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CERTAIN INTEGRALS INVOLVING ℵ-FUNCTIONS AND LAGUERRE POLYNOMIALS

ДЕЯКІ ІНТЕГРАЛИ, ЩО ВКЛЮЧАЮТЬ ℵ-ФУНКЦІЇ ТА ПОЛІНОМИ ЛАГЕРРА

Our aim is to establish certain new integral formulas involving ℵ-functions associated with Laguerre-type polynomials. We also show how the main results presented in paper are general by demonstrating 18 integral formulas that involve simpler known functions, e.g., the generalized hypergeometric function ${}_pF_q$ in a fairly systematic way.

Наша мета — встановити деякі нові інтегральні формули, що включають ℵ-функції, асоційовані з поліномами лагеррівського типу. Також показано, що основні результати, отримані у статті, є загальними. Для цього наведено 18 інтегральних формул, що включають більш прості відомі функції, наприклад узагальнену гіпергеометричну функцію ${}_pF_q$ в досить загальному вигляді.

1. Introduction and preliminaries. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ be sets of complex numbers, real and positive real numbers, integers and positive integers, respectively, and

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}, \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The Aleph (ℵ)-function, which is a very general higher transcendental function and was introduced by Südland et al. [15, 16], is defined by means of Mellin – Barnes type integral in the following manner (see, e.g., [8, 9]):

$$\aleph[z] = \aleph^{m,n}_{p_k,q_k,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (a_j,A_j)_{1,n}, [\delta_j(a_{j_k},A_{j_k})]_{n+1,p_k;r} \\ (b_j,B_j)_{1,m}, [\delta_j(b_{j_k},B_{j_k})]_{m+1,q_k;r} \end{matrix} \right] =$$
$$= \frac{1}{2\pi i} \int\limits_L \Omega^{m,n}_{p_k,q_k,\delta_k;r}(s) z^{-s}\, ds, \tag{1.1}$$

where $z \in \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_k,q_k,\delta_k;r}(s) := \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^r \delta_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{j_k} - B_{j_k} s) \prod_{j=n+1}^{p_k} \Gamma(a_{j_k} + A_{j_k} s)}. \tag{1.2}$$

Here Γ is the familiar Gamma function (see, e.g., [13], Section 1.1); the integration path $L = L_{i\gamma\infty}$, $\gamma \in \mathbb{R}$, extends from $\gamma - i\infty$ to $\gamma + i\infty$ with indentations, if necessary; the poles of the Gamma function $\Gamma(1 - a_j - A_j s)$, $j, n \in \mathbb{N}$, $1 \le j \le n$, do not coincide with those of $\Gamma(b_j + B_j s)$, $j, m \in \mathbb{N}$, $1 \le j \le m$; the parameters p_k, $q_k \in \mathbb{N}$ satisfy the conditions $0 \le n \le p_k$, $1 \le m \le q_k$, $\delta_k \in \mathbb{R}^+$, $1 \le k \le r$; the parameters A_j, B_j, A_{j_k}, $B_{j_k} \in \mathbb{R}^+$ and a_j, b_j, a_{j_k}, $b_{j_k} \in \mathbb{C}$; the empty product in

(1.2) (and elsewhere) is (as usual) understood to be unity. The existence conditions for the defining integral (1.1) are given as follows:

$$\varphi_\ell \in \mathbb{R}^+ \qquad \text{and} \qquad |\arg(z)| < \frac{\pi}{2}\varphi_\ell, \qquad \ell \in \mathbb{N}, \quad 1 \le \ell \le r,$$

and

$$\varphi_\ell \ge 0, \quad |\arg(z)| < \frac{\pi}{2}\,\varphi_\ell \qquad \text{and} \qquad \Re(\varsigma_\ell) + 1 < 0,$$

where

$$\varphi_\ell := \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \delta_\ell \left(\sum_{j=n+1}^{p_\ell} A_{j\ell} + \sum_{j=m+1}^{q_\ell} B_{j\ell} \right)$$

and

$$\varsigma_\ell := \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \delta_\ell \left(\sum_{j=n+1}^{p_\ell} A_{j\ell} + \sum_{j=m+1}^{q_\ell} B_{j\ell} \right) + \frac{1}{2}(p_\ell - q_\ell),$$

$$\ell \in \mathbb{N}, \quad 1 \le \ell \le r.$$

***Remark* 1.** The expression in (1.1) of the Aleph-function does not follow completely the notational convention of the Fox's H-function. Namely, in the $\aleph$-functions, the kernel $\Omega^{m,n}_{p_k,q_k,\delta_k;r}(s)$, parameter couples $\left(a_j, A_j\right)_{1,n}$, $\left(b_j, B_j\right)_{1,m}$ build the Gamma function terms exclusively in the numerator, and $\left[\delta_j \left(a_{j_k}, A_{j_k}\right)_{n+1,p_k}\right]$, $\left[\delta_j \left(b_{j_k}, B_{j_k}\right)_{n+1,q_k}\right]$ build the linear combination exclusively in the denominator, while, for the $H^{m,n}_{p,q}[z]$, both upper and lower couples of parameters $\left(a_j, A_j\right)_{1,p}$ and $\left(b_j, B_j\right)_{1,q}$ play roles in forming both numerator and denominator terms according to m and n.

