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A-CLUSTER POINTS VIA IDEALS

A-КЛАСТЕРНІ ТОЧКИ В ТЕРМІНАХ ІДЕАЛІВ

Following the line of the recent work by Savaş et al., we apply the notion of ideals to A-statistical cluster points. We get necessary conditions for the two matrices to be equivalent in a sense of $A^{\mathcal{I}}$-statistical convergence. In addition, we use Kolk's idea to define and study $\mathcal{B}^{\mathcal{I}}$-statistical convergence.

Ми продовжуємо дослідження, розпочате в нещодавній роботі Саваша та ін., і застосовуємо поняття ідеалів до A-статистичних кластерних точок. Отримано необхідні умови для того, щоб дві матриці були еквівалентними в сенсі $A^{\mathcal{I}}$-статистичної збіжності. Крім того, ми застосовуємо ідею Колка для того, щоб визначити і вивчити поняття $\mathcal{B}^{\mathcal{I}}$-статистичної збіжності.

1. Introduction and background. In [10], Fridy and Orhan introduced the concepts of statistical limit superior and inferior. In [2], Connor and Kline extended the concept of a statistical limit (cluster) point of a number sequence to a A-statistical limit (cluster) point where A is a nonnegative regular summability matrix. In [4], Demirci extended the concepts of statistical limit superior and inferior to A-statistical limit superior and inferior and given some A-statistical analogue of properties of statistical limit superior and inferior for a sequence of real numbers. In [4], Kolk generalized the idea of A-statistical convergence to $\mathcal{B}$-statistical convergence by using the idea of $\mathcal{B}$-summability (or F-convergence) due to Steiglitz [30]. More works on matrix summability can be seen from [6], where many references can be found.

On the other hand, the notion of ideal convergence was introduced first by P. Kostyrko et al. [18] as an interesting generalization of statistical convergence [7, 31]. More recent applications of ideals can be seen from [3, 13 – 15, 22 – 24, 26, 27] where more references can be found.

Naturally the purpose of this paper is to unify the above approaches and present the idea of A-summability with respect to ideal concept and make certain observations. Further, we produce $\mathcal{B}$-analogues via ideals of the results of Mursaleen and Edely [21].

First, we introduce some notation. Let $A = (a_{nk})$ denote a summability matrix which transforms a number sequence $x = (x_k)$ into the sequence Ax whose nth term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$.

The notion of a statistically convergent sequence can be defined using the asymptotic density of subsets of the set of positive integers $\mathbb{N} = \{1, 2, \ldots\}$. For any $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote

$$K(n) := |K \cap \{1, 2, \ldots, n\}|$$

and we define lower and upper asymptotic density of the set K by the formulas

$$\underline{\delta}(K) := \liminf_{n\to\infty} \frac{K(n)}{n}, \qquad \overline{\delta}(K) := \limsup_{n\to\infty} \frac{K(n)}{n}.$$

If $\underline{\delta}(K) = \overline{\delta}(K) =: \delta(K)$, then the common value $\delta(K)$ is called the asymptotic density of the set K and

$$\delta(K) = \lim_{n\to\infty} \frac{K(n)}{n}.$$

Obviously, the density $\delta(K)$ (if it exist) lie in the unit interval $[0,1]$

$$\delta\left(K\right)=\lim_{n}\frac{1}{n}K\left(n\right)=\lim_{n}\left(C_{1}\chi_{K}\right)_{n}=\lim_{n}\frac{1}{n}\sum_{k=1}^{n}\chi_{K}\left(k\right),$$

if it exists, where C_1 is the Cesaro mean of order one and χ_K is the characteristic function of the set K [8].

The notion of statistical convergence was originally defined for sequences of numbers in the paper [7] and also in [29]. We say that a number sequence $x=(x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon>0$ we have $\delta\left(K\left(\varepsilon\right)\right)=0$, where $K\left(\varepsilon\right)=\{k\in\mathbb{N}\colon\ |x_k-L|\geq\varepsilon\}$ and in such situation we will write $L=\text{st-}\lim x_k$.

Statistical convergence can be generalized by using a regular nonnegative summability matrix A in place of C_1. Following Freedman and Sember [8], we say that a set $K\subseteq\mathbb{N}$ has A-density if

$$\delta_{A}\left(K\right)=\lim_{n}\sum_{k\in K}a_{nk}=\lim_{n}\sum_{k=1}^{\infty}a_{nk}\chi_{K}\left(k\right)=\lim_{n}\left(A\chi_{K}\right)_{n}$$

exists where A is a nonnegative regular summability matrix.

