(-\phi_s(s,x)+L_0\phi(s,x)\Big) =$$

$$=\int_0^t\int_\Omega u(s,x)\phi(s,x)V(x)\,dx\,ds.$$

From now, we set $V_k := V \wedge k$ and denote by (P_k) the heat equation corresponding to the Dirichlet fractional Laplacian perturbed by $-V_k$ instead of $-V$:

$$(P_k): \begin{cases} -\dfrac{\partial u}{\partial t} = L_0 u - V_k u & \text{in} \quad \mathbb{R}^d \times (0, +\infty), \\ u(x,0) = u_0 & \text{for a.e.} \quad x \in \mathbb{R}^d. \end{cases}$$

Denote by H_k the self-adjoint operator associated to the quadratic form $\mathcal{E}^{V_k} := \mathcal{E} - V_k$ defined by

$$\mathcal{E}^{V_k} : D(\mathcal{E}^{V_k}) = W_0^{\frac{\alpha}{2},2}(\mathbb{R}^d), \qquad \mathcal{E}^{V_k}[u] = \mathcal{E} - \int\limits_{\mathbb{R}^d} u^2 V_k dx$$

and, by [11, p. 492] (Remark 1.22), we conclude that

$$u_k(t) = e^{-tH_k} u_0 = \int\limits_{\mathbb{R}^d} p_{t,k} u_0(y)\, dy, \quad t \geq 0, \tag{3}$$

is the solution of (P_k), where $p_{t,k}$ is the nonnegative heat kernel of e^{-tH_k}.

***Remark* 2.1.** Let Ω_k be a nondecreasing sequence in $\mathbb{R}^d$ such that $\Omega \subset \bigcup\limits_{k \geq 0} \Omega_k$, and let $\phi \in$ $\in C_c^\infty\big([0,T] \times \mathbb{R}^d\big)$ such that $\operatorname{supp} \phi \subset \Omega$. If V is bounded, the solution of (1) is given by the integral expression

$$u(x,t) = e^{-tL_0} u_0(x) + \int\limits_0^t \int\limits_\Omega q_{t-s}(x,y) u(s,y) V(y)\, dy\, ds,$$

where q_t is the heat kernel of the operator e^{-tL_0}, $t > 0$.

By the end of this section, we give some spectral properties of $L_0 = (-\triangle)^{\frac{\alpha}{2}}$ that will be needed in the proof of the existence part.

Definition 2.2 [9]**.** *The essential spectrum of a bounded self-adjoint operator A on a Hilbert space, usually denoted $\sigma_{\text{ess}}(A)$, is a subset of the spectrum σ, and its complement is called the discrete spectrum, so*

$$\sigma_{\text{disc}}(A) = \sigma(A) \backslash \sigma_{\text{ess}}(A).$$

For bounded domain Ω and for fixed $\alpha \in (0, \min(2,d)]$, the spectrum of the $(-\triangle)^{\frac{\alpha}{2}} \,|_\Omega$ is discrete and consists of a sequence $\{\lambda_k(\alpha)\}_{k=1}^\infty$ of eigenvalues (with finite multiplicity) written in increasing order according to their multiplicity (see, for example, [7])

$$0 < \lambda_1(\alpha) < \lambda_2(\alpha) \leq \ldots \leq \lambda_k(\alpha) \ldots \nearrow +\infty.$$

3. Existence of nonnegative solutions.

Theorem 3.1. *Assume that $c \leq c^*$. Then the heat equation* (1) *has at least one nonnegative solution. Here c^* is an universal constant* (*see, for example*, [1, 3]).

Proof. The idea of studying existence is based on studying the solution (u_k) of the approach problem (P_k), where V is replaced by the trancated potential $V_k = V \wedge k$. Considering first the radial function $\Phi(x) = |x|^{-\beta}$ for $\beta \in [0, d-\alpha]$ (β denotes the smaller root of $(d-\alpha-\beta)\beta = c$).

Let $p \in C^2(\mathbb{R})$ be a convex function satisfying $p(0) = p'(0) = 0$, such that its derivative is locally Lipschitz.

First, we will prove that for all $u_k \in D(\mathcal{E}^\alpha)$ and all $\Phi \in W^{\frac{\alpha}{2},2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we have $p'(u_k)\Phi \in D(\mathcal{E}^\alpha)$.

Since Φ is bounded in $D(\mathcal{E}^\alpha)$, we need only to prove that $p'(u_k) \in D(\mathcal{E}^\alpha)$.