***Remark* 2.** Setting $\delta_j = 1,\ j \in \mathbb{N},\ 1 \le j \le r$, in (1.1) yields the I-function (see [7]) whose Further, special case when $r = 1$ reduces to a familiar function (see [3, 4]).

Prabhaker and Suman [5] defined the following Laguerre-type polynomials $L_n^{(\alpha,\beta)}(x)$ as follows:

$$L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n + \beta + 1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, x^k}{k!\,\Gamma(\alpha k + \beta + 1)}, \tag{1.3}$$

$$n \in \mathbb{N}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > -1,$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & n \in \mathbb{N}, \end{cases} =$$

$$= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

The special case of (1.3) when $\alpha = 1$ reduces to the familiar generalized Laguerre polynomials $L_n^{(\beta)}(x)$ (see, e.g., [6], Chapter 12):

$$L_n^{(1,\beta)}(x)=\frac{\Gamma(n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{k!\,\Gamma(k+\beta+1)}=L_n^{(\beta)}(x).$$

The Konhauser polynomials of the second kind (see [12]) is defined by

$$Z_n^{\beta}(x;k)=\frac{\Gamma(kn+\beta+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{cn}{j}\frac{x^{kj}}{\Gamma(kj+\beta+1)}, \tag{1.4}$$

$$\Re(\beta)>-1,\quad k\in\mathbb{Z},\quad n\in\mathbb{N}.$$

It is easy to see that

$$L_n^{(0,\beta)}(x^k)=Z_n^{\beta}(x;k)$$

and

$$L_n^{(\beta)}(x)=Z_n^{\beta}(x;1). \tag{1.5}$$

The polynomials $Z_n^{(\alpha,\beta)}(x;k)$ are defined as follows (see [10]):

$$Z_n^{(\alpha,\beta)}(x;k)=\sum_{j=0}^{n}\frac{\Gamma(kn+\beta+1)(-1)^j x^{kj}}{j!\Gamma(kj+\beta+1)\Gamma(\alpha n-\alpha j+1)}, \tag{1.6}$$

$$\Re(\alpha)>0,\quad \Re(\beta)>-1,\quad n\in\mathbb{N},\quad k\in\mathbb{Z}.$$

We find from (1.4) and (1.6) that

$$Z_n^{(1,\beta)}(x;k)=Z_n^{\beta}(x;k).$$

When $\alpha\in\mathbb{N}$, (1.6) can be written in the following form:

$$Z_n^{(\alpha,\beta)}(x;k)=\frac{\Gamma(kn+\beta+1)}{\Gamma(\alpha n+1)}\sum_{m=0}^{n}\frac{(-\alpha n)_{\alpha m}x^{km}}{m!\Gamma(km+\beta+1)(-1)^{(\alpha-1)m}}.$$

The polynomials $L_n^{(\alpha,\beta)}(\gamma;x)$ are defined by (see [10])

$$L_n^{(\alpha,\beta)}(\gamma;x)=\sum_{r=0}^{n}\frac{\Gamma(\alpha n+\beta+1)(-1)^r x^r}{r!\Gamma(\alpha r+\beta+1)\Gamma(\gamma n-\gamma r+1)},$$

$$\min\{\Re(\alpha),\quad \Re(\gamma)\}>0,\quad \Re(\beta)>-1,\quad n\in\mathbb{N}.$$

Also we recall some properties of the Pochhammer symbol (see, e.g., [11])

$$(-x)_n=(-1)^n(x-n+1)_n, \tag{1.7}$$

$$(x+y)_n=\sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j}, \tag{1.8}$$

$$(x)_{n+m}=(x)_n(x+n)_m, \tag{1.9}$$

$$\binom{x}{n} = \frac{(-1)^n}{n!}(-x)_n, \tag{1.10}$$

where $x,\, y \in \mathbb{C}$ and $m,\, n \in \mathbb{N}_0$.

Here, in this paper, we aim to establish certain new integral formulas involving $\aleph$-function associated with the Laguerre-type polynomials. We also show how the main results presented here are general by choosing to demonstrate 18 integral formulas involving simpler known and familiar functions, for example, the generalized hypergeometric function ${}_pF_q$, in a rather systematic manner.

2. Integral formulas. Here we present certain integral formulas mainly involving the $\aleph$-functions.

Theorem 1. *Let $z,\, \lambda,\, \delta \in \mathbb{C}$ with $\min\{\Re(\delta),\, \Re(\lambda)\} > 0$ and $|z| < 1$. Also let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$ where $\gamma \in \mathbb{R}$ is the chosen number from the integration path $L_{i\gamma\infty}$ in* (1.1)*. Then the following integral formula holds true*:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right] du =$$

$$= \Gamma(\delta)\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \left| \begin{matrix} (\lambda+\delta, C), (a_j, A_j)_{1,n}, [\delta_j(a_{j_k}, A_{j_k})]_{n+1,\rho_k;r} \\ (b_j, B_j)_{1,m}, (\lambda, C), [\delta_j(b_{j_k}, B_{j_k})]_{m+1,\sigma_k;r} \end{matrix} \right. \right] \tag{2.1}$$

provided the other involved parameters are so constrained that each member can exist.