The number sequence $x=(x_k)_{k\in\mathbb{N}}$ is said to be A-statistically convergent to L if for every $\varepsilon>0$, $\delta_A\left(\{k\in\mathbb{N}\colon\ |x_k-L|\geq\varepsilon\}\right)=0$. In this case it is denoted as $\text{st}_A\text{-}\lim x_k=L$ [2, 20].

For $i=1,2,\ldots,$ let $\mathcal{B}_i=\left(b_{nk}\left(i\right)\right)$ be an infinite matrix of complex (or real) numbers. Let $\mathcal{B}$ denote the sequence of matrices $(\mathcal{B}_i)$. Then a sequence $x\in\ell_\infty$, the space of bounded sequences, is said to be $F_{\mathcal{B}}$-convergent or $\mathcal{B}$-summable to some number L if $\sum_{k=1}^{\infty}b_{nk}\left(i\right)x_k$ converges to L as n tends to ∞ uniformly for $i=1,2,\ldots$. L is said to be the $\mathcal{B}$-limit of x, written $\mathcal{B}\text{-}\lim x=L$ (denotes the generalized limit) or $(\mathcal{B}_i x)\to L$, and we say $(\mathcal{B}_i x)$ is convergent to L.

A sequence of matrices $\mathcal{B}=(\mathcal{B}_i)$ is regular (cf. [1, 30]) if and only if

(i) for each $k=1,2,\ldots,$ $\lim_{n\to\infty}b_{nk}\left(i\right)=0$ uniformly for $i=1,2,\ldots;$

(ii) $\lim_{n\to\infty}\sum_k b_{nk}\left(i\right)=1$ uniformly for $i=1,2,\ldots;$

(iii) for each $n,\ i=1,2,\ldots,$ $\sum_{k=1}|b_{nk}\left(i\right)|<\infty$, and there exists integers N,M such that $\sum_{k=1}|b_{nk}\left(i\right)|<M$ for $n\geq N$ and all $i=1,2,\ldots$.

In [17], Kolk introduced the following:

An index set K is said to have $\mathcal{B}$-density $\delta_{\mathcal{B}}\left(K\right)$ equal to d, if the characteristic sequence of K is $\mathcal{B}$-summable to d, i.e.,

$$\lim_{n}\sum_{k\in K}b_{nk}\left(i\right)=d,\quad\text{uniformly in}\quad i,$$

where by an index set we mean a set $K=\{k_i\}\subset\mathbb{N}$, $k_i<k_{i+1}$ for all i. For $\mathcal{B}=\mathcal{B}_1$, it is reduced to uniform statistical convergence [25].

Let $\mathcal{R}^+$ denote the set of all regular methods $\mathcal{B}$ with $b_{nk}\left(i\right)\geq 0$ for all n,k and i.

Let $\mathcal{B}\in\mathcal{R}^+$. A sequence $x=(x_k)$ is called $\mathcal{B}$-statistically convergent to the number L, if for every $\varepsilon>0$

$$\delta_{\mathcal{B}}\left(\{k\in\mathbb{N}\colon\ |x_k-L|\geq\varepsilon\}\right)=0$$

and we write $\text{st}_{\mathcal{B}}\text{-}\lim x=L$.

The notion of statistical convergence was further generalized in the paper [18, 19] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a nonempty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e., $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal.

Recall the generalization of statistical convergence from [18, 19].

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)_{k\in\mathbb{N}}$ be a sequence of points in a metric space (X, ρ). We say that the sequence x is $\mathcal{I}$-convergent (or $\mathcal{I}$-converges) to a point $\xi \in X$, and we denote it by $\mathcal{I}\text{-}\lim x = \xi$, if for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{k \in \mathbb{N}\colon \rho(x_k, \xi) \geq \varepsilon\} \in \mathcal{I}.$$

This generalizes the notion of usual convergence, which can be obtained when we take for $\mathcal{I}$ the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. A sequence is statistically convergent if and only if it is $\mathcal{I}_\delta$-convergent, where $\mathcal{I}_\delta := \{K \subset \mathbb{N}\colon \delta(K) = 0\}$ is the admissible ideal of the sets of zero asymptotic density.