Note that p' is locally Lipschitz, i.e., there exists $M > 0$ such that

$$\big|p'(u_k(x)) - p'(u_k(y))\big| \le M\big|u_k(x) - u_k(y)\big|.$$

Thereby we derive

$$\mathcal{E}^\alpha\big[p'(u_k)\big] < \infty.$$

Taking now $p'(u_k)\Phi$ as a test function, we obtain

$$\int\limits_\delta^t\int \frac{\partial u_k}{\partial t} \times p'(u_k)\Phi + \int\limits_\delta^t\int p'(u_k)\Phi L_0 u_k = \int\limits_\delta^t\int V_k u_k p'(u_k)\Phi.$$

Thus, we get

$$\int p(u_k(t))\Phi + \int\limits_\delta^t\int p'(u_k(s))\Phi L_0 u_k = \int\limits_\delta^t\int V_k u_k p'(u_k)\Phi + \int p(u_k(\delta))\Phi.$$

Since the function p is convex, we have

$$p'(u_k(x))\big(u_k(x) - u_k(y)\big) \ge \big(p(u_k)(x) - p(u_k)(y)\big).$$

Therefore,

$$\int p'(u_k(x))\Phi(x)L_0 u_k(x)\,dx =$$

$$= C_{n,s}PV\int\int \frac{p'(u_k(x))\Phi(x)\big(u_k(x) - u_k(y)\big)}{|x-y|^{d+\alpha}}\,dy\,dx \ge$$

$$\ge C_{n,s}PV\int\int \Phi(x)\frac{\big(p(u_k)(x) - p(u_k)(y)\big)}{|x-y|^{d+\alpha}}\,dy\,dx =$$

$$= \int \Phi(x)L_0 p(u_k(x))\,dx.$$

Hence,

$$\int p(u_k(t))\Phi + \int\limits_\delta^t\int \Phi(x)L_0 p(u_k(x,s))\,dx\,ds \le$$

$$\leq \int_{\delta}^{t}\int V_k u_k p'(u_k)\Phi + \int p(u_k(\delta))\Phi.$$

Replace $p(r)$ by a sequence $p_m(r)$ satisfying the hypotheses, for p and converging to $|r|$ as $m \longrightarrow \longrightarrow \infty$. Note that $p'(r) = \dfrac{r}{|r|}$.

First, we see that

$$\left|\int_{\delta}^{t}\int \Phi(x)L_0 p_m(u_k)\,dx\,ds - \int_{\delta}^{t}\int \Phi(x)L_0(u_k)\,dx\,ds\right| \leq$$

$$\leq \|\Phi\|_\infty \int_{\delta}^{t}\int \big|L_0 p_m(u_k) - L_0(u_k)\big| dx\,ds \leq$$

$$\leq |\Phi\|_\infty \big\|L_0 p_m(u_k) - L_0(u_k)\big\|_{L^1\left(\mathbb{R}^d\times[\delta,t]\right)} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty,$$

and consequently, we obtain the limiting inequality

$$\int u_k(t)\Phi\,dx + \int_{\delta}^{t}\int \Phi(x).L_0(u_k)\,dx\,ds \leq$$

$$\leq \int_{\delta}^{t}\int V_k(x)u_k(s)\Phi\,dx\,ds + \int u_k(\delta)\Phi\,dx. \tag{4}$$

We want to let $\delta \longrightarrow 0$. First we claim that

$$\int u_k(\delta)\Phi\,dx \longrightarrow \int u_0(x)\Phi\,dx.$$

Since the operator $(-\triangle)^{\frac{\alpha}{2}}$ is self-adjoint on $D\Big((-\triangle)^{\frac{\alpha}{2}}\Big)$, so, by using the Trotter–Kato theorem and the spectral theorem, we have

$$e^{-\delta(L_0-V_k)}u_0 = \lim_{m\longrightarrow+\infty}\left(e^{-\frac{\delta L_0}{m}}e^{(\frac{\delta}{m})V_k}\right)^m u_0 \leq e^{\delta\lambda}e^{-\delta L_0}u_0,$$

then $\|V_k\|_\infty \leq \lambda$ by the positivity preserving property of $\{e^{-\delta L_0}\}$. It follows that

$$e^{-\delta L_0}u_0(x) \leq u_k(\delta) = e^{-\delta(L_0-V_k)}u_0(x) = e^{-\delta L_0}e^{\delta V_k}u_0(x).$$

Thus,

$$\int(e^{-\delta L_0}u_0)\Phi \leq \int u_k(\delta)\Phi = \int e^{-\delta(L_0-V_k)}u_0\Phi \leq e^{\delta\lambda}\int(e^{-\delta L_0}u_0)\Phi,$$

whence

$$\int(e^{-\delta L_0}u_0)\Phi = \int(e^{-\delta L_0}\Phi)u_0 \longrightarrow \int \Phi u_0$$

as $\delta \longrightarrow 0$, as asserted.

