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B. Půža (Brno Univ. Technology, Czech Republic),
A. Rontó (Inst. Math. Czech Acad. Sci., Brno, Czech Republic),
M. Rontó (Inst. Math., Univ. Miskolc, Hungary),
N. Shchobak (Brno Univ. Technology, Czech Republic)

ON SOLUTIONS OF NONLINEAR BOUNDARY-VALUE PROBLEMS THE COMPONENTS OF WHICH VANISH AT CERTAIN POINTS

ПРО РОЗВ'ЯЗКИ НЕЛІНІЙНИХ КРАЙОВИХ ЗАДАЧ, КОМПОНЕНТИ ЯКИХ В ДЕЯКИХ ТОЧКАХ ОБЕРТАЮТЬСЯ В НУЛЬ

We show how an appropriate parametrization technique and successive approximations can help to investigate nonlinear boundary-value problems for systems of differential equations under the condition that the components of solutions vanish at some unknown points. The technique can be applied to nonlinearities involving the signs of absolute value and positive or negative parts of functions under various types of boundary conditions.

Показано, як відповідна процедура параметризації та послідовні наближення допомагають досліджувати нелінійні крайові задачі для систем диференціальних рівнянь за умови, що компоненти розв'язків обертаються в нуль у деяких невідомих точках. Ця процедура може бути застосована до нелінійностей, що включають знаки абсолютних величин та додатні або від'ємні частини функцій для різних типів граничних умов.

1. Introduction and problem setting. The question on finding solutions of nonlinear differential equations possessing a prescribed number of zeroes inside the given interval is interesting from many points of view (see, e. g., [1, 3 – 5, 11], and the references therein). This is a rather complicated problem and its investigation is generally based on considerations of purely qualitative character which usually do not provide a way to obtain approximations to the solution in question. Further difficulties arise when the equation is studied under nonlinear boundary conditions.

The aim of this paper is to show that this question can be efficiently treated by further extesions of numerical-analytic techniques based upon successive approximations suggested at first by A. M. Samoilenko [20, 21] for the periodic problem. Based on the schemes with interval divisions developed in [12 – 15, 17], we construct here a suitable version of this approach for finding solutions with a given number of zeroes.

We focus on the system of n nonlinear ordinary differential equations

$$u'(t) = f\big(t, [u(t)]_+, [u(t)]_-\big), \quad t \in [a, b], \tag{1.1}$$

where $[u]_\pm$ for any $u = \operatorname{col}(u_1, \ldots, u_n)$ stands for the vector $\operatorname{col}([u_1]_\pm, \ldots, [u_n]_\pm)$, and $[s]_+ :=$ $:= \max\{s, 0\}$, $[s]_- := \max\{-s, 0\}$ for any real s. System (1.1) will be studied under the nonlinear two-point boundary conditions of the general form

$$g(u(a), u(b)) = d. \tag{1.2}$$

The functions $f : [a, b] \times \Omega \times \Omega \to \mathbb{R}^n$, $g : \Omega \times \Omega \to \mathbb{R}^n$ are assumed to be continuous in their domain of definition, the choice of $\Omega \subset \mathbb{R}^n$ will be concretized later. Since $[u]_+ - [u]_- = u$ and $[u]_+ + [u]_- = |u|$, system (1.1) includes, e. g., Fučík type equations

$$u''(t) = \alpha(t)[u(t)]_+ + \beta(t)[u(t)]_- + q(t, u(t)), \quad t \in [a, b], \tag{1.3}$$

equations of Emden – Fowler type

$$u''(t) = p(t)\,|u(t)|^\lambda\, u(t) + r(t), \quad t \in [a, b], \tag{1.4}$$

and various other systems of the form

$$u'(t) = h(t, u(t), |u(t)|), \quad t \in [a, b]\,.$$

In what follows we are looking for continuously differentable solutions $u = \operatorname{col}(u_1, u_2, \ldots, u_n)$ of (1.1), (1.2) each component of which vanishes at some point from (a, b) and has prescribed signs around it (see Section 2). The numerical-analytic approach [6, 8 – 11, 18] allows one to approximate such solutions of problem (1.1), (1.2) and, moreover, rigorously prove their existence using the results of computation [7, 16].

The form of system (1.1) is motivated, in particular, by equations of type (1.3), (1.4), where the terms of type $[u]_\pm,\ |u|$ bring about additional difficulties for the practical realization of our scheme due to the need of analytic integration of expressions depending on multiple parameters. We shall see that, in the case of solutions of the kind mentioned above, the construction of approximations is simplified and any additional approximation of integrands may not be needed.

2. Solutions with fixed signs on subintervals. For the convenience of notation, we introduce two definitions (cf. [11]).

Definition 2.1. *Let* $\{\sigma_0, \sigma_1\} \subset \{-1, 1\}$ *and* t_1 *be a point from* (a, b). *We say that a function* $u\colon [a, b] \to \mathbb{R}$ *is of type* $(\sigma_0, \sigma_1; t_1)$ *if* $u(t_1) = 0$ *and*

$$\sigma_{k-1} u(t) > 0 \qquad \textit{for} \quad t \in (t_{k-1}, t_k), \quad k = 1, 2,$$

where $t_0 := a,\ t_2 := b$.

Let us suppose that $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ and $t_1, t_2, \ldots, t_n$ are such that

$$a =: t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} := b. \tag{2.1}$$

Definition 2.2. *We say that a vector-function* $u = \operatorname{col}(u_1, u_2, \ldots, u_n)\colon [a, b] \to \mathbb{R}^n$ *is of type* $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *if every* u_k, $k = 1, 2, \ldots, n$, *is of type* $(\sigma_{k0}, \sigma_{k1}; t_k)$.

In what follows, we will look for solutions of (1.1) possessing the last mentioned property with certain $t_1, t_2, \ldots, t_n$. Assumption (2.1) on the ordering of zeroes is no loss of generality because the equations in (1.1) can always be renumbered accordingly.

Before applying the iteration techniques for finding this kind of solutions, it is convenient to simplify the terms involving the positive and negative parts of a function using the information known for its sign. For this purpose, put

$$j_\sigma := \frac{1}{2}(\sigma + 1) \tag{2.2}$$

for any $\sigma \in \{-1, 1\}$ and define the function $\tilde{f}\colon [a, b] \times D \to \mathbb{R}^n$ by setting

$$\begin{aligned} \tilde{f}(t, u_1, \ldots, u_n) := f\big(&t, j_{\sigma_{11}} u_1, \ldots, j_{\sigma_{k-1,1}} u_{k-1}, j_{\sigma_{k0}} u_k, j_{\sigma_{k+1,0}} u_{k+1}, \ldots, j_{\sigma_{n0}} u_n, \\ &-j_{-\sigma_{11}} u_1, \ldots, -j_{-\sigma_{k-1,1}} u_{k-1}, -j_{-\sigma_{k0}} u_k, -j_{-\sigma_{k+1,0}} u_{k+1}, \ldots, -j_{-\sigma_{n0}} u_n\big) \end{aligned} \tag{2.3}$$

for $u = (u_i)_i^n$ from $\Omega,\ t \in [t_{k-1}, t_k],\ k = 1, 2, \ldots, n + 1$.

Lemma 2.1. *Let* $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ *be fixed. Any* $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of* (1.1) *is a solution of the system*

$$u'(t) = \tilde{f}(t, u(t)), \quad t \in [a, b], \tag{2.4}$$

where $\tilde{f}$ *is given by* (2.3)*. Conversely, any* $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of* (2.4) *satisfies* (1.1)*.*

Proof. Let $u = (u_i)_{i=1}^n$ be a solution of (1.1) having type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$. Since (2.1) is assumed on $t_1, t_2, \ldots, t_n$, it follows from the definition that

$$\operatorname{sign} u_i(t) = s_{ik}, \qquad t \in (t_{k-1}, t_k), \quad k = 1, 2, \ldots, n+1, \tag{2.5}$$

where $S = (s_{ik})$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n+1$,

$$S := \begin{pmatrix} \sigma_{10} & \sigma_{11} & \sigma_{11} & \ldots & \sigma_{11} & \sigma_{11} \\ \sigma_{20} & \sigma_{20} & \sigma_{21} & \ldots & \sigma_{21} & \sigma_{21} \\ \sigma_{10} & \sigma_{30} & \sigma_{30} & \ldots & \sigma_{31} & \sigma_{31} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{n-1,0} & \sigma_{n-1,0} & \sigma_{n-1,0} & \ldots & \sigma_{n-1,1} & \sigma_{n-1,1} \\ \sigma_{n0} & \sigma_{n0} & \sigma_{n0} & \ldots & \sigma_{n0} & \sigma_{n1} \end{pmatrix}.$$

By (2.2), we have $j_{\sigma_{ij}} = (\sigma_{ij} + 1)/2$, $-j_{-\sigma_{ij}} = (\sigma_{ij} - 1)/2$, which, together with (2.5), implies that u satisfies (2.4). The converse implication is obvious.

Lemma 2.1 is proved.

Note that, in contrast to (1.1), the expression on the right-hand side of the new system (2.4) does not contain positive or negative parts of a function: instead of $[u_i]_+$ and $[u_i]_-$, one finds there either u_i, $-u_i$ or 0, depending on the subinterval considered.