The concept of $A^{\mathcal{I}}$-statistically convergent was studied in [28] and the following definition was given:

Definition 1. *Let $A = (a_{nk})$ be a nonnegative regular matrix. A sequence $(x_k)_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$*

$$\left\{ n \in \mathbb{N}\colon \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\}$. In this case we write $L = \mathcal{I}\text{-st}_A\text{-}\lim x_k$.

By $\mathcal{I}\text{-st}_A$ we denote the set of all $A^{\mathcal{I}}$-statistically convergent sequences.

We say that a set $K \subseteq \mathbb{N}$ has $A^{\mathcal{I}}$-density if

$$\delta_{A^{\mathcal{I}}}(K) := \mathcal{I}\text{-}\lim_n \sum_{k\in K} a_{nk} = \mathcal{I}\text{-}\lim_n \sum_{k=1}^{\infty} a_{nk}\chi_K(k) = \mathcal{I}\text{-}\lim_n (A\chi_K)_n,$$

exists where A is a nonnegative regular summability matrix. Then a sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for each $\varepsilon > 0$ the set $K(\varepsilon)$ has $A^{\mathcal{I}}$-density zero, where $K(\varepsilon) = \{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\}$.

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $A^{\mathcal{I}}$-statistically convergent is the A-statistical convergence introduced by [2, 20]. Also $A^{\mathcal{I}}$-density coincides with usual A-density in [8].

2. Consistency of $A^{\mathcal{I}}$-statistical convergence. In this section we study the concepts of $A^{\mathcal{I}}$-statistical cluster points. The result are analogues to those given by Demirci [5]. These notions generalize the notions of A-statistical cluster points. Also we get necessary conditions on the matrices A and B so that A and B are equivalent in the $A^{\mathcal{I}}$-statistical convergence sense.

Following the line of Savaş et al. [28] we now introduce the following definition using ideals.

Definition 2. *Let $\mathcal{I}$ be an ideal of $\mathcal{P}(\mathbb{N})$. A number L is said to be an $A^{\mathcal{I}}$-statistical cluster point of the number sequence $x = (x_k)$ if for each $\varepsilon > 0$, $\delta_{A^{\mathcal{I}}}(K_\varepsilon) \neq 0$, where $K_\varepsilon = \{k \in \mathbb{N}\colon |x_k - L| < \varepsilon\}$. We denote the set of all $A^{\mathcal{I}}$-statistically cluster points of x by $\Gamma_{A^{\mathcal{I}}}(x)$.*

Note that the statement $\delta_{A^{\mathcal{I}}}(K_\varepsilon) \neq 0$ means that either $\delta_{A^{\mathcal{I}}}(K_\varepsilon) > 0$ or K_ε fails to have $A^{\mathcal{I}}$-density.

***Remark* 1.** If $\mathcal{I} = \mathcal{I}_f$ and $A = (C_1)$, then the above Definition 2 yields the usual definition of A-statistical cluster point of the number sequence introduced by [9].

Definition 3. *If* $\mathcal{I}\text{-st}_A \supset \mathcal{I}\text{-st}_B$, *$A$ is said to be stronger than B in the $\mathcal{I}$-statistical convergence sense.*

Definition 4. *Matrices A and B are called consistent in the $\mathcal{I}$-statistical convergence sense if* $\mathcal{I}\text{-st}_A\text{-}\lim x = \mathcal{I}\text{-st}_B\text{-}\lim x$ *whenever* $x \in \mathcal{I}\text{-st}_A \cap \mathcal{I}\text{-st}_B$. *If A is stronger than B in the $\mathcal{I}$-statistical convergence sense and consistent with B in the $\mathcal{I}$-statistical convergence sense, then write* $A \overset{\mathcal{I}\text{-st}}{\supset} B$. *If* $A \overset{\mathcal{I}\text{-st}}{\supset} B$ *and* $B \overset{\mathcal{I}\text{-st}}{\supset} A$, *are called equivalent in the $\mathcal{I}$-statistical convergence sense. In this case it is denoted as* $A \overset{\mathcal{I}\text{-st}}{\sim} B$ (*see* [12]).

Throughout this section $A = (a_{nk})$ and $B = (b_{nk})$ will denote nonnegative regular summability matrices.

Theorem 1. *If the condition*

$$\mathcal{I}\text{-}\limsup_n \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0 \tag{1}$$

holds, then $\delta_{A^{\mathcal{I}}}(K) = 0$ *if and only if* $\delta_{B^{\mathcal{I}}}(K) = 0$ *for every* $K \subseteq \mathbb{N}$.

