UDC 512.5

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A NOTE ON STRONGLY SPLIT LIE ALGEBRAS*

ЗАУВАЖЕННЯ ЩОДО СИЛЬНО РОЗЩЕПЛЕНИХ АЛГЕБР ЛІ

Split Lie algebras are maybe the most known examples of graded Lie algebras. Since an important category in the class of graded algebras is the category of strongly graded algebras, we introduce, in a natural way, the category of strongly split Lie algebras $\mathfrak{L}$ and show that if $\mathfrak{L}$ is centerless, then $\mathfrak{L}$ is the direct sum of split ideals each of which is a split-simple strongly split Lie algebra.

Розщеплені алгебри Лі є мабуть найбільш відомим прикладом градуйованих алгебр Лі. Оскільки важливою категорією в класі градуйованих алгебр є категорія сильно градуйованих алгебр, ми вводимо (природним чином) категорію сильно розщеплених алгебр Лі $\mathfrak{L}$ і доводимо, що у випадку, коли $\mathfrak{L}$ не має центра, $\mathfrak{L}$ є прямою сумою розщеплених ідеалів, кожний з яких є просто-розщепленою сильно розщепленою алгеброю Лі.

1. Introduction and preliminaries. We begin by noting that, troughout this short note, all of the Lie algebras are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

By the one hand, let us recall that given a Lie algebra L and a fixed maximal Abelian subalgebra H of L, we can consider for any linear functional $\alpha\colon H \longrightarrow \mathbb{K}$, the *root space* of L associated to α as the linear subspace

$$L_\alpha = \{v_\alpha \in L\colon [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}.$$

The elements $\alpha \in H^*$ satisfying $L_\alpha \neq 0$ are called *roots* of L with respect to H and if we denote by $\Lambda := \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$, it is said that L is a *split Lie algebra* (with respect to H), if

$$L = H \oplus \left(\bigoplus_{\alpha\in\Lambda} L_\alpha\right).$$

It is also said that Λ is the *root system* of L, being Λ called *symmetric* if $\Lambda = -\Lambda$.

Let us focuss for a while on the concept of split-ideal in the framework of split Lie algebras. Observe that the set of linear mappings $\{\mathrm{ad}(h)\colon h \in H\}$, where $\mathrm{ad}(h)\colon L \to L$ is defined by $\mathrm{ad}(h)(v) = [h, v]$, is a commuting set of diagonalizable endomorphisms. Hence, given any ideal I of L, since I is invariant under this set, we can write

$$I = (I \cap H) \oplus \left(\bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha)\right). \quad (1)$$

From here, if $I \cap H \neq 0$, then I adopts a split like expression (respect to $I \cap H$). This motivate us to introduce the concept of split-ideal as follows. An ideal I of a split Lie algebra L is called a *split-ideal* if $I \cap H \neq 0$. The split Lie algebra L will be called *split-simple* if $[L, L] \neq 0$ and L has not proper split-ideals.

* Suppoted by the PCI of the UCA "Teoria de Lie y Teoria de Espasios de Banach by the PAI wiht ptoject numbers FQM298, FQM7156 and by the project of the Spanish Ministerio de Education y Ciencia MTM2016-76327C31P.

By the other hand, we also recall that a *graded algebra*

$$A = \bigoplus_{g\in G} A_g,$$

that is, A is the direct sum of linear subspaces indexed by the elements in an Abelian group $(G,+)$ in such a way that $A_gA_h \subset A_{g+h}$, is called a *strongly graded algebra* if the condition $A_gA_h = A_{g+h}$ holds for any $g,h \in G$ (see [2, 3]).

Since Jacobi identity shows that in any split Lie algebra L we have $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ for any $\alpha,\beta \in \Lambda \cup \{0\}$ and the fact $H = L_0$ holds, we have that L becomes a graded Lie algebra by means of the Abelian free group generated by Λ. Taking into account the above observations we introduce the category of strongly split Lie algebras as follows.