The construction of $\tilde{f}$ is rather easy and proceeds by changing the relevant terms in (1.1) according to their sign. Namely, all the occurrences of $[u_i(t)]_+$ in (1.1) are replaced by $u_i(t)$ if $t \in [a, t_i]$, $\sigma_{i0} = 1$ or $t \in (t_i, b]$, $\sigma_{i1} = 1$, and by 0 in all remaining cases. Similarly, the term $[u_i(t)]_-$ is replaced by $-u_i(t)$ if $t \in [a, t_i]$, $\sigma_{i0} = -1$ or $t \in (t_i, b]$, $\sigma_{i1} = -1$, and by 0 otherwise. For example, if system (1.1) has the form

$$\begin{aligned} u_1'(t) &= p_{11}(t)[u_1(t)]_+ + p_{12}(t)[u_1(t)]_- + q_1(u_1(t), |u_2(t)|), \\ u_2'(t) &= p_{21}(t)[u_2(t)]_+ + p_{22}(t)[u_2(t)]_- + q_2(u_1(t), u_2(t)), \quad t \in [a, b], \end{aligned} \tag{2.6}$$

and we take, e. g., $\sigma_{10} = 1$, $\sigma_{11} = -1$, $\sigma_{20} = -1$, $\sigma_{21} = 1$, then the corresponding system (2.4) is written as

$$\begin{aligned} u_1'(t) &= p_{11}(t)\, u_1(t) + q_1(u_1(t), -u_2(t)), \\ u_2'(t) &= -p_{22}(t)\, u_2(t) + q_2(u_1(t), u_2(t)) \end{aligned} \tag{2.7}$$

for $t \in [a, t_1]$,

$$\begin{aligned} u_1'(t) &= -p_{12}(t)\, u_1(t) + q_1(u_1(t), -u_2(t)), \\ u_2'(t) &= -p_{22}(t)\, u_2(t) + q_2(u_1(t), u_2(t)) \end{aligned} \tag{2.8}$$

for $t \in [t_1, t_2]$, and

$$u_1'(t) = -p_{12}(t)\, u_1(t) + q_1(u_1(t), u_2(t)),$$
$$u_2'(t) = p_{21}(t)\, u_2(t)_- + q_2(u_1(t), u_2(t)) \tag{2.9}$$

for $t \in [t_2, b]$ (recall that $a < t_1 < t_2 < b$). The assertion of Lemma 2.1 in this case means that, if we restrict our consideration to solutions $u = \mathrm{col}(u_1, u_2)$ of type $[(1, -1; t_1), (-1, 1; t_2)]$ in the sense of Definition 2.2, then the original system (2.6), on the relevant subintervals, can be rewritten equivalently as (2.7)–(2.9).

***Remark* 2.1.** It is not difficult to verify that formula (2.3) for $\tilde{f}$ can be represented alternatively as

$$\tilde{f}(t, u) = f\left(t, \frac{1}{2}(M_k + I)u(t), \frac{1}{2}(M_k - I)u(t)\right), \tag{2.10}$$

where $u = (u_i)_i^n$ is from Ω, $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$, I is the unit matrix, and

$$M_k := \mathrm{diag}\,\big(\sigma_{11}, \sigma_{21}, \ldots, \sigma_{k-1,1}, \sigma_{k0}, \sigma_{k+1,0}, \ldots, \sigma_{n0}\big). \tag{2.11}$$

Equality (2.10) implies, in particular, that possible occurrences of $|u_i|$ in the original system are replaced by the ith component of $M_k u$ in $\tilde{f}$ on $[t_{k-1}, t_k]$.

Using Remark 2.1 in the example above, it is easy to write down system (2.7)–(2.9) on the three intervals directly because, in this case, in view of (2.11), the matrices $M_1 = \mathrm{diag}(\sigma_{10}, \sigma_{20})$, $M_2 = \mathrm{diag}(\sigma_{11}, \sigma_{20})$, $M_3 = \mathrm{diag}(\sigma_{11}, \sigma_{21})$ have the form

$$M_1 = \mathrm{diag}(1, -1), \qquad M_2 = \mathrm{diag}(-1, -1), \qquad M_3 = (-1, 1).$$

Therefore, on $[t_{k-1}, t_k]$, $1 \le k \le 3$, the occurrences of $[u_i]_+$ (resp., $[u_i]_-$) in (2.6) are replaced by $(1/2)[(M_k u)_i + u_i]$ (resp., $(1/2)[(M_k u)_i - u_i]$), and $|u_2|$ by $(M_k u)_2$.

3. Parametrization and auxiliary problems. We fix certain $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset$ $\subset \{-1, 1\}$ and focus on finding solutions of (1.1) which are of type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots$ $\ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ for *some* $t_1,\ t_2, \ldots, t_n$ from (a, b). The values of $t_1,\ t_2, \ldots, t_n$ are *a priori* unknown and have to be determined along with u. Without loss of generality we assume that these points are ordered as indicated in (2.1).

The idea that we will use suggests to replace the boundary-value problem (2.4), (1.2) by a suitable family of "model-type" problems with separated boundary conditions. The construction of these problems is very simple. We "freeze" the values of $u = \mathrm{col}(u_1, u_2, \ldots, u_n)$ at points (2.1) by formally putting

$$u(t_k) = z^{(k)}, \quad k = 0, 1, \ldots, n+1, \tag{3.1}$$

where $z^{(k)} = \mathrm{col}\big(z_1^{(k)}, z_1^{(k)}, \ldots, z_n^{(k)}\big)$, and consider the restrictions of system (2.4) to each of the intervals $[t_0, t_1]$, $[t_1, t_2], \ldots, [t_n, t_{n+1}]$. This leads us to the $n+1$ two-point boundary-value problems on the respective subintervals

$$u'(t) = \tilde{f}(t, u(t)), \quad t \in [t_{k-1}, t_k], \tag{3.2}$$

$$u(t_{k-1}) = z^{(k-1)}, \quad u(t_k) = z^{(k)}, \tag{3.3}$$

where $k = 1, 2, \ldots, n+1$, and $\tilde{f}$ is given by (2.3). We fix certain nonempty bounded sets

$$\Omega_k \subset \mathbb{R}^n, \quad k = 0, 1, \ldots, n+1, \tag{3.4}$$

and treat the vectors $z^{(j)}$ appearing in (3.1), (3.3), and (3.4) as parameters with values in Ω_j, $j = 0, 1, \ldots, n+1$. Using the family of problems (3.2), (3.3), we will study $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ solutions $u = \operatorname{col}(u_1, u_2, \ldots, u_n)$ of problem (1.1), (1.2) whose values at the unknown points (2.1) lie in the corresponding sets (3.4), i. e., such that

$$u(t_k) \in \Omega_k, \quad k = 0, 1, 2, \ldots, n+1. \tag{3.5}$$

After the simplification of the original system (1.1) using the sign properties of solutions, we will impose restrictions needed to apply our method directly to the transformed system (2.4). Conditions on $\tilde{f}$ will be assumed over certain sets which are a somewhat wider than sets (3.4) fixed above.

Given sets (3.4), for any $k = 1, 2, \ldots, n+1$, introduce the sets

$$\Omega_{k-1,k} := \big\{(1-\theta)\xi + \theta\eta : \xi \in \Omega_{k-1},\ \eta \in \Omega_k,\ \theta \in [0,1]\big\}. \tag{3.6}$$

It is clear that $\Omega_{k-1,k}$ is constituted by all possible straight line segments joining points of Ω_{k-1} with the points of Ω_k. Further on, we shall need the componentwise $\varrho^{(k)}$-neighbourhoods of $\Omega_{k-1,k}$, $k = 1, \ldots, n+1$:

$$\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}), \quad k = 1, 2, \ldots, n+1, \tag{3.7}$$

where

$$\mathcal{O}_\varrho(\Omega) := \bigcup_{\xi \in \Omega} \mathcal{O}_\varrho(\xi) \tag{3.8}$$

and $\mathcal{O}_\varrho(\xi) := \big\{\nu \in \mathbb{R}^n : |\nu - \xi| \le \varrho\big\}$ for any $\Omega \subset \mathbb{R}^n$, $\varrho \in \mathbb{R}^n_+$, $\xi \in \Omega$. The values of $\varrho^{(k)}$, $k = 1, 2, \ldots, n+1$, to be used in (3.7) will be chosen later. The conditions to be formulated in the sequel (see Section 4) are assumed over sets (3.7) with the respect to the space variables.

4. Assumptions. To study solutions of the auxiliary problems (3.2), (3.3) with $z^{(j)} \in \Omega_j$, $j = 0, 1, \ldots, n+1$, we use suitable parametrised successive approximations constructed analytically on the subintervals $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$. Since we are looking for solutions of type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ with the given $\{\sigma_{i0}, \sigma_{i1} : i = 1, 2, \ldots, n\} \subset \{-1, 1\}$ and some unknown $t_1, \ldots, t_n$, we assume that the set Ω_0 is chosen so that

$$\Pi_i \Omega_0 \subset \sigma_{i0}\mathbb{R}_+, \quad i = 1, 2, \ldots, n, \tag{4.1}$$

where $\Pi_i\Omega := \{s_i : (s_1, \ldots, s_i, \ldots, s_n) \in \Omega \text{ for some } s_1, \ldots, s_n\}$.

