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DIFFERENTIAL-GEOMETRIC STRUCTURE AND THE LAX – SATO INTEGRABILITY OF A CLASS OF DISPERSIONLESS HEAVENLY TYPE EQUATIONS

ДИФЕРЕНЦІАЛЬНО-ГЕОМЕТРИЧНА СТРУКТУРА ТА ІНТЕГРОВНІСТЬ ЛАКСА – САТО ДЛЯ ОДНОГО КЛАСУ БЕЗДИСПЕРСІЙНИХ РІВНЯНЬ НЕБЕСНОГО ТИПУ

This short communication is devoted to the study of differential-geometric structure and the Lax – Sato integrability of the reduced Shabat-type, Hirota, and Kupershmidt heavenly equations.

Це коротке повідомлення присвячено вивченню диференціально-геометричної структури та інтегровності Лакса – Сато для редукованих небесних рівнянь типу Шабата, Хіроти та Купершмідта.

1. Introduction. We study differential-geometric structure and the Lax – Sato integrability of a class of dispersionless hydrodynamic equations including the reduced Shabat-type, Hirota and Kupershmidt heavenly equations, based on the before devised [3, 6] Lie-algebraic integrability scheme. It is demonstrated that their compatibility conditions coincide with the corresponding heavenly type equations under consideration. It is shown that all these equations originate in this way and can be represented as the Lax – Sato compatibility conditions for specially constructed loop vector fields on toroidal manifolds.

2. The reduced Shabat-type heavenly equations. ***2.1. The first reduced Shabat-type heavenly equation.*** The entitled above equation [1] reads as

$$u_{yt} + u_t u_{xy} - u_{xt} u_y = 0 \tag{1}$$

for a function $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$, where $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. To show the Lax – Sato integrability of the equation (1), take a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{\operatorname{diff}}^*(\mathbb{T}^1)$ in the following form:

$$\tilde{l} = \left(\frac{u_t^{-2}}{\lambda + 1} + \frac{u_t^2 - u_y^2}{u_y^2 u_t^2} + \frac{u_y^{-2}}{\lambda} \right) dx,$$

where $\lambda \in \mathbb{C} \backslash \{0, -1\}$ and $\tilde{\mathcal{G}}$ is This element generates two independent hierarchies of Casimir functionals $\gamma^{(1)}, \gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions are given by the following asymptotic expansions:

$$\nabla \gamma^{(1)}(l) \sim u_t + O(\mu^2),$$

as $\lambda + 1 := \mu \to 0$, and

$$\nabla \gamma^{(2)}(l) \sim u_y + O(\mu^2),$$

as $\lambda := \mu \to 0.$ Having put now, by definition,

$$\nabla h^{(t)}(l) := \mu(\mu^{-2}\nabla\gamma^{(1)}(l))_-\big|_{\mu=\lambda+1}, \qquad \nabla h^{(y)}(l) := \mu(\mu^{-2}\nabla\gamma^{(2)}(l))_-\big|_{\mu=\lambda},$$

one easily ensues from the compatibility condition

$$\partial\tilde{A}^{(y)}/\partial t - \partial\tilde{A}^{(t)}/\partial y = [\tilde{A}^{(y)}, \tilde{A}^{(t)}], \tag{2}$$

for a set of the vector fields

$$\tilde{A}^{(t)} := \nabla h^{(t)}(l)\frac{\partial}{\partial x}, \qquad \tilde{A}^{(y)} := \nabla h^{(y)}(l)\frac{\partial}{\partial x} \tag{3}$$

a compatible Lax – Sato representation as the following system of vector field equations:

$$\frac{\partial\psi}{\partial t} + \frac{u_t}{\lambda+1}\frac{\partial\psi}{\partial x} = 0, \qquad \frac{\partial\psi}{\partial y} + \frac{u_y}{\lambda}\frac{\partial\psi}{\partial x} = 0, \tag{4}$$

satisfied for $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$, any $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and all $\lambda \in \mathbb{C}\backslash\{0, -1\}$.

