UDC 512.5

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PARAMETERS FOR RAMANUJAN'S FUNCTION $\chi(q)$ OF DEGREE FIVE AND THEIR EXPLICIT EVALUATION

ПАРАМЕТРИ ФУНКЦІЇ РАМАНУДЖАНА $\chi(q)$ П'ЯТОГО СТУПЕНЯ ТА ЇХ ЯВНЕ ЗНАХОДЖЕННЯ

We study the ratios of parameters for Ramanujan's function $\chi(q)$ and their explicit values.

Вивчаються відношення параметрів функції Рамануджана $\chi(q)$ та їх явні значення.

1. Introduction. In Chapter 16 of his second notebook [1], Ramanujan develops the theory of Theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} =$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \quad |ab| < 1, \tag{1.1}$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ldots$.

Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty$$

and

$$\chi(q) := (-q;q^2)_\infty.$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1$, $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ for any positive integer n, and $|x| < 1$.

Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right),$$

where $0 < x < 1$.

Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.2}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.2). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

The function $\chi(q)$ is intimately connected to Ramanujan's class invariants G_n an g_n which are defined by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q), \qquad g_n = 2^{-1/4}q^{-1/24}\chi(-q),$$

where $q = e^{-\pi\sqrt{n}}$ and n is a positive rational number. Since from [1, p. 56] (Entry 12(v), (vi))

$$\chi(q) = 2^{1/6}\left\{\alpha(1-\alpha)q^{-1}\right\}^{-1/24},$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}\alpha^{-1/24}q^{-1/24}.$$

Nipen Saikia [4] introduce the parameter $I_{m,n}$ which is defined as

$$I_{m,n} := \frac{\chi(q)}{q^{(-m+1)/24}\chi(q^m)}, \quad q = e^{-\pi\sqrt{n/m}}, \tag{1.3}$$

where m and n are positive real numbers.

In Section 3, we study the modular relation between $I_{5,n}$ and I_{5,k^2n}, their explicit evaluations of $I_{5,n}$ for $n = 2, 3, 4, 5, 7$ and 11.

2. Preliminary results.

Lemma 2.1 [3]. *If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^{10})}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + (PQ)^2 + \left(\frac{5}{PQ}\right)^2 + 16\left(\frac{P}{Q} - \frac{Q}{P}\right) =$$

$$= 2\left(P^2 + \frac{5}{P^2}\right) + 2\left(Q^2 + \frac{5}{Q^2}\right) + 4. \qquad (2.1)$$

Lemma 2.2 ([2, p. 233], Ch. 25, Entry 66). *If* $P = \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{15})}$, *then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \qquad (2.2)$$

Lemma 2.3 [3]. *If* $P := \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^5)\varphi(q^{20})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^5)\varphi(q^4)}$, *then*

$$Q^4 + \frac{1}{Q^4} - 112\left(Q^3 + \frac{1}{Q^3}\right) + 1440\left(Q^2 + \frac{1}{Q^2}\right) - 3184\left(Q + \frac{1}{Q}\right) + 7316 =$$

$$= 8\left(P + \frac{1}{P}\right)\left[22\left(Q^2 + \frac{1}{Q^2}\right) - 31\left(Q + \frac{1}{Q}\right) + 170\right] - 2\left(P^2 + \frac{5^2}{P^2}\right) \times$$

$$\times \left[3\left(Q^2 + \frac{1}{Q^2}\right) + 24\left(Q + \frac{1}{Q}\right) + 64\right] + 4\left(P^3 + \frac{5^3}{P^3}\right)\left[\left(Q + \frac{1}{Q}\right) + 4\right]. \qquad (2.3)$$