Letting $\delta \longrightarrow 0$ in (4), we deduce

$$\int u_k(t)\Phi\,dx + \int_0^t \int \Phi(x) L_0(u_k)\,dx\,ds \leq$$

$$\leq \int_0^t \int V_k(x) u_k(s)\Phi\,dx\,ds + \int u_0(x)\Phi\,dx. \tag{5}$$

Our aim now is to estimate the LHS of (5). By using [3] (Lemma 2.2), we recall that $\Phi(x) = |x|^{-\beta}$ is the unique solution for the equation

$$L_0\Phi(x) = c|x|^{-\alpha}|x|^{-\beta} = c|x|^{-\alpha}\Phi(x) \quad \text{in the sense of distributions,}$$

where

$$c = 2^\alpha \frac{\Gamma\left(\dfrac{\alpha+\beta}{2}\right)\Gamma\left(\dfrac{d-\beta}{2}\right)}{\Gamma\left(\dfrac{d-(\alpha+\beta)}{2}\right)\Gamma\left(\dfrac{\beta}{2}\right)},$$

which implies

$$L_0\Phi(x) = c|x|^{-\alpha}\Phi(x) \geq V_k(x)\Phi(x).$$

Therefore,

$$\int u_k L_0\Phi\,dx \geq \int V_k(x) u_k \Phi\,dx \quad \text{in the sense of distributions,}$$

and consequently, from (4) we deduce

$$\int u_k(t)\Phi\,dx + \int_0^t \int V_k u_k \Phi\,dx\,ds \leq \int u_k(t)\Phi\,dx + \int_0^t \int \Phi(x) L_0(u_k)\,dx\,ds \leq$$

$$\leq \int_0^t \int V_k u_k \Phi\,dx\,ds + \int u_0(x)\Phi\,dx.$$

Thereby we derive that

$$\int u_k(t)\Phi\,dx \leq \int u_0(x)\Phi\,dx.$$

Let

$$v_k(t) = u_k(t)\Phi.$$

Therefore, if

$$\int u_0(x)\Phi(x)\,dx < \infty,$$

we have

$$\|v_k\|_{L^1(\mathbb{R}^d)} < M \quad \text{for all} \quad k.$$

Then (v_k) is bounded in L^1 and by the weak compactness in L^1, there exists a subsequence still called (v_k) such that (v_k) converge weakly to $v \in L^1(\Omega)$ $\forall \Omega \subset \mathbb{R}^d$ and, consequently, a subsequence

(u_k) converging to a function $\overline{u}$ in $L^1(\Omega)$ and for, every $\Omega \subset \mathbb{R}^d$, we have

$$\|v_k V_k - vV\|_{L^1} = \|v_k V_k - v_k V + v_k V - vV\|_{L^1} \leq$$
$$\leq \big\|v_k(V_k - V)\big\|_{L^1} + \big\|(v_k - v)V\big\|_{L^1}.$$

Hence,

$$\big\|u_k V_k \Phi - \overline{u} V \Phi\big\|_{L^1} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow +\infty.$$

On the other hand, by hypothesis (cf. (3)), (u_k) is an increasing sequence, thus (u_k) increases to $\overline{u}(x,t)$, and $\overline{u}$ is a solution of (1) in the sense of distributions.

References

1. *Ali Ben Amor, Kenzizi T.* The heat equation for the Dirichlet fractional Laplacian with negative potentials: Eistence and blow-up of nonnegative solutions // Acta Math. Sin. (Engl. Ser.). – 2017.
2. *Bogdan K., Byczkowski T.* Potential theory of Schrödinger operator on bounded Lipschitz domains // Stud. Math. – 1999. – **133**, № 1. – P. 53 – 92.
3. *Ali Beldi, Nedra Belhajrhouma, Ali Ben Amor.* Pointwise estimates for the ground states of singular Dirichlet fractional Laplacian // J. Phys. A: Math. and Theor. – 2013. – **46(44)**.
4. *Bogdan K., Byczkowski T., Kulczycki T., Ryznar M., Song R., Vondraček Z.* Potential analysis of stable processes and its extensions // Lect. Notes Math. – 2009. – **1980**.
5. *Baras P., Goldstein J. A.* The heat equation with a singular potential // Trans. Amer. Math. Soc. – 1984. – **284**, № 1. – P. 121 – 139.
6. *Baras P., Goldstein J. A.* Remarks on the inverse square in quantum mechanics // North-Holland Math. Stud. – 1984. – **92**. – P. 31 – 35.
7. *Blumenthal R., Getoor R.* The asymptotic distribution of the eigenvalues for a class of Markov operators // Pacific J. Math. – 1959. – **9**. – P. 399 – 408.
8. *Berg C., Forst G.* Potential theory on locally compact abelian groups. – New York: Springer-Verlag, 1975.
9. *Kubrusly C. S.* Spectral theory of operators on Hilbert spaces. – Springer Sci.+Business Media, LLC, 2012.
10. *Cabré X., Martel Y.* Existence versus explosion instantanée pour des equations de la chaleur linéaires avec potentiel singulier // C. R. Math. Acad. Sci. Paris. – 1999. – **329**, № 11. – P. 973 – 978.
11. *Kato T.* Perturbation theory for linear operators // Classics Math. – Berlin: Springer-Verlag, 1995.
12. *Goldstein J. A., Zhang Q. S.* On a degenerate heat equation with a singular potential // J. Funct. Anal. – 2001. – **186**. – P. 342 – 359.
13. *Yosida K.* Functional analysis // Classics Math. – Berlin: Springer-Verlag, 1995.

Received 09.08.16,
after revision — 18.04.17