2.2. The second reduced Shabat-type heavenly equation. The entitled above equation [1] reads as

$$u_{yy} - u_{xt}u_y + u_t u_{xy} = 0 \tag{5}$$

for a function $u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R})$, where $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. In this case for demonstrating the Lax – Sato integrability of the equation (5) we will take a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{\operatorname{diff}}^*(\mathbb{T}^1)$ as

$$\tilde{l} = (\lambda u_y^{-2} + 2(u_t + u_y^2)u_y^{-3} + \lambda^{-1}u_t(3u_t + 4u_y)u_y^{-4})dx,$$

giving rise to two independent Casimir functionals $\gamma^{(1)}, \gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions are given by the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim -\lambda u_y + u_t + O(1/\lambda^2),$$

$$\nabla\gamma^{(2)}(l) \sim \lambda u_y - (u_t + u_y) + O(1/\lambda^2),$$

as $\lambda \to \infty$. Having put now, by definition,

$$\nabla\gamma^{(t)}(l) := (\lambda\nabla\gamma^{(1)}(l))|_+ = \lambda u_t - \lambda^2 u_y,$$

$$\nabla\gamma^{(y)}(l) := -(\lambda\nabla\gamma^{(1)}(l) + \lambda\nabla\gamma^{(2)}(l)|_+ = \lambda u_y,$$

we construct the vector fields

$$\tilde{A}^{(t)} := \nabla h^{(t)}(l)\frac{\partial}{\partial x} = (\lambda u_t - \lambda^2 u_y)\frac{\partial}{\partial x},$$

$$\tilde{A}^{(y)} := \nabla h^{(y)}(l)\frac{\partial}{\partial x} = \lambda u_y\frac{\partial}{\partial x},$$

satisfying the compatibility condition (2), entailing the heavenly equation (1). Moreover, this compatibility condition (2) is, evidently, equivalent to the following Lax – Sato vector field representation

$$\frac{\partial\psi}{\partial t} + (\lambda u_t - \lambda^2 u_y)\frac{\partial\psi}{\partial x} = 0, \qquad \frac{\partial\psi}{\partial y} + \lambda u_y\frac{\partial\psi}{\partial x} = 0, \tag{6}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$, any $(t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and all $\lambda \in \mathbb{C}$.

The obtained above results can be formulated as the following theorem.

Theorem 1. *The reduced Shabat-type heavenly equations* (1) *and* (5) *are completely integrable Hamiltonian flows equivalent to the Lax – Sato vector field compatibility conditions* (4) *and* (6), *respectively.*

3. The Hirota heavenly equation. The Hirota equation describes [2, 4] three-dimensional Veronese webs and reads as

$$\alpha u_x u_{yt} + \beta u_y u_{xt} + \gamma u_t u_{xy} = 0 \tag{7}$$

for any evolution parameters $t, y \in \mathbb{R}$ and the spatial variable $x \in \mathbb{T}^1$, where α, β and $\gamma \in \mathbb{R}$ are arbitrary constants, satisfying the numerical constraint

$$\alpha + \beta + \gamma = 0.$$

To demonstrate the Lax-type integrability of the Hirota equation (7) we choose a seed vector field $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{\mathrm{diff}}^*(\mathbb{T}^1)$ in the rational form

$$\tilde{l} = \left(\frac{u_x^2}{u_t^2(\lambda + \alpha)} - \frac{u_x^2(u_y^2 + u_t^2)}{2\alpha u_t^2 u_y^2} + \frac{u_x^2}{u_y^2(\lambda - \alpha)} \right) dx.$$

