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COEFFICIENT ESTIMATES FOR TWO SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

КОЕФІЦІЄНТНІ ОЦІНКИ ДЛЯ ДВОХ ПІДКЛАСІВ АНАЛІТИЧНИХ ТА БІУНІВАЛЕНТНИХ ФУНКЦІЙ

We introduce two new subclasses of the class σ of analytic and bi-univalent functions in the open unit disk U. Furthermore, we obtain the estimates for the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions from these new subclasses.

Введено два нових підкласи класу σ аналітичних та біунівалентних функцій у відкритому одиничному крузі U. Крім того, отримано оцінки для перших двох коефіцієнтів Тейлора–Маклорена $|a_2|$ та $|a_3|$ для функцій із цих нових підкласів.

1. Introduction. Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $\mathbb{C}$ being, as usual, the set of complex numbers. We also denote by S the subclass of all functions in A which are univalent in U. Some of the important and well-investigated subclasses of the univalent function class S include (for example) the class $S^*(\alpha)$ of starlike functions of order α in U and the class $K(\alpha)$ of convex functions of order α in U. By definition, we have

$$S^*(\alpha) := \left\{ f \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \le \alpha < 1, \quad z \in U \right\}$$

and

$$K(\alpha) := \left\{ f \in S : f'(0) \ne 0, \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad 0 \le \alpha < 1, \quad z \in U \right\}.$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function φ, which (by definition) is analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for all $z \in U$, such that $f(z) = g(\varphi(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

By using the method of differential subordination Obradovic [21] gave some criteria for univalence expressing by $\Re\{f'(z)\} > 0$, for the linear combinations

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{1}{f'(z)}.$$

In [24] Silverman investigated an expression involving the quotient of the analytic representations of convex and starlike functions. Precisely, for $0 < b \le 1$ he considered the class

$$G_b := \left\{ f \in A \colon \; \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \quad z \in U \right\}$$

and proved that $G_b \subset S^*(2/(1+\sqrt{1+8b})$.

For each $f \in S$, the Koebe one-quarter theorem [11] ensures the image of U under f contains a disk of radius $1/4$. Thus every univalent function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \qquad z \in U,$$

and

$$f(f^{-1}(w)) = w, \qquad |w| < r_0(f), \qquad r_0(f) \leq \frac{1}{4}.$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \ldots .$$

A function $f \in A$ is said to be bi-univalent in U if f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions in U given by (1). The familiar Koebe function is not a member of σ because it maps the unit disk U univalently onto the entire complex plane minus a slit along the line $-\frac{1}{4}$ to $-\infty$. Hence the image domain does not contain the unit disk U.

In 1985 Louis de Branges [3] proved the celebrated Bieberbach Conjecture which states that, for each $f(z) \in S$ given by the Taylor–Maclaurin series expansion (1), the following coefficient inequality holds true:

$$|a_n| \leq n \qquad (n \in N - \{1\}),$$

N being the set of positive integers. The class of analytic bi-univalent functions was first introduced and studied by Lewin [15], where it was proved that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Brannan and Taha [6] and Taha [29] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For further historical account of functions in the class σ, see the work by Srivastava et al. [26] (see also [5, 6]). In fact, the above-cited recent pioneering work of Srivastava et al. [26] has apparently revived the study of analytic and bi-univalent functions in recent years; it was followed by such works as those by Frasin and Aouf [12], Xu et al. [31, 32], Hayami [14], and others (see, for example, [1, 2, 7–10, 13, 16–19, 22, 23, 25, 27, 28, 30]).

In the present investigation, we derive estimates on the initial coefficients $|a_2|$ and $|a_3|$ of two new subclass of the bi-univalent function class σ.

2. Coefficient estimates. In the section, it is assumed that ϕ is an analytic function with positive real part in the unit disk U, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \qquad B_1 > 0.$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \qquad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n, \quad z \in U. \tag{2}$$

It is well known that (see [20, p. 172])

$$|b_1| \leq 1, \qquad |b_2| \leq 1 - |b_1|^2, \qquad |c_1| \leq 1, \qquad |c_2| \leq 1 - |c_1|^2. \tag{3}$$

By a simple calculation, we have

$$\phi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \ldots, \quad z \in U, \tag{4}$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \ldots, \quad w \in U. \tag{5}$$

Definition 1. *A function $f \in \sigma$ is said to be in the class $H_\sigma^\lambda(\phi)$, $\lambda \geq 1$ if the following subordinations holds*:

$$\lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\lambda)\frac{1}{f'(z)} \prec \phi(z), \qquad \lambda \geq 1, \quad z \in U,$$

and

$$\lambda \left(1 + \frac{wg''(w)}{g'(w)}\right) + (1-\lambda)\frac{1}{g'(w)} \prec \phi(w), \qquad \lambda \geq 1, \quad w \in U,$$

where $g(w) := f^{-1}(w)$.