Proof. Let $\mathcal{L}_1$ be the left-hand side of (2.1). Then, using (1.1) and changing the order of the double integrals, which is guaranteed under the given conditions, we obtain

$$\mathcal{L}_1 = \frac{1}{2\pi i}\int_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s)z^{-s}\left\{\int_0^1 u^{\lambda+Cs-1}(1-u)^{\delta-1}\,du\right\} ds. \tag{2.2}$$

Recall the familiar Beta function $B(x,\, y)$ which is defined by and expressed in terms of the Gamma function as follows (see, e.g., [13, p. 8]):

$$B(x,\, y) = \begin{cases} \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt, & \min\{\Re(x),\, \Re(y)\} > 0, \\ \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, & x,\, y \in \mathbb{C}\setminus\mathbb{Z}_0^-, \end{cases}$$

to evaluate the inner integral in (2.2) to yield

$$\mathcal{L}_1 = \Gamma(\delta)\frac{1}{2\pi i}\int_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s)z^{-s}\frac{\Gamma(\lambda+Cs)}{\Gamma(\lambda+\delta+Cs)}\,ds. \tag{2.3}$$

Finally, interpreting the right-hand side of (2.3) in terms of the definition (1.1), we arrive at the right-hand side of (2.1).

Theorem 2. *Let $z,\, \lambda,\, \delta \in \mathbb{C}$ with $\min\{\Re(\delta),\, \Re(\lambda)\} > 0$ and $|z| < 1$. Also let $x,\, t \in \mathbb{R}$ with $x \geq t$. Further, let $C \in \mathbb{R}^+$ and $\Re(\lambda) > -C\,\gamma$ where $\gamma \in \mathbb{R}$ is the chosen number from the integration path $L_{i\gamma\infty}$ in* (1.1)*. Then the following integral formula holds true*:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right]du = \Gamma(\delta)(x-t)^{\delta+\lambda-1}\times$$

$$\times\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta,C),(a_j,A_j)_{1,n},[\delta_j(a_{j_k},A_{j_k})]_{n+1,\rho_k;r}\\(b_j,B_j)_{1,m},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{m+1,\sigma_k;r}\end{matrix}\right] \tag{2.4}$$

provided the other involved parameters are so constrained that each member can exist.

Proof. Let $\mathcal{L}_2$ be the left-hand side of (2.4) and change the variable u into $v = \dfrac{u-t}{x-t}$. Similarly as in the proof of Theorem 1, we can obtain

$$\mathcal{L}_2 = \frac{(x-t)^{\delta+\lambda-1}}{2\pi i}\int_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s)z^{-s}(x-t)^{Cs}\left\{\int_0^1(1-v)^{\delta-1}v^{\lambda+Cs-1}\,dv\right\}ds =$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{1}{2\pi i}\int_L \Omega^{m,n}_{\rho_k,\sigma_k,\delta_k;r}(s)z^{-s}(x-t)^{Cs}\frac{\Gamma(\lambda+Cs)}{\Gamma(\delta+\lambda+Cs)}\,ds.$$

Now it is easy to see that, in view of the definition (1.1), the last equality is interpreted into the right-hand side of (2.4).

Theorem 3. *Let* $z,\ \nu,\ \mu\in\mathbb{C}$ *with* $\min\{\Re(\nu),\ \Re(\mu)\}>0$ *and* $x\in\mathbb{R}^+$. *Also let* $C\in\mathbb{R}^+$ *and* $\Re(\lambda)>-C\,\gamma$ *where* $\gamma\in\mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following integral formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\aleph^{m,n}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right]dt =$$

$$= x^{\nu+\mu-1}\Gamma(\nu)\aleph^{m,n+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\mu+\nu,C),(a_j,A_j)_{1,n},[\delta_j(a_{j_k},A_{j_k})]_{n+1,\rho_k;r}\\(b_j,B_j)_{1,m},(\mu,C),[\delta_j(b_{j_k},B_{j_k})]_{m+1,\sigma_k;r}\end{matrix}\right] \tag{2.5}$$

provided the other involved parameters are so constrained that each member can exist.

Proof. A similar argument as in the proof of either Theorem 1 or Theorem 2 can establish the result (2.5). So we choose to omit the details of its proof.

For the sequel of the above theorems, we need the following formula (see [1]) which is recalled in Lemma 1.

Lemma 1. *Let* $\min\{\Re(a),\ \Re(c),\ \Re(\zeta),\ \Re(\xi)\}>0$, $\min\{\Re(b),\ \Re(d)\}>-1$ *and* $h,\ m,\ n\in\mathbb{N}$. *Then the following formula holds true*:

$$L_n^{(a,b)}(\xi;x)L_m^{(c,d)}(\zeta;x) = \sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(an+b+1)\Gamma(cm+d+1)}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\times$$

$$\times\frac{(-x)^h}{\Gamma(\xi(n-k)+1)\Gamma(ak+b+1)\Gamma(c(h-k)+d+1)}. \tag{2.6}$$

Theorem 4. *Let* $z,\ \delta,\ \lambda\in\mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda)\}>0$ *and* $|z|<1$. *Also let* $\min\{\Re(\sigma),\ \Re(\xi),\ \Re(\zeta)\}>0$ *and* $\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\}>-1$. *Further, let* $C\in\mathbb{R}^+$ *and* $\Re(\lambda)>-C\,\gamma$

where $\gamma \in \mathbb{R}$ is the chosen number from the integration path $L_{i\gamma\infty}$ in (1.1)*. Then the following formula holds true*:

$$\int\limits_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\zeta;\sigma(1-u)) L_n^{(c',d')}(\xi;\sigma(1-u))\times$$

$$\times\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right]du = \sum_{h=0}^{m+n} \Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\Gamma(\delta+h)\,\sigma^h\times$$