***Remark* 4.1.** Due to the nature of the problem under consideration, in addition to (4.1), it is natural to suppose that the sets Ω_0, $\Omega_1, \ldots, \Omega_{n+1}$ have the properties

$$\Pi_i \Omega_j \subset v_{ji}\mathbb{R}_+, \qquad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, n+1, \tag{4.2}$$

where $v_k = (v_{ki})_{i=1}^n$ are defined as

$$v_0 := (\sigma_{10}, \sigma_{20}, \ldots, \sigma_{n0}),$$

$$v_k = (\sigma_{11}, \sigma_{21}, \ldots, \sigma_{k-1,1}, 0, \sigma_{k+1,0}, \sigma_{n0}), \quad k = 1, 2, \ldots, n,$$

$$v_{n+1} := (\sigma_{11}, \sigma_{21}, \ldots, \sigma_{n1}).$$

Although relations (4.2) are useful because they exclude from consideration sets which cannot contain the values of solutions with type $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$, it is enough to assume in the sequel condition (4.1) fixing the signs of the solution at the initial subinterval.

Two assumptions on the function $\tilde{f}$ appearing in (2.4), (3.2) will be needed.

Assume that there exist nonnegative vectors $\varrho^{(1)},\ \varrho^{(2)},\ldots,\ \varrho^{(n+1)}$ such that

$$\varrho^{(k)} \geq \frac{t_k - t_{k-1}}{4}\delta_{[t_{k-1},t_k],\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})}(\tilde{f}) \tag{4.3}$$

for all $k = 1, 2, \ldots, n+1$, where

$$\delta_{[\alpha,\beta],\Omega}(\tilde{f}) := \max_{(t,u)\in[\alpha,\beta]\times\Omega} \tilde{f}(t,u) - \min_{(t,u)\in[\alpha,\beta]\times\Omega} \tilde{f}(t,u) \tag{4.4}$$

for $a < \alpha < \beta < b$ and a closed $\Omega \subset \mathbb{R}^n$.

Fix certain $\varrho^{(1)},\ \varrho^{(2)},\ldots,\ \varrho^{(n+1)}$ for which (4.3) holds, consider the sets $\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}),\ k = 1, 2, \ldots, n+1$, and suppose that, with some nonnegative matrices $K_k,\ k = 1, 2, \ldots, n+1$, the function $\tilde{f}$ satisfies the Lipschitz condition

$$\left|\tilde{f}(t, y_1) - \tilde{f}(t, y_2)\right| \leq K_k\, |y_1 - y_2| \tag{4.5}$$

for $t \in [t_{k-1}, t_k],\ \{y_1, y_2\} \subset \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}),\ k = 1, 2, \ldots, n+1$. Finally, assume that

$$r(K_k) < \frac{10}{3(t_k - t_{k-1})} \tag{4.6}$$

for all $k = 1, 2, \ldots, n+1$.

Remark **4.2.** When looking for solutions vanishing at certain points (which is the case, in particular, for the class of solutions defined in Section 2), the direct verification of condition (4.6) is impossible because the values of $t_1,\ t_2, \ldots, t_n$ are unknown. Obviously, the fulfilment of (4.6) is guaranteed if

$$\max_{1\leq k\leq n+1} r(K_k) < \frac{10}{3(b-a)}. \tag{4.7}$$

It does make sense, however, to keep inequalities (4.6) because they may lead one to conditions considerably weaker than (4.7) if some estimates for $t_1, t_2, \ldots, t_n$ are available (see Section 6).

Remark **4.3.** In order to verify condition (4.3) on $\varrho^{(0)}, \ldots, \varrho^{(n+1)}$, it is needed to compute maximal and minimal values of the function $\tilde{f}$ over $\varrho^{(k)}$-neighbourhoods of sets $\Omega_{k-1,k},\ k = 1, 2, \ldots, n+1$, constructed according to (3.6). One may use computer software for this purpose. It is convenient to specify suitable sets

$$\Omega^{(k)} \supset \Omega_{k-1,k}, \quad k = 1, 2, \ldots, n+1, \tag{4.8}$$

of simpler structure (e. g., parallelepipeds: if $\Omega^{(k)}$ is a parallelepiped, then, by (3.8), the set $\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$ is a parallelepiped as well) and use the inequality $\delta_{[\alpha,\beta],\tilde{\Omega}}(\tilde{f}) \geq \delta_{[\alpha,\beta],\Omega}(\tilde{f})$ for any $\tilde{\Omega} \supset \Omega$, which is an immediate consequence of (4.4). Then the fulfilment of (4.3) is guaranteed if

$$\varrho^{(k)} \geq \frac{t_k - t_{k-1}}{4}\delta_{[t_{k-1},t_k],\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})}(\tilde{f}) \tag{4.9}$$

for $k = 1, 2, \ldots, n+1$. The same observation concerns the Lipschitz condition (4.5), which may be easier to check on the set $\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$ instead of $\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}),\ k = 1, 2, \ldots, n+1$.

5. Successive approximations and determining equations. As we have seen above, the question on $\big[(\sigma_{10}, \sigma_{11}; t_1), (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ solutions of the boundary-value problem (1.1), (1.2) reduces to the same problem (1.2) for equation (2.4), where the function $\tilde{f}$ is constructed according to (2.3). To treat problem (2.4), (1.2), we can use the approach of [15, 17] using properties of the auxiliary problems (3.2), (3.3). From now on till the end of the paper we assume that conditions (4.3), (4.5), and (4.6) are satisfied.

Let us define the parametrized recurrence sequences of functions $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$, $m = 0, 1, \ldots,$ by putting

$$u_0^{(k)}(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) :=$$

$$:= \left(1 - \frac{t - t_{k-1}}{t_k - t_{k-1}}\right) z^{(k-1)} + \frac{t - t_{k-1}}{t_k - t_{k-1}} z^{(k)}, \tag{5.1}$$

$$u_m^{(k)}(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) :=$$

$$:= u_0^{(k)}(t, z^{(k-1)}, z^{(k)}) + \int\limits_{t_{k-1}}^{t} \tilde{f}\big(s, u_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big)ds -$$

$$- \frac{t - t_{k-1}}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big)ds \tag{5.2}$$

for all $m = 1, 2, \ldots,$ $z^{(0)} \in \Omega_0,$ $z^{(k)} \in \Omega_k,$ $t \in [t_{k-1}, t_k],$ $k = 1, 2, \ldots, n+1$. We recall that $t_0 = a$, $t_{n+1} = b$, while the intermediate time instants $t_1, \ldots, t_n$ are treated as unknown parameters.

It is clear that every function $u_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)$, $m = 0, 1, \ldots,$ satisfies conditions (3.3) independently of the choice of $z^{(k-1)}$ and $z^{(k)}$:

$$u_m^{(k)}\big(t_{k-1}, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = z^{(k-1)}, \qquad u_m^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = z^{(k)}. \tag{5.3}$$

The sequences given by (5.1), (5.2) are helpful for the investigation of the auxiliary problems (3.2), (3.3) and, ultimately, of the given problem (1.1), (1.2).

Theorem 5.1. *Assume* (4.3), (4.5), *and* (4.6). *Then, for any fixed* $z^{(k)} \in \Omega_k,$ $k = 0, 1, \ldots, n+1$:

1. *Functions* (5.2) *are continuously differentiable on* $t \in [t_{k-1}, t_k],$ $k = 1, \ldots, n+1,$ *and the inclusion*

$$\Big\{u_m^{(k)}(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) : t \in [t_{k-1}, t_k]\Big\} \subset \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}) \tag{5.4}$$

holds.

2. *The limit*

$$\lim_{m\to\infty} u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) =: u_\infty^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$$

exists uniformly in $t \in [t_{k-1}, t_k],$ $k = 1, 2, \ldots, n+1$.

3. *The limit functions satisfy the separated two-point boundary conditions*

$$\begin{gathered} u_\infty^{(k)}\big(t_{k-1}, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = z^{(k-1)}, \\ u_\infty^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = z^{(k)}. \end{gathered} \tag{5.5}$$

4. *The function* $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ *is the unique continuously differentiable solution of the integral equation*

$$u(t) = u_0^{(k)}(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k) +$$

$$+\int\limits_{t_{k-1}}^{t} \tilde{f}(s, u(s))ds - \frac{t - t_{k-1}}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \tilde{f}(s, u(s))ds, \quad t \in [t_{k-1}, t_k], \tag{5.6}$$

with values in $\mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})$.

5. *For any* $m \geq 0$, *the following estimate holds:*

$$\big|u_\infty^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) - u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)\big| \leq$$

$$\leq \frac{5}{9}\alpha_1(t, t_{k-1}, t_k)\, Q_k^m \,(I - Q_k)^{-1}\, \delta_{[t_{k-1}, t_k], \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k})}(f),$$

where $Q_k := (3/10)(t_k - t_{k-1})K_k$,

$$\alpha_1(t, t_{k-1}, t_k) := 2\,(t - t_{k-1})\left(1 - \frac{t - t_{k-1}}{t_k - t_{k-1}}\right)$$

for $t \in [t_{k-1}, t_k]$.