Theorem 1. *If f given by* (1) *is in the class $H_\sigma^\lambda(\phi)$, $\lambda \geq 1$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{4(2\lambda-1)^2 B_1 + \left|(\lambda+1)B_1^2 - 4(2\lambda-1)^2 B_2\right|}}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{\lambda+1}, & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{4(2\lambda-1)^2 B_1 |B_2| + B_1 \left|(\lambda+1)B_1^2 - 4(2\lambda-1)^2 B_2\right|}{(\lambda+1)\left[4(2\lambda-1)^2 B_1 + \left|(\lambda+1)B_1^2 - 4(2\lambda-1)^2 B_2\right|\right]}, & \text{if} \quad |B_2| > B_1. \end{cases}$$

Proof. Let $f \in H_\sigma^\lambda(\phi)$, $\lambda \geq 1$. Then there are analytic functions $u, v \colon U \to U$ given by (2) such that

$$\lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\lambda)\frac{1}{f'(z)} = \phi(u(z)), \quad \lambda \geq 1, \tag{6}$$

and

$$\lambda \left(1 + \frac{wg''(w)}{g'(w)}\right) + (1-\lambda)\frac{1}{g'(w)} = \phi(v(w)), \quad \lambda \geq 1, \tag{7}$$

where $g(w) = f^{-1}(w)$. Since

$$\lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\lambda)\frac{1}{f'(z)} =$$

$$= 1 + 2(2\lambda - 1)a_2 z + \left[(9\lambda - 3)a_3 + 4(1 - 2\lambda)a_2^2\right] z^2 + \ldots$$

and

$$\lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) + (1-\lambda)\frac{1}{g'(w)} = 1 - 2(2\lambda - 1)a_2 w +$$
$$+ \left[(10\lambda - 2)a_2^2 - (9\lambda - 3)a_3\right] w^2 + \ldots,$$

it follows from (4), (5), (6) and (7) that

$$2(2\lambda - 1)a_2 = B_1 b_1, \tag{8}$$

$$(9\lambda - 3)a_3 + 4(1 - 2\lambda)a_2^2 = B_1 b_2 + B_2 b_1^2, \tag{9}$$

$$-2(2\lambda - 1)a_2 = B_1 c_1, \tag{10}$$

$$(10\lambda - 2)a_2^2 - (9\lambda - 3)a_3 = B_1 c_2 + B_2 c_1^2. \tag{11}$$

From (8) and (10), we get

$$b_1 = -c_1, \tag{12}$$

$$8(2\lambda - 1)^2 a_2^2 = B_1^2\left(b_1^2 + c_1^2\right). \tag{13}$$

By adding (11) to (9), further computations using (13) lead to

$$\left[2(1+\lambda)B_1^2 - 8(2\lambda - 1)^2 B_2\right] a_2^2 = B_1^3(b_2 + c_2), \tag{14}$$

(12), (14), together with (3), give that

$$\left|(1+\lambda)B_1^2 - 4(2\lambda - 1)^2 B_2\right| \left|a_2^2\right| \le B_1^3(1 - \left|b_1^2\right|). \tag{15}$$

From (8) and (15) we obtain

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{4(2\lambda - 1)^2 B_1 + \left|(1+\lambda)B_1^2 - 4(2\lambda - 1)^2 B_2\right|}}.$$

Next, from (11) and (9), we have

$$(1+\lambda)(9\lambda - 3)a_3 = (5\lambda - 1)B_1 b_2 + 2(2\lambda - 1)B_1 c_2 + (9\lambda - 3)B_2 b_1^2.$$

Then, in view of (3), we get

$$(1+\lambda)\left|a_3\right| \le B_1 + \left[|B_2| - B_1\right]\left|b_1^2\right|.$$

Notice that

$$\left|b_1^2\right| = \frac{4(2\lambda - 1)^2}{B_1^2}\left|a_2^2\right| \le \frac{4(2\lambda - 1)^2 B_1}{4(2\lambda - 1)^2 B_1 + \left|(1+\lambda)B_1^2 - 4(2\lambda - 1)^2 B_2\right|},$$

we obtain

$$|a_3| \leq \begin{cases} \dfrac{B_1}{\lambda+1}, & \text{if} \quad |B_2| \leq B_1, \\ \dfrac{4(2\lambda-1)^2 B_1 |B_2| + B_1 \left|(\lambda+1)B_1^2 - 4(2\lambda-1)^2 B_2\right|}{(\lambda+1)\left[4(2\lambda-1)^2 B_1 + \left|(\lambda+1)B_1^2 - 4(2\lambda-1)^2 B_2\right|\right]}, & \text{if} \quad |B_2| > B_1. \end{cases}$$

Theorem 1 is proved.