$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ (b_j,B_j)_{1,\alpha},(\lambda,C)[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right] \tag{2.7}$$

provided the other involved parameters are so constrained that each member can exist. Here $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ is given by

$$\Delta^{n,m,\xi,\zeta}_{a',b',c',d'} := \sum_{k=0}^{h}\binom{h}{k}\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(-1)^h}{\Gamma(\zeta(m-h+k)+1)\Gamma(\xi(n-k)+1)}\times$$

$$\times\frac{1}{\Gamma(h+1)\Gamma(a'k+b'+1)\Gamma\left(c'\left(h-k'\right)+d'+1\right)}. \tag{2.8}$$

Proof. Let $\mathcal{L}_3$ be the left-hand side of (2.7). Then, by using (2.6), we have

$$\mathcal{L}_3 = \sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(\sigma)^h}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\times$$

$$\times\frac{(-1)^h}{\Gamma(\xi(n-k)+1)\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\times$$

$$\times\int\limits_0^1 u^{\lambda-1}(1-u)^{\delta+h-1}\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right]du. \tag{2.9}$$

Applying (2.1) to the integral in (2.9), we obtain

$$\mathcal{L}_3 = \sum_{h=0}^{m+n}\sum_{k=0}^{h}\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)(\sigma)^h}{\Gamma(h-k+1)\Gamma(\zeta(m-h+k)+1)\Gamma(k+1)}\times$$

$$\times\frac{(-1)^h}{\Gamma(\xi(n-k)+1)\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\Gamma(\delta+h)\times$$

$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\,\middle|\,\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\ (b_j,B_j)_{1,\alpha},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right]. \tag{2.10}$$

Finally, it is easy to see that the expression in (2.10) corresponds with the right-hand side of (2.7).

Here we present five integral formulas asserted in Theorems 5 – 9, without their proofs, because each of their proofs would run parallel to that of Theorem 4.

Theorem 5. *Let* z, δ, $\lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda)\} > 0$ *and* $|z| < 1$. *Also let* $\min\{\Re(\sigma), \Re(\xi), \Re(\zeta)\} > 0$ *and* $\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1$. *Further, let* $x, t \in \mathbb{R}$ *with* $x \geq t$, $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} L_m^{(a',b')}(\zeta;\sigma(u-t)) L_n^{(c',d')}(\xi;\sigma(u-t)) \times$$

$$\times \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \sum_{h=0}^{m+n} \Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\,\sigma^h \times$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right]$$

provided the other involved parameters are so constrained that each member can exist. Here $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ *is given as in* (2.8).

Theorem 6. *Let* z, μ, $\nu \in \mathbb{C}$ *with* $\min\{\Re(\mu), \Re(\nu)\} > 0$ *and* $|z| < 1$. *Also let* $\min\{\Re(\sigma), \Re(\xi), \Re(\zeta)\} > 0$ *and* $\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1$. *Further, let* $x \in \mathbb{R}^+$, $C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_0^x t^{\nu-1}(x-t)^{\mu-1} L_m^{(a',b')}(\zeta;\sigma(x-t)) L_n^{(c',d')}(\xi;\sigma(x-t)) \times$$

$$\times \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right] dt = x^{\mu+\nu-1} \sum_{h=0}^{m+n} \Delta^{n,m,\xi,\zeta}_{a',b',c',d'}\,\sigma^h\, x^h \times$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\mu, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix} \right]$$

provided the other involved parameters are so constrained that each member can exist. Here $\Delta^{n,m,\xi,\zeta}_{a',b',c',d'}$ *is given as in* (2.8).

Theorem 7. *Let* z, δ, $\lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda)\} > 0$ *and* $|z| < 1$. *Also let* $\min\{\Re(\sigma), \Re(\xi), \Re(\zeta)\} > 0$ *and* $\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1$. *Further, let* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\zeta;\sigma(1-u)) L_n^{(c',d')}(\xi;\sigma(1-u)) \times$$

$$\times \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right] du = \sum_{h=0}^{m+n} \nabla^{n,m,\xi,\zeta}_{a',b',c',d'}\Gamma(\delta+h)\,\sigma^h \times$$

$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\left|\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\(b_j,B_j)_{1,\alpha},(\lambda,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right.\right] \tag{2.11}$$

provided the other involved parameters are so constrained that each member can exist. Here the $\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}$ *is given by*

$$\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}:=\frac{\Gamma(a'n+b'+1)\Gamma\left(c'm+d'+1\right)}{\Gamma\left(\zeta m+1\right)\Gamma\left(\xi n+1\right)}\times$$
$$\times\sum_{k=0}^{h}\left[\binom{h}{k}\frac{(-1)^{h-\zeta(h-k)-\xi k}(-\zeta m)_{\zeta(h-k)}(-\xi n)_{\xi k}}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right]. \tag{2.12}$$