It follows from (5.6) that the function $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big),\ k = 1, 2, \ldots, n+1$, is the unique solution of the Cauchy problem for the system

$$u'(t) = \tilde{f}(t, u(t)) + \frac{1}{t_k - t_{k-1}}\Delta^{(k)}(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k), \tag{5.7}$$

$$u(t_{k-1}) = z^{(k-1)}, \tag{5.8}$$

where $\Delta^{(k)} : \Omega_{k-1} \times \Omega_k \times (a, b)^2 \to \mathbb{R}^n,\ k = 1, \ldots, n+1$, is defined by the formula

$$\Delta^{(k)}(\xi, \eta, s_0, s_1) := \eta - \xi - \int\limits_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_\infty^{(k)}(s, \xi, \eta, s_0, s_1)\big)ds \tag{5.9}$$

for all $\xi \in \Omega_{k-1},\ \eta \in \Omega_k$, and $\{s_0, s_1\} \subset (a, b)$.

The proof proceeds by analogy to [17] (Theorem 1) and [15] (Theorem 5.1). The starting point is to establish inclusion (5.4).

It is natural to expect that the limit functions $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big),\ k = 1, 2, \ldots, n+1$, of the iterations (5.2) on the subintervals $t \in [t_{k-1}, t_k]$ will help one to formulate criteria of solvability of the original problem (1.1), (1.2). It turns out that it is the functions

$$\Delta^{(k)} : \Omega_{k-1} \times \Omega_k \times (a, b)^2 \to \mathbb{R}^n, \quad k = 1, 2, \ldots, n+1, \tag{5.10}$$

defined according to equalities (5.9) that provide such conclusions. Indeed, Theorem 5.1 guarantees that under the conditions assumed, the functions $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) : [t_{k-1}, t_k] \to \mathbb{R}^n$,

$k = 1, 2, \ldots, n+1$, are well defined for all $(z^{(k-1)}, z^{(k)}) \in \Omega_{k-1} \times \Omega_k$, $(t_{k-1}, t_k) \in (a, b)^2$. Therefore, by putting

$$u_\infty\bigl(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, t_2, \ldots, t_n\bigr) := u_\infty^{(k)}\bigl(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\bigr) \tag{5.11}$$

for $t \in [t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$, we obtain a function $u_\infty(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots$ $\ldots, t_n) \colon [a, b] \to \mathbb{R}^n$. This function is obviously continuous at the points t_k, $k = 1, 2, \ldots, n$, because, by (5.5),

$$u_\infty^{(k)}\bigl(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\bigr) = u_\infty^{(k+1)}\bigl(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\bigr).$$

Along with (3.2), consider the equations with constant forcing terms

$$u'(t) = \tilde{f}(t, u(t)) + \frac{1}{t_k - t_{k-1}} \mu^{(k)}, \quad t \in [t_{k-1}, t_k], \tag{5.12}$$

under the initial conditions

$$u(t_{k-1}) = z^{(k-1)}, \tag{5.13}$$

where $\mu^{(k)} = \operatorname{col}(\mu_1^{(k)}, \mu_2^{(k)}, \ldots, \mu_n^{(k)})$, $k = 1, 2, \ldots, n+1$, are control parameters. Then, similarly to [19] (Theorem 2), one obtains the following theorem.

Theorem 5.2. *Assume* (4.3), (4.5), *and* (4.6). *Let* $z^{(j)} \in \Omega_j$, $j = 0, 1, \ldots, n+1$, *be fixed. Then, for the solutions of the Cauchy problems* (5.12), (5.13) *to have the properties*

$$u(t_k) = z^{(k)}, \quad k = 1, 2, \ldots, n+1, \tag{5.14}$$

it is neccessary and sufficient that $\mu^{(k)}$ *have the form*

$$\mu^{(k)} = \Delta^{(k)}\bigl(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\bigr), \quad k = 1, 2, \ldots, n+1, \tag{5.15}$$

in which case the solution of (5.12), (5.13) *coincides with* $u_\infty^{(k)}\bigl(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\bigr)$ *for any* $k = 1, 2, \ldots, n+1$.

The next statement establishes the relation of function (5.11) to solutions of the original problem (1.1), (1.2) in the terms of zeroes of functions (5.10). Recall that Ω_0 is chosen so that (4.1) holds.

Theorem 5.3. *Let* (4.3), (4.5), *and* (4.6) *hold. Then the function*

$$u_\infty(\cdot, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n) \colon [a, b] \to \mathbb{R}^n$$

is a continuously differentiable solution of the boundary-value problem (1.1), (1.2) *if and only if the vectors* $z^{(k)}$, $k = 0, 1, 2, \ldots, n+1$, *and the points* $t_1, \ldots, t_n$ *satisfy the system of* $n(n+2)$ *numerical determining equations*

$$\Delta^{(k)}\bigl(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\bigr) = 0, \quad k = 1, 2, \ldots, n+1, \tag{5.16}$$

$$g\bigl(u_\infty^{(1)}(a, z^{(0)}, z^{(1)}, a, t_1), u_\infty^{(n+1)}(b, z^{(n)}, z^{(n+1)}, t_n, b)\bigr) = d. \tag{5.17}$$

Furthermore, this solution has type $\bigl[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\bigr]$.

Proof. The statement is proved similarly to [19] (Theorem 3). We use Lemma 2.1 and take into account the choice of the domain Ω_0 according to (4.1), which, by virtue of the unique solvability of the Cauchy problems (5.7), (5.8), excludes the existence of solutions not possessing the prescribed property $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$.

Theorem 5.3 is proved.

Finally, the following analogue of [19] (Theorem 4) shows that the determining equations (5.16), (5.17) detect all possible solutions of the problem (1.1), (1.2) with graphs lying in the domains specified.

Theorem 5.4. *Let* (4.3), (4.5), *and* (4.6) *hold. If there exist some* $t_1, \ldots, t_n$ *from* (a, b) *and* $z^{(j)} \in \Omega_j,\ j = 0, 1, \ldots, n+1,$ *that satisfy the determining equations* (5.16), (5.17), *then the function*

$$u^*(t) = u_\infty\big(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n\big), \quad t \in [a, b], \tag{5.18}$$

is a $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ *solution of the boundary-value problem* (1.1), (1.2). *Conversely, if problem* (1.1), (1.2) *has a solution* $u^*(\cdot)$ *of type* $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots$ $\ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$, *which, in addition, satisfies the conditions*

$$u^*(t_j) \in \Omega_j, \quad j = 0, 1, \ldots, n+1,$$

$$\big\{u^*(t) : t \in [t_{k-1}, t_k]\big\} \subset \mathcal{O}_{\varrho^{(k)}}(\Omega_{k-1,k}), \quad k = 1, 2, \ldots, n+1,$$

then the system of determining equations (5.16), (5.17) *is satisfied with the same* $t_1, \ldots, t_n$, *and*

$$z^{(j)} := u^*(t_j), \quad j = 0, 1, \ldots, n+1.$$

Moreover, the solution $u^*(\cdot)$ *necessarily has form* (5.18) *with these values of parameters.*

***Remark* 5.1.** In the case of $\big[(\sigma_{10}, \sigma_{11}; t_1),\ (\sigma_{20}, \sigma_{21}; t_2), \ldots, (\sigma_{n0}, \sigma_{n1}; t_n)\big]$ solutions, the parameters $z^{(1)}, z^{(2)}, \ldots, z^{(n)}$ in the auxiliary two-point problems (3.2), (3.3) have the form

$$z^{(k)} = \operatorname{col}\big(z_1^{(k)}, \ldots, z_{k-1}^{(k)}, 0, z_{k+1}^{(k)}, \ldots, z_n^{(k)}\big), \quad k = 1, 2, \ldots, n, \tag{5.19}$$

and, therefore, system (5.16), (5.17) involves $n(n+1)$ variables.

6. Computation of approximate solutions. Although Theorem 5.4 describes theoretically all the solutions of problem (2.4), (1.2) with graphs contained in the given region, its direct application is difficult because the form of the limit functions of sequences and (5.1), (5.2) is usually unknown and, as a consequence, the determining equations (5.16), (5.17) can rarely be written down explicitly. The complication can be overcome in a customary way (see, e. g., [6, 12] and references therein) if we replace in (5.11) the unknown limit $u_\infty^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ by an iteration $u_m^{(k)}(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k),\ k = 1, 2, \ldots, n+1,$ of form (5.2) for a fixed m. In this way, we obtain the function

$$u_m\big(t, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n\big) := u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) \tag{6.1}$$

for $t \in [t_{k-1}, t_k],\ k = 1, 2, \ldots, n+1$. We see that (6.1) is an approximate version of the unknown function (5.11). Its values can be found explicitly for all values of the parameters involved. Considering function (6.1), we arrive in a natural way to the mth approximate system of determining equations

$$\Delta_m^{(k)}\big(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = 0, \quad k = 1, 2, \ldots, n+1, \tag{6.2}$$

$$g\big(u_m^{(1)}\big(a, z^{(0)}, z^{(1)}, a, t_1\big), u_m^{(n+1)}(b, z^{(n)}, z^{(n+1)}, t_n, b)\big) = d, \tag{6.3}$$

where, by a direct analogy to (5.9), the functions $\Delta_m^{(k)} : \Omega_{k-1} \times \Omega_k \times (a,b)^2 \to \mathbb{R}^n$, $k = 1, \ldots, n+1$, are defined as

$$\Delta_m^{(k)}(\xi, \eta, s_0, s_1) := \eta - \xi - \int\limits_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_m^{(k)}(s, \xi, \eta, s_0, s_1)\big) ds \tag{6.4}$$

for $\xi \in \Omega_{k-1}$, $\eta \in \Omega_k$, and $\{s_0, s_1\} \subset (a,b)$. Note that, unlike system (5.16), (5.17), the mth approximate system (6.2), (6.3) contains only terms involving the functions $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$, $k = 1, 2, \ldots, n+1$, which are computable explicitly.