Definition 1.1. *A split Lie algebra $\mathfrak{L}$ with set of nonzero roots Λ is called a strongly split Lie algebra if $H = \sum_{\alpha\in\Lambda}[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ and given $\alpha,\beta \in \Lambda$ such that $\alpha+\beta \in \Lambda$, then we have $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = \mathfrak{L}_{\alpha+\beta}$.*

As examples of strongly split Lie algebra we can consider the finite dimensional semisimple Lie algebras, the Lie H^*-algebras, the locally finite split Lie algebras, the split graded Lie algebras with only integer roots and the split Lie algebras considered in [1] among other classes of Lie algebras (see [4–6]).

2. Main results. In the following, $\mathfrak{L}$ denotes a strongly split Lie algebra with a symmetric root system and $\mathfrak{L} = H \oplus (\bigoplus_{\alpha\in\Lambda} \mathfrak{L}_\alpha)$ the corresponding root spaces decomposition.

Definition 2.1. *Let $\alpha \in \Lambda$ and $\beta \in \Lambda$ be two nonzero roots. We say that α is connected to β if there exists a family $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ satisfying the following conditions:*

1) $\alpha_1 = \alpha$,

2) $\{\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \ldots, \alpha_1+\ldots+\alpha_{n-1}\} \subset \Lambda$,

3) $\alpha_1+\alpha_2+\ldots+\alpha_n = \epsilon_\beta\beta$ *for some* $\epsilon_\beta \in \{\pm 1\}$.

We also say that $\{\alpha_1,\ldots,\alpha_n\}$ is a connection from α to β.

It is straightforward to verify the relation connection is an equivalence relation. In particular, $\alpha \sim -\alpha$. So we can consider the quotient set $\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\}$. Now, for any $[\alpha] \in \Lambda/\sim$ we are going to consider the linear subspace $\mathfrak{L}_{[\alpha]} := H_{[\alpha]} \oplus V_{[\alpha]}$, where $H_{[\alpha]} := \sum_{\beta\in[\alpha]}[\mathfrak{L}_\beta, \mathfrak{L}_{-\beta}] \subset H$ and $V_{[\alpha]} := \bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta$.

Proposition 2.1. *Any $\mathfrak{L}_{[\alpha]}$ is an ideal of $\mathfrak{L}$. If furthermore $\mathfrak{L}$ is centerless, then $\mathfrak{L}_{[\alpha]}$ is split-simple.*

Proof. We begin by showing that

$$[\mathfrak{L}_{[\alpha]}, \mathfrak{L}] = \left[H_{[\alpha]} \oplus V_{[\alpha]}, H \oplus \left(\bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta\right) \oplus \left(\bigoplus_{\gamma\notin[\alpha]} \mathfrak{L}_\gamma\right)\right] \subset \mathfrak{L}_{[\alpha]}.$$

Clearly $[H_{[\alpha]}, H \oplus (\bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta)] + [V_{[\alpha]}, H] \subset V_{[\alpha]}$.

Since in case $[\mathfrak{L}_\delta, \mathfrak{L}_\tau] \neq 0$ for some $\delta,\tau \in \Lambda$ with $\delta+\tau \neq 0$, the connections $\{\delta,\tau\}$ and $\{\delta,\tau,-\delta\}$ imply $[\delta] = [\delta+\tau] = [\tau]$, we get $[V_{[\alpha]}, \bigoplus_{\beta\in[\alpha]} \mathfrak{L}_\beta] \subset \mathfrak{L}_{[\alpha]}$ and

$$\left[V_{[\alpha]}, \quad \bigoplus_{\gamma \notin [\alpha]} \mathfrak{L}_\gamma\right] = 0. \tag{2}$$

This fact and Jacobi identity finally give us

$$\left[H_{[\alpha]}, \quad \bigoplus_{\gamma \notin [\alpha]} \mathfrak{L}_\gamma\right] = 0. \tag{3}$$

We have showed $\mathfrak{L}_{[\alpha]}$ is an ideal of $\mathfrak{L}$. Since equation (3) implies $[H_{[\gamma]}, V_{[\alpha]}] = 0$ for any $[\gamma] \neq [\alpha]$ we get $H_{[\alpha]} \neq 0$. From here, we can also assert that $\mathfrak{L}_{[\alpha]}$ is a strongly split Lie algebra with the split decomposition

$$\mathfrak{L}_{[\alpha]} = H_{[\alpha]} \oplus \left(\bigoplus_{\beta \in [\alpha]} \mathfrak{L}_\beta \right).$$