The corresponding gradients for the Casimir invariants $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, $j = \overline{1, 2}$, are given by the following asymptotic expansions:

$$\nabla \gamma^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(1)}(l) \mu^j, \tag{8}$$

as $\lambda + \alpha := \mu \to 0$, and

$$\nabla \gamma^{(2)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(2)}(l) \mu^j, \tag{9}$$

as $\lambda - \alpha = \mu \to 0$. For the first case (8) one easily obtains that

$$\nabla \gamma^{(1)}(l) \sim -2\gamma \frac{u_t}{u_x} + O(\mu^2),$$

and for the second one (9) one obtains

$$\nabla \gamma^{(2)}(l) \sim 2\beta \frac{u_y}{u_x} + O(\mu^2),$$

where we took into account that the two Hamiltonian flows on $\tilde{\mathcal{G}}^*$:

$$\frac{d\tilde{l}}{dy} = ad^*_{\nabla h^{(y)}(\tilde{l})} \tilde{l}, \qquad \frac{d\tilde{l}}{dt} = ad^*_{\nabla h^{(t)}(\tilde{l})} \tilde{l}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$ hold for the following conservation laws gradients:

$$\nabla h^{(t)}(l) := \mu(\mu^{-2} \nabla \gamma^{(1)}(l))_- \big|_{\mu = \lambda + \alpha} = \frac{-2\gamma}{\lambda + \alpha} \frac{u_t}{u_x},$$

$$\nabla h^{(y)}(l) := \mu(\mu^{-2} \nabla \gamma^{(2)}(l))_- \big|_{\mu = \lambda - \alpha} = \frac{2\beta}{\lambda - \alpha} \frac{u_y}{u_x}.$$

It is easy now to check that the compatibility condition (2) for the set of vector fields (3) gives rise to the Hirota heavenly equation (7), whose equivalent Lax – Sato vector field representation reads as a system of the linear vector field equations

$$\frac{\partial\psi}{\partial t} - \frac{2\gamma u_t}{u_x(\lambda+\alpha)}\frac{\partial\psi}{\partial x} = 0, \qquad \frac{\partial\psi}{\partial y} + \frac{2\beta u_y}{u_x(\lambda-\alpha)}\frac{\partial\psi}{\partial x} = 0, \tag{10}$$

satisfied for $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1;\mathbb{C})$ for all $(y,t;x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$. Thus the obtained result can be formulated as the next theorem.

Theorem 2. *The Hirota heavenly equation* (7) *is a completely integrable Hamiltonian flow equivalent to the Lax – Sato vector field compatibility condition* (10).

4. The Kupershmidt hydrodynamic system. These two compatible to each other hydrodynamic systems [5, 7] read as

$$\begin{gathered} 3v_y - 6uv_x + 6u_xv + 6uu_y - 6u^2u_x - 2u_t = 0, \\ -12v_x + 6u_y - 12uu_x = 0, \\ 6uv_{xx} - 3v_{xy} - 6u_{xx}v - 6u_xu_y + 6u^2u_{xx} - 6uu_{xy} + 12uu_x^2 + 2u_{xt} = 0, \\ 6v_{xx} + 6uu_{xx} - 3u_{xy} + 6u_x^2 = 0 \end{gathered} \tag{11}$$

for smooth functions $(u,v) \in C^\infty(\mathbb{R};\mathbb{R}^2)$ with respect to evolution parameters $t,y \in \mathbb{R}$ and the spatial variable $x \in \mathbb{T}^1$. Its Lax-type integrability stems from a seed vector field $\bar{l} \in \bar{\mathcal{G}}^*$, where $\bar{\mathcal{G}}$ denotes the holomorphic in $\lambda \in \mathbb{S}^1_\pm$ Lie algebra $\bar{\mathcal{G}} := \mathrm{diff}_{\mathrm{hol}}(\ \mathbb{C}\times\mathbb{T}^1) \subset \mathrm{diff}(\ \mathbb{C}\times\mathbb{T}^1)$ of the diffeomorphism group $\mathrm{Diff}(\mathbb{C}\times\mathbb{T}^1)$, and

$$\bar{l} = [\lambda(v_x + 2uu_x) + \lambda^2 u_x]dx + [(v+u^2) + 2\lambda u + \lambda^2]d\lambda.$$