The approximate solutions of the original problem are obtained as usual (see, e.g., [6, 12]) by substituting into (6.1) roots of the corresponding approximate determining system (6.2), (6.3). The approximations, according to the approach described here, are constructed by "gluing" together the curves obtained on every single interval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n+1$. This gluing is smooth.

Lemma 6.1. *If* $z^{(k)} \in \Omega_k$, $k = 0, 1, 2, \ldots, n+1$, *satisfy equations* (6.2) *for a certain* m, *then the corresponding function* (6.1) *is continuosly differentiable on* $[a,b]$.

Proof. Fix $z^{(j)}$, $j = 0, 1, \ldots, n+1$, put $v := u_m\big(\cdot, z^{(0)}, z^{(1)}, z^{(2)}, \ldots, z^{(n+1)}, t_1, \ldots, t_n\big)$, and consider the values of v around t_k for a fixed $k = 1, 2, \ldots, n$. By (6.1), it is enough to check only $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ and $u_m^{(k+1)}\big(\cdot, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big)$.

Indeed, it follows immediately from (5.2) that

$$\begin{aligned} v'(t_k-) = \tilde{f}\big(t_k, u_{m-1}^{(k)}(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big) + \\ + \frac{1}{t_k - t_{k-1}} \Delta_m^{(k)}\big(z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} v'(t_k+) = \tilde{f}\big(t_k, u_{m-1}^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1})\big) + \\ + \frac{1}{t_{k+1} - t_k} \Delta_m^{(k+1)}\big(z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big). \end{aligned} \tag{6.6}$$

Since $z^{(j)}$, $j = 0, 1, \ldots, n+1$, are supposed to satisfy (6.2), equalities (6.5), (6.6) imply that

$$\begin{aligned} v'(t_k-) &= \tilde{f}\big(t_k, x_{m-1}^{(k)}(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k)\big), \\ v'(t_k+) &= \tilde{f}\big(t_k, x_{m-1}^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1})\big). \end{aligned} \tag{6.7}$$

However, in view of (5.3), we have

$$u_{m-1}^{(k)}\big(t_k, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) = u_{m-1}^{(k+1)}\big(t_k, z^{(k)}, z^{(k+1)}, t_k, t_{k+1}\big) = z^{(k)},$$

which, together with (6.7), yields $v'(t_k-) = v'(t_k+)$.

Lemma 6.1 is proved.

The solvability of the determining system (5.16), (5.17) can be analyzed similarly to [7, 16] using topological degree methods [2] by studying some its approximate versions (6.2), (6.3) (this subject is not treated here).

A special note should be made on the verification of the assumptions of Section 4. Namely, both relations (4.3), which should be satisfied by $\varrho^{(1)}, \varrho^{(2)}, \ldots, \varrho^{(n+1)}$, and inequalities (4.6) for $r(K_j)$, $j = 1, \ldots, n+1$, depend upon the unknown $t_1, t_2, \ldots, t_n$. Although one can replace the subintervals by the entire $[a, b]$ (cf. (4.7)), this would lead to more restrictive conditions. Another, better opportunity is to use preliminary results of computation according to the scheme described above.

Indeed, it is always expedient to start computations directly *before* checking conditions (4.3), (4.6) because, by doing so, we may obtain a preliminary information on the space localization of solutions and, as a consequence, a useful hint how to choose the regions where the conditions should be verified. This concerns both the choice of the sets Ω_k, $k = 0, 1, \ldots, n+1$, with respect to the space variables and intervals containing zeroes of solutions.

Suppose that we start computation directly and try to solve approximate determining equations. If the computation shows reasonable, in some sense, results and we get certain approximate values $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n$ of $t_1, t_2, \ldots, t_n$, these values are natural to be used to set restrictions of the form

$$T_k^- \leq t_k \leq T_k^+, \quad k = 1, 2, \ldots, n, \tag{6.8}$$

by choosing appropriately the bounds T_k^-, T_k^+, $k = 1, 2, \ldots, n$. Perhaps, the simplest choice is to put

$$T_k^- := \max\left\{a, \hat{t}_k - \frac{b-a}{n+1}\right\},$$

$$T_k^+ := \min\left\{\hat{t}_k + \frac{b-a}{n+1}, b\right\}$$

for $k = 1, 2, \ldots, n$, however, finer estimates may be available in concrete situations. Knowing estimates of form (6.8), instead of (4.3), we can verify the relations

$$\varrho^{(k)} \geq \frac{T_k^+ - T_{k-1}^-}{4} \delta_{[T_{k-1}^-, T_k^+], \mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})}(\tilde{f}), \tag{6.9}$$

where $T_0^- = T_0^+ = a$, $T_{n+1}^- = T_{n+1}^+ = b$ and $\Omega^{(k)}$, $k = 1, \ldots, n+1$, are suitably chosen sets satisfying (4.8). Similarly, instead of (4.6), we will check the condition

$$r(K_k) < \frac{10}{3\left(T_k^+ - T_{k-1}^-\right)}, \tag{6.10}$$

where K_k is the Lipschitz matrix for the restriction of $\tilde{f}$ to $[t_{k-1}, t_k] \times \mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$, $k = 1, 2, \ldots, n+$ $+ 1$. Condition (6.10) is, of course, preferable to (4.7).

Assuming (6.8), we formally make the problem more difficult since $t_1, t_2, \ldots, t_n$ should satisfy additional inequalities and cannot be arbitrary any more. However, with a reasonable choice of bounds based on the results of computation, inequalities (6.8), in fact, say only that we restrict ourselves to looking for the unknown time instants in the regions where we have reasons to believe they are.

7. An illustrative example. We demonstrate the approach described above on a model example of the three-dimensional system

$$\begin{aligned} u_1'(t) &= u_2(t)u_3(t) - t^2 + \frac{67}{10}t - \frac{387}{100}, \\ u_2'(t) &= |u_3(t)|\, u_2(t) + q_2(t), \\ u_3'(t) &= |u_1(t)| + q_3(t) \end{aligned} \tag{7.1}$$

for $t \in [0,1]$ with

$$q_2(t) := \begin{cases} t^2 - \dfrac{6}{5}t + \dfrac{33}{25} & \text{if} \quad t \in [0, t_3], \\ -t^2 + \dfrac{6}{5}t + \dfrac{17}{25} & \text{if} \quad t \in [t_3, 1], \end{cases} \tag{7.2}$$

and

$$q_3(t) := \begin{cases} -\dfrac{11}{4}t^2 + \dfrac{71}{20}t + \dfrac{2}{5} & \text{if} \quad t \in [0, t_1], \\ \dfrac{11}{4}t^2 - \dfrac{71}{20}t + \dfrac{8}{5} & \text{if} \quad t \in [t_1, 1], \end{cases} \tag{7.3}$$

where t_1 and t_3, $t_1 < t_3$, are unknown points from the interval $(0,1)$. System (7.1) will be considered under the two-point nonlinear boundary conditions

$$u_1^2(0) - u_2^2(1) = 0, \qquad u_2(0)\, u_3(1) = -\frac{2}{25}, \qquad u_1(0) - u_3(1) = \frac{2}{5}. \tag{7.4}$$

Let us set the problem on finding $\big[(1,-1;t_1), (-1,1;t_2), (-1,1;t_3)\big]$ solutions of (7.1), (7.4), where t_2 is a point lying between t_1 and t_3. The values of time instants t_1, t_2, and t_3, where the sign changes of the respective components of u occur, are to be determined.

It can be verified directly by computation that, for $t_1 = 1/5$, $t_2 = 2/5$, $t_3 = 4/5$, the function $u^* = (u_i^*)_{i=1}^3$ with the components

$$u_1^*(t) = \frac{11}{4}t^2 - \frac{71}{20}t + \frac{3}{5}, \qquad u_2^*(t) = t - \frac{2}{5}, \qquad u_3^*(t) = t - \frac{4}{5} \tag{7.5}$$

is a solution of the boundary-value problem (7.1), (7.4). This solution, as is easy to see, has type $\big[(1,-1;1/5),\ (-1,1;2/5), (-1,1;4/5)\big]$ in the sense of Definition 2.2.