Suppose now $\mathfrak{L}$ is centerless and let us show $\mathfrak{L}_{[\alpha]}$ is split-simple. Consider a split-ideal I of $\mathfrak{L}_{[\alpha]}$. By equation (1) we can write $I = (I \cap H_{[\alpha]}) \oplus (\bigoplus_{\beta \in [\alpha]} (I \cap \mathfrak{L}_\beta))$ with $I \cap H_{[\alpha]} \neq 0$. For any $0 \neq h \in I \cap H_{[\alpha]}$, the fact $\mathfrak{L}$ is centerless gives us there exists $\beta \in [\alpha]$ such that $[h, \mathfrak{L}_\beta] \neq 0$. From here we get $[I \cap H_{[\alpha]}, \mathfrak{L}_\beta] = \mathfrak{L}_\beta$ and so $0 \neq \mathfrak{L}_\beta \subset I$.

Given now any $\delta \in [\alpha] \setminus \{\pm\beta\}$, the fact that β and δ are connected allows us to take a connection $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ from β to δ. Since $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 \in \Lambda$ we have $[\mathfrak{L}_{\alpha_1}, \mathfrak{L}_{\alpha_2}] = \mathfrak{L}_{\alpha_1+\alpha_2} \subset I$ as consequence of $\mathfrak{L}_{\alpha_1} = \mathfrak{L}_\beta \subset I$. In a similar way $[\mathfrak{L}_{\alpha_1+\alpha_2}, \mathfrak{L}_{\alpha_3}] = \mathfrak{L}_{\alpha_1+\alpha_2+\alpha_3} \subset I$ and we finally get by following this process that $\mathfrak{L}_{\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n} = \mathfrak{L}_{\epsilon_\delta \delta} \subset I$ for some $\epsilon_\delta \in \pm 1$. From here we have $H_{[\alpha]} \subset I$ and as consequence, taking also into account that equation (3) allows us to assert $[H_{[\alpha]}, \mathfrak{L}_\delta] = \mathfrak{L}_\delta$ for any $\delta \in [\alpha]$, that $V_{[\alpha]} \subset I$. We have showed $I = \mathfrak{L}_{[\alpha]}$ and so $\mathfrak{L}_{[\alpha]}$ is split-simple.

Proposition 2.1 is proved.

Theorem 2.1. *Any centerless strongly split Lie algebra is the direct sum of split-ideals, each one being a split-simple strongly split Lie algebra.*

Proof. Since we can write the disjoint union $\Lambda = \bigcup_{[\alpha] \in \Lambda/\sim} [\alpha]$ we have $\mathfrak{L} = \sum_{[\alpha] \in \Lambda/\sim} \mathfrak{L}_{[\alpha]}$. Let us now verify the direct character of the sum: given $x \in \mathfrak{L}_{[\alpha]} \cap \sum_{\substack{[\beta] \in \Lambda/\sim \\ \beta \nsim \alpha}} \mathfrak{L}_{[\beta]}$ since by equations (2) and (3) we have $[\mathfrak{L}_{[\alpha]}, \mathfrak{L}_{[\beta]}] = 0$ for $[\alpha] \neq [\beta]$, we obtain

$$\left[x, \mathfrak{L}_{[\alpha]}\right] + \left[x, \sum_{\substack{[\beta] \in \Lambda/\sim \\ \beta \nsim \alpha}} \mathfrak{L}_{[\beta]}\right] = 0.$$

From here $[x, \mathfrak{L}] = 0$ and so $x = 0$, as desired. Consequently we can write

$$\mathfrak{L} = \bigoplus_{[\alpha] \in \Lambda \backslash \sim} \mathfrak{L}_{[\alpha]}.$$

Finally, Proposition 2.1 completes the proof.

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Received 15.01.14,
after revision — 03.05.18