If we set

$$\lambda = 1, \phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \ldots, \quad z \in U,$$

in Definition 1 of the bi-univalent function class $H_\sigma^\lambda(\phi)$, we obtain the class of bi-convex functions $H_\sigma(\phi)$ given by Definition 2 below.

Definition 2. *A function $f \in \sigma$ is said to be in the class $H_\sigma(\phi)$, if the following conditions hold true*:

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad z \in U,$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \phi(w), \quad w \in U,$$

where $g(w) := f^{-1}(w)$.

Using the parameter setting of Definition 2 in the Theorem 1, we get the following corollary.

***Corollary* 1.** *Let the function $f \in H_\sigma(\phi)$, be given by* (1). *Then*

$$|a_2| \leq 1 \qquad \textit{and} \qquad |a_3| \leq 1.$$

This is a special case of Theorem 4.1 (with $\beta = 0$) in Brannan and Taha [6].

Definition 3. *A function $f \in \sigma$ is said to be in the class $K_\sigma(\phi)$ if and only if*

$$\frac{1 + \dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} \prec \phi(z), \qquad \frac{1 + \dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} \prec \phi(w),$$

where $g(w) := f^{-1}(w)$.

Theorem 2. *If f given by* (1) *is in the class $K_\sigma(\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{B_1 + |B_2|}},$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{4}, & \textit{if} \quad B_1 \leq \dfrac{1}{4}, \\ \left[1 - \dfrac{1}{4B_1}\right] \dfrac{B_1^3}{B_1 + |B_2|} + \dfrac{B_1}{4}, & \textit{if} \quad B_1 > \dfrac{1}{4}. \end{cases}$$

Proof. Let $f \in K_\sigma(\phi)$. Then there are analytic functions $u, v\colon U \to U$ given by (2) such that

$$\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = \phi(u(z)) \tag{16}$$

and

$$\frac{1+\dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = \phi(v(w)), \tag{17}$$

where $g(w) = f^{-1}(w)$. Since

$$\frac{1+\dfrac{zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}} = 1 + a_2 z + 4\left(a_3 - a_2^2\right) z^2 + \ldots$$

and

$$\frac{1+\dfrac{wg''(w)}{g'(w)}}{\dfrac{wg'(w)}{g(w)}} = 1 - a_2 w - 4\left(a_3 - a_2^2\right) w^2 + \ldots,$$

it follows from (4), (5), (16) and (17) that

$$a_2 = B_1 b_1, \tag{18}$$

$$4\left(a_3 - a_2^2\right) = B_1 b_2 + B_2 b_1^2, \tag{19}$$

$$-a_2 = B_1 c_1, \tag{20}$$

$$-4\left(a_3 - a_2^2\right) = B_1 c_2 + B_2 c_1^2. \tag{21}$$

From (18) and (20), we get

$$b_1 = -c_1 \tag{22}$$

and

$$2a_2^2 = B_1^2\left(b_1^2 + c_1^2\right). \tag{23}$$

By adding (19) to (21), further computations using (23) lead to

$$2B_2 a_2^2 = -B_1^3(b_2 + c_2), \tag{24}$$

(22), (24), together with (3), give that

$$|B_2|\left|a_2^2\right| \le B_1^3(1 - \left|b_1^2\right|). \tag{25}$$

From (18) and (25) we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{B_1 + |B_2|}}.$$

Next, from (19) and (21), we have

$$8a_3 = 8a_2^2 + B_1(b_2 - c_2). \tag{26}$$

From (3), (18), (22) and (26), it follows that

$$|a_3| \le a_2^2 + \frac{B_1}{4}\left(1 - \left|b_1^2\right|\right) = \left(1 - \frac{1}{4B_1}\right)a_2^2 + \frac{B_1}{4} \le$$

$$\le \begin{cases} \dfrac{B_1}{4}, & \text{if} \quad B_1 \le \dfrac{1}{4}, \\ \left[1 - \dfrac{1}{4B_1}\right]\dfrac{B_1^3}{B_1 + |B_2|} + \dfrac{B_1}{4}, & \text{if} \quad B_1 > \dfrac{1}{4}. \end{cases}$$

Theorem 2 is proved.

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