Let us use the approach described above. It is clear that (7.1) is a particular case of (1.1) with $a = 0$, $b = 1$, $n = 3$, and f of the form

$$f(t, x_1, x_2, x_3, y_1, y_2, y_3) := \begin{pmatrix} (x_2 - y_2)(x_3 - y_3) - t^2 + \dfrac{67}{10}t - \dfrac{387}{100} \\ (x_3 + y_3)(x_2 - y_2) + q_2(t) \\ x_1 + y_2 + q_3(t) \end{pmatrix}, \tag{7.6}$$

and, hence, the preceding argument is applicable. This explicit form (7.6) of f is, however, not needed for writing down the corresponding system (2.4) because the function $\tilde f = (\tilde f_i)_{i=1}^3$ determining (2.4) can be constructed as in Remark 2.1 by using matrices (2.11):

$$\begin{aligned} M_1 &= \operatorname{diag}(\sigma_{10}, \sigma_{20}, \sigma_{30}), \qquad M_2 = \operatorname{diag}(\sigma_{11}, \sigma_{20}, \sigma_{30}), \\ M_3 &= \operatorname{diag}(\sigma_{11}, \sigma_{21}, \sigma_{30}), \qquad M_4 = \operatorname{diag}(\sigma_{11}, \sigma_{21}, \sigma_{31}). \end{aligned} \tag{7.7}$$

Since, in our case, $\sigma_{10} = 1,\ \sigma_{11} = -1,\ \sigma_{20} = -1,\ \sigma_{21} = 1,\ \sigma_{30} = -1,\ \sigma_{31} = 1$, equalities (7.7) yield

$$M_1 = \operatorname{diag}(1, -1, -1), \qquad M_2 = \operatorname{diag}(-1, -1, -1),$$

$$M_3 = \operatorname{diag}(-1, 1, -1), \qquad M_4 = \operatorname{diag}(-1, 1, 1).$$

Then $|u_i|,\ i = 1, 3$, on the kth interval $[t_{k-1}, t_k],\ 1 \le k \le 4$, where $t_0 = 0$ and $t_4 = 1$, should be replaced by the ith component of $M_k u$. By doing so, we obtain

$$\tilde{f}_1(t, u_1, u_2, u_3) = u_2 u_3 - t^2 + \frac{67}{10} t - \frac{387}{100} \tag{7.8}$$

for all $t \in [0, 1]$, while $\tilde{f}_2,\ \tilde{f}_3$ on the relevant subintervals are defined as follows:

$$\begin{aligned} \tilde{f}_2(t, u_1, u_2, u_3) &= -u_2 u_3 + t^2 - \frac{6}{5} t + \frac{33}{25}, \\ \tilde{f}_3(t, u_1, u_2, u_3) &= u_1 - \frac{11}{4} t^2 + \frac{71}{20} t + \frac{2}{5} \end{aligned} \tag{7.9}$$

for $t \in [0, t_1]$,

$$\begin{aligned} \tilde{f}_2(t, u_1, u_2, u_3) &= -u_2 u_3 + t^2 - \frac{6}{5} t + \frac{33}{25}, \\ \tilde{f}_3(t, u_1, u_2, u_3) &= -u_1 + \frac{11}{4} t^2 - \frac{71}{20} t + \frac{8}{5} \end{aligned} \tag{7.10}$$

for $t \in [t_1, t_3]$ (the equations have the same form on $[t_1, t_2]$ and $[t_2, t_3]$), and

$$\begin{aligned} \tilde{f}_2(t, u_1, u_2, u_3) &= u_2 u_3 - t^2 + \frac{6}{5} t + \frac{17}{25}, \\ \tilde{f}_3(t, u_1, u_2, u_3) &= -u_1 + \frac{11}{4} t^2 - \frac{71}{20} t + \frac{8}{5} \end{aligned} \tag{7.11}$$

for $t \in [t_3, 1]$. The system (2.4) corresponding to (7.1) thus has the form

$$u_i'(t) = \tilde{f}_i(t, u_1(t), u_2(t), u_3(t)), \qquad i = 1, 2, 3, \quad t \in [t_{k-1}, t_k], \quad \le 1 \le k \le 4, \tag{7.12}$$

with $(\tilde{f}_i)_{i=1}^3$ given by the respective equalities (7.8)–(7.11), and we pass from (7.1), (7.4) to problem (7.12), (7.4).

In order to apply the techniques described above, we need to choose suitable domains and verify the conditions. Let us choose the sets $\Omega_0,\ \Omega_1, \ldots, \Omega_4$ in (3.4) as follows:

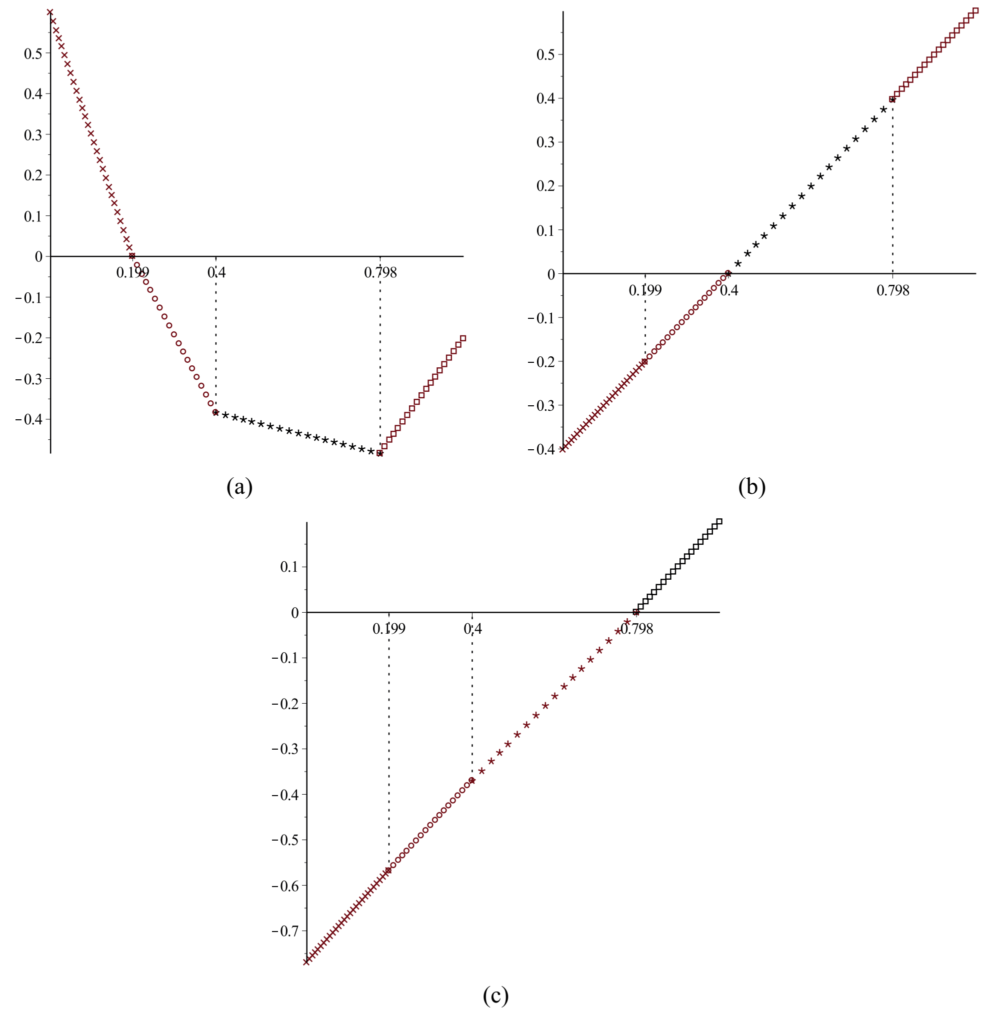


Fig. 1. Zeroth approximation: first component (*a*), second component (*b*), and third component (*c*).

$$
\begin{aligned}
&\Omega_0 = \big\{(u_1, u_2, u_3) : 0.5 \le u_1 \le 0.7,\ -0.6 \le u_2 \le -0.3,\ -0.95 \le u_3 \le -0.6\big\},\\
&\Omega_1 = \big\{(u_1, u_2, u_3)\ :\ -0.1 \le u_1 \le 0.1,\ -0.3 \le u_2 \le -0.1,\ -0.75 \le u_3 \le -0.5\big\},\\
&\Omega_2 = \big\{(u_1, u_2, u_3) : -0.5 \le u_1 \le -0.25,\ -0.1 \le u_2 \le 0.1,\ -0.5 \le u_3 \le -0.3\big\},\\
&\Omega_3 = \big\{(u_1, u_2, u_3) : -0.55 \le u_1 \le -0.3,\ 0.3 \le u_2 \le 0.5,\ -0.1 \le u_3 \le 0.1\big\},\\
&\Omega_4 = \big\{(u_1, u_2, u_3) : -0.3 \le u_1 \le -0.1,\ 0.5 \le u_2 \le 0.7,\ 0.1 \le u_3 \le 0.3\big\}.
\end{aligned}
\tag{7.13}
$$

This choice is motivated by the fact that the zeroth approximate determining system (i.e., (6.2), (6.3) with $m = 0$) has roots lying in these sets, see the second column in Table 1. Figures 1 (a)–(c) present the graph of the zeroth approximation $U_0 = (U_{0i})_{i=1}^3$. Recall that, in order to obtain it, only functions (5.1) are used, and no iteration is yet carried out. We see that this piecewise linear function provides quite reasonable approximate values of the parameters (in particular, of the time instants