Theorem 8. *Let* $z,\,\delta,\,\lambda\in\mathbb{C}$ *with* $\min\{\Re(\delta),\,\Re(\lambda)\}>0$ *and* $|z|<1$. *Also let* $\min\{\Re(\sigma),\,\Re(\xi),\,\Re(\zeta)\}>0$ *and* $\min\{\Re(a'),\,\Re(b'),\,\Re(c'),\,\Re(d')\}>-1$. *Further, let* $x,\,t\in\mathbb{R}$ *with* $x\geq t$, $C\in\mathbb{R}^+$, *and* $\Re(\lambda)>-C\,\gamma$ *where* $\gamma\in\mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_t^x(x-u)^{\delta-1}(u-t)^{\lambda-1}L^{(a',b')}_m(\zeta;\sigma(u-t))L^{(c',d')}_n(\xi;\sigma(u-t))\times$$
$$\times\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right]du=\Gamma(\delta)(x-t)^{\delta+\lambda-1}\sum_{h=0}^{m+n}\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}\,\sigma^h\times$$
$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\left|\begin{matrix}(\lambda+\delta+h,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\(b_j,B_j)_{1,\alpha},(\lambda+h,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right.\right] \tag{2.13}$$

provided the other involved parameters are so constrained that each member can exist. Here $\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}$ *is given as in* (2.12).

Theorem 9. *Let* $z,\,\mu,\,\nu\in\mathbb{C}$ *with* $\min\{\Re(\mu),\,\Re(\nu)\}>0$ *and* $|z|<1$. *Also let* $\min\{\Re(\sigma),\,\Re(\xi),\,\Re(\zeta)\}>0$ *and* $\min\{\Re(a'),\,\Re(b'),\,\Re(c'),\,\Re(d')\}>-1$. *Further, let* $x\in\mathbb{R}^+$, $C\in\mathbb{R}^+$, *and* $\Re(\lambda)>-C\,\gamma$ *where* $\gamma\in\mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_0^x(t)^{\nu-1}(x-t)^{\mu-1}L^{(a',b')}_m(\zeta;\sigma(x-t))L^{(c',d')}_n(\xi;\sigma(x-t))\times$$
$$\times\aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right]dt=x^{\mu+\nu-1}\sum_{h=0}^{m+n}\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}\,\sigma^h\,x^h\times$$
$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\left|\begin{matrix}(\mu+\nu,C),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\(b_j,B_j)_{1,\alpha},(\mu,C),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right.\right] \tag{2.14}$$

provided the other involved parameters are so constrained that each member can exist. Here $\nabla^{n,m,\,\xi,\,\zeta}_{a',b',c',d'}$ *is given as in* (2.12).

3. Special cases. It is noted that the results in Section 2 are general enough to be specialized to yield a large number of simpler integral formulas. Here we choose to present the following formulas.

***Corollary* 1.** *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u)) L_n^{(c',d')}(\sigma(1-u)) \times$$

$$\times \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[zu^{-C}\right] du = \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!} \sum_{h=0}^{m+n} \sigma^h\,\Gamma(\delta+h)\times$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right] \times$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix}\right].$$

provided the other involved parameters are so constrained that each member can exist.

Proof. Setting $\zeta = \xi = 1$ in (2.11), after a little simplification, we get the desired result.

***Corollary* 2.** *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x,\ t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1,$$

$C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int\limits_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1) \times$$

$$\times \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(u-t)^{-C}\right] du = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!} \times$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right] \times$$

$$\times \aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, [\delta_j(a_{j_k}, A_{j_k})]_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), [\delta_j(b_{j_k}, B_{j_k})]_{\alpha+1,\sigma_k;r} \end{matrix}\right]$$

provided the other involved parameters are so constrained that each member can exist.

Proof. Setting $a' = c' = \xi = \zeta = 1$ in (2.13) and using (1.5) to consider $L_n^{1,b}(1;x) = Z_n^{(1,b)}(x;1)$, after a little simplification, we get the desired result.

***Corollary* 3.** *Let* $z,\ \mu,\ \nu \in \mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and* $\min\{\Re(b'),\ \Re(d')\} > -1$. *Further, let* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\left[1-\sigma(x-t)\right]^n \aleph^{\alpha,\beta}_{\rho_k,\sigma_k,\delta_k;r}\left[z(x-t)^{-C}\right]dt =$$

$$= x^{\mu+\nu-1}\sum_{h=0}^{n}(-n)_h\,\sigma^h\,x^h\times$$

$$\times\aleph^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1,\delta_k;r}\left[z\left|\begin{matrix}(\mu+\nu,p),(a_j,A_j)_{1,\beta},[\delta_j(a_{j_k},A_{j_k})]_{\beta+1,\rho_k;r}\\(b_j,B_j)_{1,\alpha},(\mu,p),[\delta_j(b_{j_k},B_{j_k})]_{\alpha+1,\sigma_k;r}\end{matrix}\right.\right]$$

provided the other involved parameters are so constrained that each member can exist.

Proof. Setting $a' = c' = 0$ and $\xi = \zeta = 1$ in (2.14), and using some suitable identities in Section 1 including (1.7)–(1.10), after a little simplification, we get the desired result.

When $\delta_1 = \ldots = \delta_r = 1$ in (1.1), the definition of the I-function is recovered (see [7]):

$$I[z] = I^{m,n}_{p_k,q_k;r}[z] =$$

$$= \aleph^{m,n}_{p_k,q_k,1;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},[1(a_{j_k},A_{j_k})]_{n+1,p_k;r} =\\(b_j,B_j)_{1,m},[1(b_{j_k},B_{j_k})]_{m+1,q_k;r}\end{matrix}\right.\right] =$$

$$= \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_k,q_k,1;r}(s)z^{-s}\,ds, \tag{3.1}$$

where $z \in \mathbb{C}\setminus\{0\}$, $i = \sqrt{-1}$ and $\Omega^{m,n}_{p_k,q_k,1;r}(s)$ is defined in (1.2), and the integration path L can be used as in (1.1). Otherwise, a new integration path for this (3.1) can be chosen. The existence conditions for the integral (3.1) can be easily deduced from those of the $\aleph$-function (1.1) with $\delta_1 = \ldots = \delta_r = 1$.