Proof. If $\delta_{A^{\mathcal{I}}}(K) = 0$, then

$$\left\{ n \in \mathbb{N} \colon \sum_{k \in K} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

for any $\delta > 0$. Since

$$|(A\chi_K)_n - (B\chi_K)_n| \leq \sum_{k \in K} |a_{nk} - b_{nk}| \leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}|,$$

we have $\mathcal{I}\text{-}\limsup_n |(A\chi_K)_n - (B\chi_K)_n| = 0$ by (1), which implies

$$\delta_{B^{\mathcal{I}}}(K) = \mathcal{I}\text{-}\lim_n \sum_{k \in K} b_{nk} = 0.$$

Sufficiency follows from the symmetry.

Hence we can get the following results from Theorem 1.

Theorem 2. *If A and B satisfy the condition* (1), *then*

(i) $\mathcal{I}\text{-st}_A = \mathcal{I}\text{-st}_B$,

(ii) $\Gamma_{A^{\mathcal{I}}}(x) = \Gamma_{B^{\mathcal{I}}}(x)$ *for a real number sequence x.*

The $\mathcal{I}$-statistical limits in (i) of Theorem 2 agree (i.e., $\mathcal{I}\text{-st}_B\text{-}\lim x = L$ implies $\mathcal{I}\text{-st}_A\text{-}\lim x = L$). Therefore, if A and B satisfy condition (1) of Theorem 1, then A and B are consistent in the $\mathcal{I}$-statistical convergence sense.

Note that the support sets generated by nonnegative summability methods A and B can be used to determine when, if a sequence x is both $A^{\mathcal{I}}$-statistically convergent and $B^{\mathcal{I}}$-statistically convergent, the $A^{\mathcal{I}}$-statistical and $B^{\mathcal{I}}$-statistical limits of x agree. In [2], Connor and Kline, using the „$\beta\mathbb{N}$ program" have shown that A and B assign the same statistical limit to x if $K_A \cap K_B \neq \varnothing$, where the sets K_A and K_B are the support sets of the nonnegative regular summability matrices A and B.

The next corollary shows that we have the same result under different conditions.

***Corollary* 1.** *If A and B satisfy the conditions* (1) *of Theorem* 1, *then* $A \overset{\mathcal{I}\text{-st}}{\sim} B$.

Definition 5. *The real number sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-statistically bounded if there is a number K such that $\delta_{A^{\mathcal{I}}}(\{k \in \mathbb{N} \colon |x_k| > K\}) = 0$.*

Recall that $A^{\mathcal{I}}$-statistically boundedness of real number sequences implies that $\mathcal{I}\text{-st}_A\text{-}\limsup x$ and $\mathcal{I}\text{-st}_A\text{-}\liminf x$ are finite and $\mathcal{I}\text{-st}_A\text{-}\limsup x$ and $\mathcal{I}\text{-st}_A\text{-}\liminf x$ are the greatest and least $A^{\mathcal{I}}$-statistically cluster point of such an x [16].

For any complex number sequence $x = (x_k)$ the A-statistical core of x is given by

$$\mathrm{st}_A\text{-core}\{x\} = \bigcap\nolimits_{H \in \mathcal{H}(x)} H,$$

where $\mathcal{H}(x)$ is the collection of all closed half-planes H that satisfy $\delta_A(\{k \in \mathbb{N} \colon x_k \in H\}) = 1$ (see [4]).

From Theorem 6 in [4], it is shown that for every A-statistically bounded complex number sequence $x = (x_k)$

$$\mathrm{st}_A\text{-core}\{x\} = \bigcap\nolimits_{z \in \mathbb{C}} B_x(z),$$

where $B_x(z) = \{w \in \mathbb{C} \colon |w - z| \leq \mathcal{I}\text{-st}_A\text{-}\limsup_k |x_k - z|\}$. When $A = C_1$ we shall simply write st-core instead of st_{C_1}-core (see [11]).

Recall that the core of any A-statistically bounded real number sequence x, that is, $\mathrm{st}_A\text{-core}\{x\}$, is the interval $[\mathrm{st}_A\text{-}\liminf x,\ \mathrm{st}_A\text{-}\limsup x]$ [4]. In analogy to the $\mathrm{st}_A\text{-core}\{x\}$ we first give a definition of $A^{\mathcal{I}}$-core of bounded real number sequence x as follows.