The corresponding gradients for the Casimir invariants $\gamma^{(j)} \in I(\bar{\mathcal{G}}^*)$, $j = \overline{1,2}$, are easily constructed from the determining conditions $ad^*_{\nabla h^{(j)}(\bar{l})}\bar{l} = 0, j = \overline{1,2}$, as the following asymptotic expansions:

$$\nabla\gamma^{(j)}(l) \sim \sum_{j\in\mathbb{Z}_+} \nabla\gamma_j^{(j)}(l)\lambda^{-j},$$

giving rise to the expressions

$$\begin{gathered} \nabla\gamma^{(1)}(l) \sim (2(\lambda+u), -2\lambda u_x)^\intercal + O(\lambda^{-1}), \\ \nabla\gamma^{(2)}(l) \sim (3(\lambda^2 + 2\lambda u + +u^2 + v), -3\lambda(\lambda u_x + 2uu_x + v_x))^\intercal + O(\lambda^{-1}), \end{gathered}$$

as $|\lambda| \to \infty$. Now taking into account the following Hamiltonian flows on $\tilde{\mathcal{G}}^*$:

$$d\bar{l}/dy = -ad^*_{\nabla h^{(y)}(\bar{l})}\bar{l}, \qquad d\bar{l}/dy = -ad^*_{\nabla h^{(t)}(\bar{l})}\bar{l} \tag{12}$$

with respect to the evolution parameters $y,t \in \mathbb{R}$, where, by definition,

$$\begin{gathered} \nabla h^{(y)}(\bar{l}) :=\ \nabla\gamma^{(1)}(\bar{l}))_+ = 2(\lambda+u)\partial/\partial x - 2\lambda u_x\partial/\partial\lambda, \\ \nabla h^{(t)}(\bar{l}) := \nabla\gamma^{(2)}(\bar{l}))_+ = 3(\lambda^2 + 2\lambda u + u^2 + v)\partial/\partial x - 3\lambda(\lambda u_x + 2uu_x + v_x)\partial/\partial\lambda \end{gathered}$$

are holomorphic vector fields on $\mathbb{C} \times \mathbb{T}^1$, we can easily derive the corresponding compatible Kupershmidt hydrodynamic systems (11).

It is also easy to check that the compatibility condition for a set of the vector fields (12) gives rise to the equivalent Lax – Sato vector field representation

$$\begin{gathered}\frac{\partial \psi}{\partial t} - 3(\lambda^2 + 2\lambda u + u^2 + v)\frac{\partial \psi}{\partial x} + 3\lambda(\lambda u_x + 2uu_x + v_x)\frac{\partial \psi}{\partial \lambda} = 0,\\ \frac{\partial \psi}{\partial y} - 2(\lambda + u)\frac{\partial \psi}{\partial x} + 2\lambda u_x \frac{\partial \psi}{\partial \lambda} = 0,\end{gathered} \tag{13}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$ for all $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and $\lambda \in \mathbb{C}$. The result obtained we formulate as the following theorem.

Theorem 3. *The Kupershmidt hydrodynamic heavenly type system* (11) *is representable as commuting Hamiltonian flows* (12) *on orbits of the coadjoint action of the holomorphic Lie algebra* $\bar{\mathcal{G}} = \mathrm{diff}_{\mathrm{hol}}(\mathbb{C} \times \mathbb{T}^1)$ *and are equivalent to the Lax – Sato vector field compatibility condition* (13).

5. Conclusion. As we have demonstrated above, the devised in [3, 6] Lie-algebraic scheme of studying Lax – Sato-type integrability proved to be both natural and anlytically effective, reducing the problem to describing an infinite hierarchy of commuting to each other Hamiltonian flows as the corresponding Lie – Poisson orbits of loop diffeomorphism groups on torus.

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