Then the integral formulas in Corollaries 1–3 can reduce to yield the following integral formulas involving the I-function given in Corollaries 4–6, respectively.

***Corollary* 4.** *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\sigma(1-u))L_n^{(c',d')}(\sigma(1-u))\times$$

$$\times I^{\alpha,\beta}_{\rho_k,\sigma_k;r}\left[zu^{-C}\right]du = \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h)\times$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right] \times$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right]$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 5.** *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x,\ t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1,$$

$C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1) \times$$

$$\times I^{\alpha,\beta}_{\rho_k,\sigma_k;r} \left[z(u-t)^{-C} \right] du = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!} \times$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)} \right] \times$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\lambda+h, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right]$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 6.** *Let* $z,\ \mu,\ \nu \in \mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and* $\min\{\Re(b'),\ \Re(d')\} > -1$. *Further, let* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1} \left[1-\sigma(x-t)\right]^n I^{\alpha,\beta}_{\rho_k,\sigma_k;r}\left[z(x-t)^{-C}\right] dt =$$

$$= x^{\mu+\nu-1} \sum_{h=0}^{n} (-n)_h\, \sigma^h\, x^h \times$$

$$\times I^{\alpha,\beta+1}_{\rho_k+1,\sigma_k+1;r} \left[z \,\middle|\, \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\beta}, (a_{j_k}, A_{j_k})_{\beta+1,\rho_k;r} \\ (b_j, B_j)_{1,\alpha}, (\mu, C), (b_{j_k}, B_{j_k})_{\alpha+1,\sigma_k;r} \end{matrix} \right]$$

provided the other involved parameters are so constrained that each member can exist.

Further, the special case $r = 1$ of the I-function (3.1) reduces to become the H-function (see [3, 4]). Then the formulas in Corollaries 4 – 6 reduce to yield the following integral formulas involving the H-function which are in Corollaries 7 – 9, respectively.

***Corollary* 7.** *Let* $z,\ \delta,\ \lambda,\ \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1.$$

Further, let $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u)) L_n^{(c',d')}(\sigma(1-u))\times$$

$$\times H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[zu^{-C}\right] du = \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!} \sum_{h=0}^{m+n} \sigma^h\,\Gamma(\delta+h)\times$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right]\times$$

$$\times H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\rho_1} \\ (b_j, B_j)_{1,\sigma_1}, (\lambda, C) \end{matrix}\right] \tag{3.2}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 8.** *Let* $z,\ \delta,\ \lambda \in \mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x,\ t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\} > -1,$$

$C \in \mathbb{R}^+$, *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1)\times$$

$$\times H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[z(u-t)^{-C}\right] du = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}\times$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right]\times$$

$$\times H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z \,\middle|\, \begin{matrix} (\lambda+\delta+h, C), (a_j, A_j)_{1,\rho_1} \\ (b_j, B_j)_{1,\sigma_1}, (\lambda+h, C) \end{matrix}\right] \tag{3.3}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 9.** *Let* $z,\ \mu,\ \nu \in \mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and* $\min\{\Re(b'),\ \Re(d')\} > -1$. *Further, let* $C \in \mathbb{R}^+$ *and* $\Re(\lambda) > -C\,\gamma$ *where* $\gamma \in \mathbb{R}$ *is the chosen number from the integration path* $L_{i\gamma\infty}$ *in* (1.1). *Then the following formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\left[1-\sigma(x-t)\right]^n H^{\alpha,\beta}_{\rho_1,\sigma_1}\left[z(x-t)^{-C}\right] dt =$$

$$= x^{\mu+\nu-1} \sum_{h=0}^{n} (-n)_h \, \sigma^h \, x^h H^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1} \left[z \, \middle| \, \begin{matrix} (\mu+\nu, C), (a_j, A_j)_{1,\rho_1} \\ (b_j, B_j)_{1,\sigma_1}, (\mu, C) \end{matrix} \right] \tag{3.4}$$

provided the other involved parameters are so constrained that each member can exist.

It is noted that the special case of the H-function when $A_j = 1$, $j = 1, \ldots, p$, and $B_j = 1$, $j = 1, \ldots, q$, reduces to the Meijer's G-function (see, e.g., [2], Section 8.2) as follows:

$$H^{\alpha,\beta}_{\rho_1,\sigma_1} \left[x \, \middle| \, \begin{matrix} (a_j, 1)_{1,\rho_1} \\ (b_j, 1)_{1,\sigma_1} \end{matrix} \right] = G^{\alpha,\beta}_{\rho_1,\sigma_1} \left[x \, \middle| \, \begin{matrix} (a_{\rho_1}) \\ (b_{\sigma_1}) \end{matrix} \right].$$

Then the formulas in Corollaries 7 – 9 are seen to reduce to give the corresponding integral formulas involving the Meijer's G-function (3.5) – (3.7).