Definition 6. *If x is any $A^{\mathcal{I}}$-statistically bounded real number sequence, then we define its $A^{\mathcal{I}}$-core by*

$$\left[\mathcal{I}\text{-st}_A\text{-}\liminf x,\ \mathcal{I}\text{-st}_A\text{-}\limsup x\right].$$

We use $\mathcal{I}\text{-st}_A\text{-core}(x)$ to denote $A^{\mathcal{I}}$-core of real number sequence x.

Hence we can get the following from (ii) of Theorem 2.

***Corollary* 2.** *If A and B satisfy the conditions* (1) *of Theorem* 1, *then* $\mathcal{I}\text{-st}_A\text{-core}\{x\} = \mathcal{I}\text{-st}_B\text{-core}\{x\}$ *for every bounded real sequence x.*

Let $\mathcal{I} = \mathcal{I}_f$. Then all these results imply the similar theorems for A-statistical cluster points which are investigated in [5].

3. $\mathcal{B}$-statistical convergence via ideals. In this section, we produce $\mathcal{B}$-analogues via ideals of the results of Fridy and Orhan [10].

We give some analogue definitions for the method $\mathcal{B}$.

Definition 7. *A sequence $x = (x_k)_{k \in \mathbb{N}}$ is called $\mathcal{B}^{\mathcal{I}}$-statistically convergent to the number L, if for any $\varepsilon > 0$ and $\delta > 0$*

$$\left\{ n \in \mathbb{N} \colon \sum_{k \in K(\varepsilon)} b_{nk}(i) \geq \delta \ \text{ for all } \ i = 1, 2, \ldots \right\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in \mathbb{N} \colon |x_k - L| \geq \varepsilon\}$. In this case we write $L = \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\lim x_k$.

We say that a set $K \subseteq \mathbb{N}$ has $\mathcal{B}^{\mathcal{I}}$-density if

$$\delta_{\mathcal{B}^{\mathcal{I}}}(K) := \mathcal{I}\text{-}\lim_n \sum_{k \in K} b_{nk}(i) = \mathcal{I}\text{-}\lim_n \sum_{k=1}^{\infty} b_{nk}(i)\, \chi_K(k) =$$

$$= \mathcal{I}\text{-}\lim_n (\mathcal{B}\chi_K)_n, \quad \text{uniformly for} \quad i = 1, 2, \ldots$$

exists. Then a sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be $\mathcal{B}^{\mathcal{I}}$-statistically convergent to L if for each $\varepsilon > 0$ the set $K(\varepsilon)$ has $\mathcal{B}^{\mathcal{I}}$-density zero, where $K(\varepsilon) = \{k \in \mathbb{N} \colon |x_k - L| \geq \varepsilon\}$.

Throughout the paper by $\delta_{\mathcal{B}^{\mathcal{I}}}(K) \neq 0$ we mean that either $\delta_{\mathcal{B}^{\mathcal{I}}}(K) > 0$ or K fails to have $\mathcal{B}$-density.

Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Let $\mathcal{I} = \mathcal{I}_f$, then $\mathcal{B}^{\mathcal{I}}$-statistically convergent is the $\mathcal{B}$-statistical convergence introduced by [21]. In particular, if $\mathcal{I} = \mathcal{I}_f$ and $\mathcal{B} = (C_1)$, then $\mathcal{B}^{\mathcal{I}}$-statistical convergence is reduced the usual statistical convergence. For $\mathcal{B} = (A)$, it is reduced to $A^{\mathcal{I}}$-statistical cluster point [16].

Definition 8. *Let $\mathcal{I}$ be an ideal of $\mathcal{P}(\mathbb{N})$. The number ζ is said to be $\mathcal{B}^{\mathcal{I}}$-statistical cluster point of a sequence $x = (x_k)$ if for each $\varepsilon > 0$, $\delta_{\mathcal{B}^{\mathcal{I}}}(K_\varepsilon) \neq 0$, where $K_\varepsilon = \{k \in \mathbb{N} \colon |x_k - \zeta| < \varepsilon\}$. We denote the set of all $\mathcal{B}^{\mathcal{I}}$-statistically cluster points of x by $\Gamma_{\mathcal{B}^{\mathcal{I}}}(x)$.*