Table 1. Exact values of parameters for solution (7.5) and their computed approximations

u^*		$m=0$	$m=1$	$m=2$	$m=3$
$z_1^{(0)}$	0.6	0.5987479750	0.5999603161	0.6000012161	0.5999999745
$z_2^{(0)}$	–0.4	–0.4025198245	–0.4000793836	–0.3999975678	–0.4000000510
$z_3^{(0)}$	–0.8	–0.7704273551	–0.8000737079	–0.7999810861	–0.8000000219
$z_2^{(1)}$	–0.2	–0.2020709807	–0.2001071216	–0.199996485	–0.2000000620
$z_3^{(1)}$	–0.6	–0.5684050181	–0.6000878738	–0.5999800574	–0.6000000464
$z_1^{(2)}$	–0.38	–0.3838828957	–0.3801569730	–0.3799948547	–0.3800000942
$z_3^{(2)}$	–0.4	–0.3700284637	–0.3999682608	–0.3999842474	–0.3999999656
$z_1^{(3)}$	–0.48	–0.4842219409	–0.4800381441	–0.4799998221	–0.4800000226
$z_2^{(3)}$	0.4	0.3967271760	0.4000262560	0.3999978194	0.4000000159
$z_1^{(4)}$	–0.2	–0.2025151122	–0.2000954038	–0.1999968834	–0.2000000586
$z_2^{(4)}$	0.6	0.5987479750	0.5999603161	0.6000012161	0.5999999745
$z_3^{(4)}$	0.2	0.1987479750	0.1999603161	0.2000012161	0.1999999745
t_1	0.2	0.1988295851	0.1999917615	0.2000000919	0.1999999900
t_2	0.4	0.4003180698	0.4001083934	0.3999957228	0.4000000506
t_3	0.8	0.7979815572	0.8000579299	0.7999969454	0.8000000377

t_1, t_2, and t_3). In general, the quality of approximation by U_0 grows with the number of equations (which is equal to the number of intermediate nodes).

Given sets (7.13), we need to verify conditions of Section 4 on the corresponding sets $\Omega_{0,1}, \ldots$ $\ldots, \Omega_{3,4}$ defined according to (3.6). For this purpose, we use Remark 4.3 and choose suitable parallelepipeds $\Omega^{(k)} \supset \Omega_{k-1,k}$, $k = 1, \ldots, 4$:

$$\begin{aligned}
&\Omega^{(1)} := \big\{(u_1, u_2, u_3)\colon -0.1 \le u_1 \le 0.7,\ -0.6 \le u_2 \le -0.1,\ -0.95 \le u_3 \le -0.5\big\},\\
&\Omega^{(2)} := \big\{(u_1, u_2, u_3)\colon -0.5 \le u_1 \le 0.1,\ -0.3 \le u_2 \le 0.1,\ -0.75 \le u_3 \le -0.3\big\},\\
&\Omega^{(3)} := \big\{(u_1, u_2, u_3)\colon -0.55 \le u_1 \le -0.25,\ -0.1 \le u_2 \le 0.5,\ -0.5 \le u_3 \le 0.1\big\},\\
&\Omega^{(4)} := \big\{(u_1, u_2, u_3)\colon -0.55 \le u_1 \le -0.1,\ 0.3 \le u_2 \le 0.7,\ -0.1 \le u_3 \le 0.3\big\}.
\end{aligned} \tag{7.14}$$

We are going to verify conditions (4.9) on the sets $\mathcal{O}_{\varrho^{(k)}}(\Omega^{(k)})$, $k = 1, \ldots, 4$, for which purpose the vectors $\varrho^{(1)}, \ldots, \varrho^{(4)}$ should be chosen. Let us put, for example

$$\varrho^{(1)} = \operatorname{col}(0.2, 0.2, 0.2), \qquad \varrho^{(2)} = \varrho^{(1)},$$
$$\varrho^{(3)} = \operatorname{col}(0.6, 0.2, 0.3), \qquad \varrho^{(4)} = \operatorname{col}(0.3, 0.2, 0.2). \tag{7.15}$$

Then, according to (3.8), we obtain from (7.14)

$$\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)}) = \big\{(u_1, u_2, u_3)\colon -0.3 \le u_1 \le 0.9,\ -0.8 \le u_2 \le 0.1,\ -1.15 \le u_3 \le -0.3\big\},$$
$$\mathcal{O}_{\varrho^{(2)}}(\Omega^{(2)}) = \big\{(u_1, u_2, u_3)\colon -0.7 \le u_1 \le 0.3,\ -0.5 \le u_2 \le 0.3,\ -0.95 \le u_3 \le -0.1\big\},$$
$$\mathcal{O}_{\varrho^{(3)}}(\Omega^{(3)}) = \big\{(u_1, u_2, u_3)\colon -1.15 \le u_1 \le 0.35,\ -0.3 \le u_2 \le 0.7,\ -0.8 \le u_3 \le 0.4\big\},$$
$$\mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)}) = \big\{(u_1, u_2, u_3)\colon -0.85 \le u_1 \le 0.2,\ 0.1 \le u_2 \le 0.9,\ -0.3 \le u_3 \le 0.5\big\}. \tag{7.16}$$

A direct computation using (7.8)–(7.11) shows that the Lipschitz condition (4.5) for $\tilde{f}$ holds on $\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)}), \ldots, \mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)})$, respectively, with the matrices

$$K_1 = \begin{pmatrix} 0 & 1.15 & 0.8 \\ 0 & 1.15 & 0.8 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0.95 & 0.5 \\ 0 & 0.95 & 0.5 \\ 1 & 0 & 0 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 0 & 0.8 & 0.7 \\ 0 & 0.8 & 0.7 \\ 1 & 0 & 0 \end{pmatrix}, \qquad K_4 = \begin{pmatrix} 0 & 0.5 & 0.9 \\ 0 & 0.5 & 0.9 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7.17}$$

Then, taking into account the rough approximations of $t_1,\ t_2$, and t_3 obtained at the zeroth step (the second column of Table 1), we can assume, e. g., the following bounds for the regions (6.8) where more precise values of these variables should be looked for:

$$T_1^- \le t_1 \le T_1^+, \qquad T_2^- \le t_2 \le T_2^+, \qquad T_3^- \le t_3 \le T_3^+, \tag{7.18}$$

where

$$T_1^- := 0.15, \qquad T_1^+ := 0.25, \qquad T_2^- := 0.35,$$
$$T_2^+ := 0.45, \qquad T_3^- := 0.75, \qquad T_3^+ := 0.85. \tag{7.19}$$

Assuming (7.18), we obtain from (7.17)

$$r(K_1) \approx 1.6383 < \frac{40}{3} = \frac{10}{3T_1^+},$$
$$r(K_2) \approx 1.3268 < \frac{100}{9} = \frac{10}{3(T_2^+ - T_1^-)},$$
$$r(K_3) \approx 1.3274 < \frac{20}{3} = \frac{10}{3(T_3^+ - T_2^-)},$$

Table 2. Meaning of parameters in the example

$z_1^{(0)}$	$z_2^{(0)}$	$z_3^{(0)}$	$z_2^{(1)}$	$z_3^{(1)}$	$z_1^{(2)}$	$z_3^{(2)}$	$z_1^{(3)}$	$z_2^{(3)}$	$z_1^{(4)}$	$z_2^{(4)}$	$z_3^{(4)}$
$u_1(0)$	$u_2(0)$	$u_3(0)$	$u_2(t_1)$	$u_3(t_1)$	$u_1(t_2)$	$u_3(t_2)$	$u_1(t_3)$	$u_2(t_3)$	$u_1(1)$	$u_2(1)$	$u_3(1)$

$$r(K_4) \approx 1.2311 < \frac{40}{3} = \frac{10}{3(1 - T_3^-)},$$

which means that conditions (6.10) hold. Furthermore, in view of (4.4), we have

$$\varrho^{(1)} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{T_1^+}{4} \delta_{[0,T_1^+],\mathcal{O}_{\varrho^{(1)}}(\Omega^{(1)})}(\tilde{f}) \approx \frac{0.25}{4} \begin{pmatrix} 2.6475 \\ 1.2725 \\ 1.9156 \end{pmatrix} \approx \begin{pmatrix} 0.1655 \\ 0.0795 \\ 0.1197 \end{pmatrix},$$

$$\varrho^{(2)} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{T_2^+ - T_1^-}{4} \delta_{[T_1^-,T_2^+],\mathcal{O}_{\varrho^{(2)}}(\Omega^{(2)})}(\tilde{f}) = \frac{0.3}{4} \begin{pmatrix} 2.59 \\ 0.94 \\ 1.57 \end{pmatrix} = \begin{pmatrix} 0.19425 \\ 0.0705 \\ 0.11775 \end{pmatrix},$$

$$\varrho^{(3)} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.3 \end{pmatrix} > \frac{T_3^+ - T_2^-}{4} \delta_{[T_2^-,T_3^+],\mathcal{O}_{\varrho^{(3)}}(\Omega^{(3)})}(\tilde{f}) \approx \frac{0.5}{4} \begin{pmatrix} 3.59 \\ 0.9025 \\ 1.7401 \end{pmatrix} \approx \begin{pmatrix} 0.4488 \\ 0.1128 \\ 0.2175 \end{pmatrix},$$

$$\varrho^{(4)} = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.2 \end{pmatrix} > \frac{1 - T_3^-}{4} \delta_{[T_3^-,1],\mathcal{O}_{\varrho^{(4)}}(\Omega^{(4)})}(\tilde{f}) \approx \frac{0.25}{4} \begin{pmatrix} 1.9575 \\ 0.8575 \\ 1.3656 \end{pmatrix} \approx \begin{pmatrix} 0.1223 \\ 0.0536 \\ 0.0854 \end{pmatrix}.$$

This means that conditions (6.9) are satisfied.