***Corollary* 10.** *Let* $z, \delta, \lambda, \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let*

$$\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1.$$

Then the following formula holds true:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1} L_m^{(a',b')}(\sigma(1-u)) L_n^{(c',d')}(\sigma(1-u)) \times$$

$$\times G^{\alpha,\beta}_{\rho_1,\sigma_1}[zu^{-1}]\, du = \frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\, n!} \sum_{h=0}^{m+n} \sigma^h \, \Gamma(\delta+h) \times$$

$$\times \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)} \right] \times$$

$$\times \frac{\Gamma(\lambda+\delta+h)}{\Gamma(\lambda)} G^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1} \left[z \, \middle| \, \begin{matrix} (\lambda+\delta+h), (a_{\rho_1}) \\ (b_{\sigma_1}), (\lambda) \end{matrix} \right] \tag{3.5}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 11.** *Let* $z, \delta, \lambda, \sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x, t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\{\Re(a'), \Re(b'), \Re(c'), \Re(d')\} > -1.$$

Then the following formula holds true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1) \times$$

$$\times G^{\alpha,\beta}_{\rho_1,\sigma_1}[z(u-t)^{-1}]\, du = \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\, n!} \times$$

$$\times \sum_{h=0}^{m+n} \sigma^h \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)} \right] \times$$

$$\times\frac{\Gamma(\lambda+\delta+h)}{\Gamma(\lambda+h)}G^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z\left|\begin{matrix}(\lambda+\delta+h),(a_{\rho_1})\\(b_{\sigma_1}),(\lambda+h)\end{matrix}\right.\right] \tag{3.6}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 12.** *Let* $z,\ \mu,\ \nu,\ \sigma\in\mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\}>0$ *and* $|z|<1$. *Also let* $x\in\mathbb{R}^+$ *and* $\min\{\Re(b'),\ \Re(d')\}>-1$. *Then the following formula holds true*:

$$\int\limits_0^x t^{\nu-1}(x-t)^{\mu-1}\,[1-\sigma(x-t)]^n\,G^{\alpha,\beta}_{\rho_1,\sigma_1}[z(x-t)^{-1}]\,dt=$$

$$=x^{\mu+\nu-1}\sum_{h=0}^{n}(-n)_h\,\sigma^h\,x^hG^{\alpha,\beta+1}_{\rho_1+1,\sigma_1+1}\left[z\left|\begin{matrix}(\mu+\nu),(a_{\rho_1})\\(b_{\sigma_1}),(\mu)\end{matrix}\right.\right] \tag{3.7}$$

provided the other involved parameters are so constrained that each member can exist.

Here, replacing $\sigma_1,\ a_j,\ b_j$ by $\sigma_1+1,\ 1-a_j,\ 1-b_j$ with $b_1=0$, respectively, and letting $\alpha=1$ in the H-function is seen to yield the Wright's generalized hypergeometric function ${}_p\Psi_q$ (see, *e.g.*, [14, p. 50]):

$$H^{1,\rho_1}_{\rho_1,\sigma_1+1}\left[-x\left|\begin{matrix}(1-a_j,A_j)_{1,p}\\(0,1),(1-b_j,B_j)_{1,q}\end{matrix}\right.\right]={}_{\rho_1}\Psi_{\sigma_1}\left[\begin{matrix}(a_j,A_j)_{1,p}\,;\\(b_j,B_j)_{1,q}\,;\end{matrix}x\right]. \tag{3.8}$$

Then, applying the relation (3.8) to the formulas (3.2), (3.3) and (3.4) yields the following corresponding integral formulas involving the Wright's generalized hypergeometric function ${}_p\Psi_q$ (3.9)–(3.11).

***Corollary* 13.** *Let* $z,\ \delta,\ \lambda,\ \sigma\in\mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\}>0$ *and* $|z|<1$. *Also let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\}>-1.$$

Then the following formula holds true:

$$\int\limits_0^1 u^{\lambda-1}(1-u)^{\delta-1}L^{(a',b')}_m(\sigma(1-u))L^{(c',d')}_n(\sigma(1-u))\times$$

$$\times{}_{\rho_1}\Psi_{\sigma_1}[zu^{-p}]\,du=\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h)\times$$

$$\times\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right]\times$$

$$\times{}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\lambda+\delta+h,p),(a_j,A_j)_{1,\rho_1}\\(b_j,B_j)_{1,\sigma_1},(\lambda,p)\end{matrix};z\right] \tag{3.9}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 14.** *Let* $z,\ \delta,\ \lambda,\ \sigma\in\mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\}>0$ *and* $|z|<1$. *Also let* $x,\ t\in\mathbb{R}$ *with* $x\geq t$. *Further, let*

$$\min\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\}>-1.$$

Then the following formula holds true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1)\times$$

$$\times {}_{\rho_1}\Psi_{\sigma_1}[z(u-t)^{-p}]\,du = \Gamma(\delta)(x-t)^{\delta+\lambda-1}\frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\,n!}\times$$

$$\times\sum_{h=0}^{m+n}\sigma^h\sum_{k=0}^{h}\binom{h}{k}\left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right]\times$$

$$\times {}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\lambda+\delta+h,p),(a_j,A_j)_{1,\rho_1}\\(b_j,B_j)_{1,\sigma_1},(\lambda+h,p)\end{matrix};z\right] \tag{3.10}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 15.** *Let* $z,\ \mu,\ \nu,\ \sigma\in\mathbb{C}$ *with* $\min\{\Re(\mu),\ \Re(\nu),\ \Re(\sigma)\}>0$ *and* $|z|<1$. *Also let* $x\in\mathbb{R}^+$ *and* $\min\{\Re(b'),\ \Re(d')\}>-1$. *Then the following formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\,[1-\sigma(x-t)]^n\,{}_{\rho_1}\Psi_{\sigma_1}[z(x-t)^{-p}]\,dt=$$