Note that for $\mathcal{B} = (A)$ in Definition 8, we get $A^{\mathcal{I}}$-statistical cluster point [16]. For $\mathcal{B} = (C_1)$ and $\mathcal{I} = \mathcal{I}_f$, these are reduced to the usual statistical cluster point [9]. For a number sequence $x = (x_k)$, we write

$$M_g = \{g \in \mathbb{R} \colon \delta_{\mathcal{B}^{\mathcal{I}}}\{k \colon x_k > g\} \neq 0\} \quad \text{and} \quad M^f = \{f \in \mathbb{R} \colon \delta_{\mathcal{B}^{\mathcal{I}}}\{k \colon x_k < f\} \neq 0\}.$$

Then we define the $\mathcal{B}$-statistical limit superior and $\mathcal{B}$-statistical limit inferior of x as follows:

$$\mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\limsup x = \begin{cases} \sup M_g, & M_g \neq \varnothing, \\ -\infty, & M_g = \varnothing, \end{cases}$$

and

$$\mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\liminf x = \begin{cases} \inf M^f, & M^f \neq \varnothing, \\ +\infty, & M^f = \varnothing. \end{cases}$$

Definition 9. *The real number sequence $x = (x_k)$ is said to be $\mathcal{B}^{\mathcal{I}}$-statistically bounded if there is a number K such that*

$$\delta_{\mathcal{B}^{\mathcal{I}}}(\{k \in \mathbb{N} \colon |x_k| > K\}) = 0.$$

The next statement is an analogue of Theorem 2.7 of [21].

Theorem 3. (a) *If $\beta = \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\limsup x$ is finite, then for each $\varepsilon > 0$*

$$\delta_{\mathcal{B}^{\mathcal{I}}}(\{k \in \mathbb{N} \colon x_k > \beta - \varepsilon\}) \neq 0 \qquad \text{and} \qquad \delta_{\mathcal{B}^{\mathcal{I}}}(\{k \in \mathbb{N} \colon x_k > \beta + \varepsilon\}) = 0. \tag{2}$$

Conversely, if (2) *holds for each $\varepsilon > 0$ then $\beta = \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\limsup x$.*

(b) *If $\alpha = \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\liminf x$ is finite, then for each $\varepsilon > 0$,*

$$\delta_{\mathcal{B}^{\mathcal{I}}}(\{k \in \mathbb{N} \colon x_k < \alpha + \varepsilon\}) \neq 0 \qquad \text{and} \qquad \delta_{\mathcal{B}^{\mathcal{I}}}(\{k \in \mathbb{N} \colon x_k < \alpha - \varepsilon\}) = 0. \tag{3}$$

Conversely, if (3) *holds for each $\varepsilon > 0$, then $\alpha = \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\liminf x$.*

By Definition 8 we see that Theorem 3 can be interpreted by saying that $\mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\limsup x$ and $\mathcal{I}$-$\text{st}_{\mathcal{B}}\text{-}\liminf x$ are the greatest and the least $\mathcal{B}^{\mathcal{I}}$-statistically cluster points of x.

The next theorem reinforces this observation.

Theorem 4. *For every real sequence x,*

$$\mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\liminf x \leq \mathcal{I}\text{-st}_{\mathcal{B}}\text{-}\limsup x.$$

Proof. First consider the case in which $\mathcal{I}$-st$_\mathcal{B}$-$\limsup x = -\infty$. Hence we have $M_g = \varnothing$, so for every $g \in \mathbb{R}$, $\delta_{\mathcal{B}^\mathcal{I}} \{k : x_k > g\} = 0$ which implies that $\delta_{\mathcal{B}^\mathcal{I}} \{k : x_k \leq g\} = 1$, so for every $f \in \mathbb{R}$, $\delta_{\mathcal{B}^\mathcal{I}} \{k : x_k < f\} \neq 0$. Hence, $\mathcal{I}$-st$_\mathcal{B}$-$\liminf x = -\infty$.