Thus, taking account the observation of Section 6, we conclude that the scheme based on Theorems 5.1 – 5.4 is applicable provided that bounds (7.18) for t_1, t_2, and t_3 are assumed. Note that, as the computation shows, the true values of these variables indeed satisfy estimates (7.18). This situation is generic: when using this kind of computational schemes, it is always natural to choose the sets in the conditions *after* we get some notion of where we are going to find the values of the unknowns in the course of computation.

The scheme is now implemented as follows. We use equalities (5.1), (5.2) to construct the corresponding functions

$$u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)\colon [t_{k-1}, t_k] \to \mathbb{R}^3, \qquad 1 \le k \le 4, \quad m \ge 0. \tag{7.20}$$

These functions depend on the 12 scalar parameters listed in Table 2 and on the unknown time instants t_1, t_2, and t_3. By Theorem 5.1, functions (7.20) form convergent sequences as $m \to \infty$.

Note that, according to Section 5, the function $u_m^{(k)}\big(\cdot, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big)$ is an approximation to the solution of the kth auxiliary two-point problem (3.2), (3.3) on the respective subintervals $[t_{k-1}, t_k]$, $1 \le k \le 4$. For this example, system (3.2), (3.3) means the following four problems:

Eq. (7.12) on $[0, t_1]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2,\ \tilde{f}_3$ from (7.9) under the conditions

$$\begin{aligned} &u_1(0) = z_1^{(0)}, \qquad u_2(0) = z_2^{(0)}, \qquad u_3(0) = z_3^{(0)}, \\ &u_1(t_1) = 0, \qquad u_2(t_1) = z_2^{(1)}, \qquad u_3(t_1) = z_3^{(1)}; \end{aligned} \tag{7.21}$$

Eq. (7.12) on $[t_1, t_2]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2,\ \tilde{f}_3$ from (7.10) under the conditions

$$\begin{aligned} &u_1(t_1) = 0, \qquad u_2(t_1) = z_2^{(1)}, \qquad u_3(t_1) = z_3^{(1)}, \\ &u_1(t_2) = z_1^{(2)}, \qquad u_2(t_2) = 0, \qquad u_3(t_2) = z_3^{(2)}; \end{aligned} \tag{7.22}$$

Eq. (7.12) on $[t_2, t_3]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2,\ \tilde{f}_3$ from (7.10) under the conditions

$$\begin{aligned} &u_1(t_2) = z_1^{(2)}, \qquad u_2(t_2) = 0, \qquad u_3(t_2) = z_3^{(2)}, \\ &u_1(t_3) = z_1^{(3)}, \qquad u_2(t_3) = z_2^{(3)}, \qquad u_3(t_3) = 0; \end{aligned} \tag{7.23}$$

Eq. (7.12) on $[t_3, 1]$ with $\tilde{f}_1$ from (7.8) and $\tilde{f}_2,\ \tilde{f}_3$ from (7.11) under the conditions

$$\begin{aligned} &u_1(t_3) = z_1^{(3)}, \qquad u_2(t_3) = z_2^{(3)}, \qquad u_3(t_3) = 0, \\ &u_1(1) = z_1^{(4)}, \qquad u_2(1) = z_2^{(4)}, \qquad u_3(1) = z_3^{(4)}. \end{aligned} \tag{7.24}$$

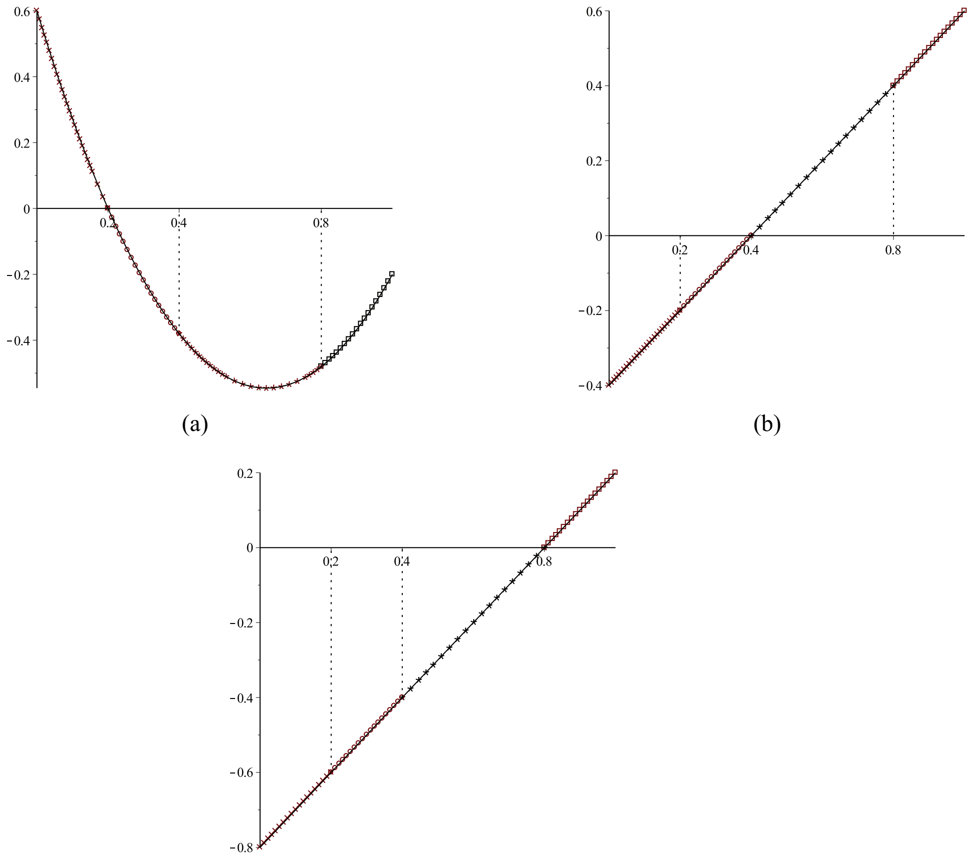
The auxiliary problems (7.21) – (7.24) are however not treated directly in the course of computation, which involves functions (7.20) only. Approximate solutions of the given problem (7.1), (7.4) are constructed, on the respective subintervals, in the form

$$U_m(t) := u_m^{(k)}\big(t, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big), \qquad t \in [t_{k-1}, t_k], \quad k = 1, \ldots, 4,$$

where m is fixed and $z^{(j)},\ j = 0, \ldots, 4$, are vectors of form (5.19) satisfying the mth approximate determining system (6.2), (6.3):

$$\begin{gathered} z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} \tilde{f}\big(s, u_m^{(k)}(s, z^{(k-1)}, z^{(k)}, t_{k-1}, t_k\big) ds = 0, \quad k = 1, 2, \ldots, 4, \\ \big(u_1^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0)\big)^2 - \big(u_2^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1)\big)^2 = 0, \\ u_2^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0)\, u_3^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1) = -\frac{2}{25}, \\ u_1^{(1)}(0, z^{(0)}, z^{(1)}, 0, t_0) - u_3^{(4)}(1, z^{(3)}, z^{(4)}, t_3, 1) = \frac{2}{5}. \end{gathered} \tag{7.25}$$

In order to determine the values of parameters on step m, equations (7.25) are solved numerically for $z^{(j)} \in \Omega_j,\ j = 0, \ldots, 4$, and $t_i \in [T_i^-, T_i^+],\ i = 1, 2, 3$. An initial hint for the region where the roots should be looked for is obtained by using the zeroth approximation $(m = 0)$, the graphs of which are shown on Fig. 1 (a) – (c). We have used Maple 14 to carry out all the computations.



(a) (b)

(c)

Fig. 2. Exact solution (solid line) and third approximation: first component (*a*), second component (*b*), and third component (*c*).

The numerical values of the 15 unknown parameters obtained from (7.25) for the first three steps of iteration are shown in Table 1. We see that the approximate values from the third iteration are very close to the exact ones.

The graphs of the respective components of the exact solution (7.5) and the approximate $\big[(1,-1;t_1),(-1,1;t_2),(-1,1;t_3)\big]$ solution $U_3=(U_{3i})_{i=1}^3$ of problem (7.1), (7.4) corresponding to the numerical values from Table 1 are shown on Fig. $2\,(a)-(c)$. The curves corresponding to the subintervals $[t_{k-1},t_k]$, $k=1,\ldots,4$, with the values of $t_1,\ t_2$, and t_3 computed on the third step are drawn with different symbols.

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