$$= x^{\mu+\nu-1}\sum_{h=0}^{n}(-n)_h\,\sigma^h\,x^h\,{}_{\rho_1+1}\Psi_{\sigma_1+1}\left[\begin{matrix}(\mu+\nu,p),(a_j,A_j)_{1,\rho_1}\\(b_j,B_j)_{1,\sigma_1},(\mu,p)\end{matrix};z\right] \tag{3.11}$$

provided the other involved parameters are so constrained that each member can exist.

Also, choosing $p=1;\ \alpha=1,\ \beta=2,\ \rho_1=\sigma_1=2;\ A_j=B_j=1;\ b_1=0$ and replace $a_1,\ a_2,\ b_2$ into $1-a_1,\ 1-a_2,\ 1-b_2$, respectively, the H-function reduces to the Gaussian hypergeometric function ${}_2F_1$ as follows:

$$H_{2,2}^{1,2}\left[x\,\middle|\,\begin{matrix}(1-a_1,1),(1-a_2,1)\\(0,1),(1-b_2,1)\end{matrix}\right]=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_2)}{}_2F_1[a_1,a_2;b_2;-x]. \tag{3.12}$$

Then applying the relation (3.12) to the formulas (3.2), (3.3) and (3.4) is seen to yield the following results (3.13)–(3.15) whose integrands and resulting formulas contain ${}_2F_1$ and the generalized hypergeometric function ${}_3F_2$, respectively.

***Corollary* 16.** *Let* $z,\ \delta,\ \lambda,\ \sigma\in\mathbb{C}$ *with* $\min\{\Re(\delta),\ \Re(\lambda),\ \Re(\sigma)\}>0$ *and* $|z|<1$. *Also let*

$$\min\left\{\Re(a'),\ \Re(b'),\ \Re(c'),\ \Re(d')\right\}>-1.$$

Then the following formula holds true:

$$\int_0^1 u^{\lambda-1}(1-u)^{\delta-1}L_m^{(a',b')}(\sigma(1-u))L_n^{(c',d')}(\sigma(1-u))\times$$

$$\times{}_2F_1[a_1,a_2;b_2;zu^{-1}]\,du=\frac{\Gamma(a'n+b'+1)\Gamma(c'm+d'+1)}{m!\,n!}\sum_{h=0}^{m+n}\sigma^h\,\Gamma(\delta+h)\times$$

$$\times(\delta+h)_\lambda \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{h-k}(-n)_k}{\Gamma(a'k+b'+1)\Gamma(c'(h-k)+d'+1)}\right] \times$$

$$\times {}_3F_2[\lambda+\delta+h, a_1, a_2; b_2, \lambda; z] \tag{3.13}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 17.** *Let* z, δ, λ, $\sigma \in \mathbb{C}$ *with* $\min\{\Re(\delta), \Re(\lambda), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* x, $t \in \mathbb{R}$ *with* $x \geq t$. *Further, let*

$$\min\left\{\Re(a'),\, \Re(b'),\, \Re(c'),\, \Re(d')\right\} > -1.$$

Then the following formula holds true:

$$\int_t^x (x-u)^{\delta-1}(u-t)^{\lambda-1} Z_m^{(1,b')}(\sigma(u-t);1) Z_n^{(1,d')}(\sigma(u-t);1)\times$$

$$\times {}_2F_1[a_1, a_2; b_2; z(u-t)^{-1}]\, du =$$

$$= \Gamma(\delta)(x-t)^{\delta+\lambda-1} \frac{\Gamma(n+b'+1)\Gamma(m+d'+1)}{m!\, n!} \times$$

$$\times \sum_{h=0}^{m+n} \sigma^h\, (\lambda+h)_\delta \sum_{k=0}^{h} \binom{h}{k} \left[\frac{(-m)_{(h-k)}(-n)_k}{\Gamma(k+b'+1)\Gamma((h-k)+d'+1)}\right] \times$$

$$\times {}_3F_2[\lambda+\delta+h, a_1, a_2; b_2, \lambda+h; z] \tag{3.14}$$

provided the other involved parameters are so constrained that each member can exist.

***Corollary* 18.** *Let* z, μ, ν, $\sigma \in \mathbb{C}$ *with* $\min\{\Re(\mu), \Re(\nu), \Re(\sigma)\} > 0$ *and* $|z| < 1$. *Also let* $x \in \mathbb{R}^+$ *and* $\min\{\Re(b'), \Re(d')\} > -1$. *Then the following formula holds true*:

$$\int_0^x t^{\nu-1}(x-t)^{\mu-1}\, [1-\sigma(x-t)]^n\, {}_2F_1[a_1, a_2; b_2; z(x-t)^{-1}]\, dt =$$

$$= x^{\mu+\nu-1}(\mu)_\nu \sum_{h=0}^{n} (-n)_h\, \sigma^h\, x^h {}_3F_2[\mu+\nu, a_1, a_2; b_2, \mu; z] \tag{3.15}$$

provided the other involved parameters are so constrained that each member can exist.

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