The case in which $\mathcal{I}$-st$_\mathcal{B}$-$\limsup x = +\infty$ needs no proof, so we next assume that $\beta = \mathcal{I}$-st$_\mathcal{B}$-$\limsup x$ is finite, and let $\alpha = \mathcal{I}$-st$_\mathcal{B}$-$\liminf x$. Given $\varepsilon > 0$ we show that $\beta + \varepsilon \in M^f$, so that $\alpha \leq \beta + \varepsilon$. By Theorem 3(a), $\delta_{\mathcal{B}^\mathcal{I}} \left\{k : x_k > \beta + \dfrac{\varepsilon}{2}\right\} = 0$, since $\beta = \sup \{g \in \mathbb{R} : \delta_{\mathcal{B}^\mathcal{I}}\{k : x_k > g\} \neq$ $\neq 0\}$. This implies $\delta_{\mathcal{B}^\mathcal{I}} \left\{k : x_k \leq \beta + \dfrac{\varepsilon}{2}\right\} = 1$, which, in turn, gives $\delta_{\mathcal{B}^\mathcal{I}} \{k : x_k < \beta + \varepsilon\} = 1$. Hence $\beta + \varepsilon \in M^f$, and since ε is arbitrary this proves that $\alpha \leq \beta$.

***Remark* 2.** If $\mathcal{I}$-st$_A$-$\lim x$ exists, then a sequence x is $A^\mathcal{I}$-statistically bounded.

Note that $\mathcal{B}^\mathcal{I}$-statistical boundedness of real number sequences implies that $\mathcal{I}$-st$_\mathcal{B}$-$\limsup$ and $\mathcal{I}$-st$_\mathcal{B}$-$\liminf$ are finite, so that properties (a) and (b) of Theorem 3 hold good.

Theorem 5. *The $\mathcal{B}^\mathcal{I}$-statistically bounded sequence x is $\mathcal{B}^\mathcal{I}$-statistically convergent if and only if* $\mathcal{I}$-st$_\mathcal{B}$-$\liminf x = \mathcal{I}$-st$_\mathcal{B}$-$\limsup x$.

Proof. We prove the *necessity* first. Let $L = \mathcal{I}$-st$_\mathcal{B}$-$\lim x$ and $\varepsilon > 0$. Then

$$\delta_{\mathcal{B}^\mathcal{I}} \left(\{k \in \mathbb{N} : x_k > L + \varepsilon\}\right) = 0 \text{ and } \delta_{\mathcal{B}^\mathcal{I}} \left(\{k \in \mathbb{N} : x_k < L - \varepsilon\}\right) = 0.$$

So for any $g \geq L + \varepsilon$ and $f < L - \varepsilon$, the sets $\delta_{\mathcal{B}^\mathcal{I}} (M_g) = 0$ and $\delta_{\mathcal{B}^\mathcal{I}} \left(M^f\right) = 0$. We conclude $\sup \{g : \delta_{\mathcal{B}^\mathcal{I}} (M_g) \neq 0\} \leq L + \varepsilon$ and $\inf \left\{f : \delta_{\mathcal{B}^\mathcal{I}} \left(M^f\right) \neq 0\right\} \geq L - \varepsilon$. Combining with Theorem 4, we conclude that $L = \mathcal{I}$-st$_\mathcal{B}$-$\liminf x = \mathcal{I}$-st$_\mathcal{B}$-$\limsup x$.

To prove *sufficiency,* suppose that $L = \mathcal{I}$-st$_\mathcal{B}$-$\liminf x = \mathcal{I}$-st$_\mathcal{B}$-$\limsup x$ and x be $\mathcal{B}^\mathcal{I}$-statistical bounded. Then for $\varepsilon > 0$, by (2) and (3), we have

$$\delta_{\mathcal{B}^\mathcal{I}} \left(\left\{k : x_k > L + \frac{\varepsilon}{2}\right\}\right) = 0 \quad \text{and} \quad \delta_{\mathcal{B}^\mathcal{I}} \left(\left\{k : x_k < L - \frac{\varepsilon}{2}\right\}\right) = 0.$$

We conclude that $L = \mathcal{I}$-st$_\mathcal{B}$-$\lim x$.

We state the following result without proof, since the result can be established using same the technique applied for the Theorems 3.3 and 3.4 of [21].

Theorem 6. (i) *If number sequence x is bounded from above and $\mathcal{B}$-summable to the number* $L = \mathcal{I}$-st$_\mathcal{B}$-$\limsup x$, *then x is $\mathcal{B}^\mathcal{I}$-statistical convergent to L.*

(ii) *If number sequence x is bounded from below and $\mathcal{B}$-summable to the number* $L = \mathcal{I}$-st$_\mathcal{B}$-$\liminf x$, *then x is $\mathcal{B}^\mathcal{I}$-statistical convergent to L.*

Let $\mathcal{I} = \mathcal{I}_f$. Then all these results in Section 3 imply the similar theorems for $\mathcal{B}$-statistical convergence which are investigated in [21].

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