Lemma 2.4 [3]. *If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^5)}{\varphi(q^{25})}$, *then*

$$\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} + 5\left(PQ + \frac{5}{PQ}\right) + 5\left(Q^2 + \frac{5}{P^2}\right) =$$

$$= P^2Q^2 + \frac{5^2}{P^2Q^2} + 15. \qquad (2.4)$$

Lemma 2.5 [3]. *If* $P := \dfrac{\phi(q)\phi(q^7)}{\phi(q^5)\phi(q^{35})}$ *and* $Q := \dfrac{\phi(q)\phi(q^{35})}{\phi(q^5)\phi(q^7)}$, *then*

$$Q^4 - \frac{1}{Q^4} - 14\left[\left(Q^3 + \frac{1}{Q^3}\right) - \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3} =$$

$$= 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) - \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\}. \qquad (2.5)$$

Lemma 2.6 [3]. *If* $P = \dfrac{\phi(q)\phi(q^{11})}{\phi(q^5)\phi(q^{55})}$ *and* $Q = \dfrac{\phi(q)\phi(q^{55})}{\phi(q^5)\phi(q^{11})}$, *then*

$$Q^6 + \frac{1}{Q^6} + 33\left(Q^5 + \frac{1}{Q^5}\right) - 99\left(Q^4 + \frac{1}{Q^4}\right) +$$

$$+1529\left(Q^3 + \frac{1}{Q^3}\right) - 1683\left(Q^2 + \frac{1}{Q^2}\right) + 8800\left(Q + \frac{1}{Q}\right) =$$

$$= 6534 + \left(P^5 + \frac{5^5}{P^5}\right) - 11\left\{\left(P^4 + \frac{5^4}{P^4}\right)\left(Q + \frac{1}{Q}\right) - \left(P^3 + \frac{5^3}{P^3}\right)\left[11 + 4\left(Q^2 + \frac{1}{Q^2}\right)\right] -\right.$$

$$- \left(P^2 + \frac{5^2}{P^2}\right)\left[18 - 56\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right) - 8\left(Q^3 + \frac{1}{Q^3}\right)\right] -$$

$$- \left(P + \frac{5}{P}\right)\left[324 - 126\left(Q + \frac{1}{Q}\right) + 160\left(Q^2 + \frac{1}{Q^2}\right) - 18\left(Q^3 + \frac{1}{Q^3}\right) +\right.$$

$$\left.\left. +9\left(Q^4 + \frac{1}{Q^4}\right)\right] - \left(P^3 + \frac{5^3}{P^3}\right)\left[11 + 4\left(Q^2 + \frac{1}{Q^2}\right)\right]\right\}. \tag{2.6}$$

Lemma 2.7 [7, p. 56; 5]**.**

$$\frac{f^6(-q)}{qf^6(-q^5)} = \frac{\varphi^4(-q)}{\varphi^4(-q^5)}\left\{\frac{5\varphi^2(-q^5) - \varphi^2(-q)}{\varphi^2(-q^5) - \varphi^2(-q)}\right\}. \tag{2.7}$$

Lemma 2.8 ([1, p. 39], Ch. 16, Entry 24(iii))**.**

$$\chi(q) = \frac{\varphi(q)}{f(q)}. \tag{2.8}$$

Lemma 2.9 [4]**.** *We have*

$$I_{m,1} = 1. \tag{2.9}$$

Lemma 2.10 [4]**.** *We get*

$$I_{m,n}I_{m,1/n} = 1. \tag{2.10}$$

3. General theorems and explicit evaluations of $I_{m,n}$.

Theorem 3.1. *If* $P := q^{1/3}\dfrac{\chi(q)\chi(q^2)}{\chi(q^5)\chi(q^{10})}$ *and* $Q := q^{-1/6}\dfrac{\chi(q)\chi(q^{10})}{\chi(q^5)\chi(q^2)}$, *then*

$$\left\{Q^3 + \frac{1}{Q^3}\right\}\left[\left\{P^5 + \frac{1}{P^5}\right\} + 8\left\{P^3 + \frac{1}{P^3}\right\} + 19\left\{P + \frac{1}{P}\right\}\right] + \left\{Q^6 + \frac{1}{Q^6}\right\} =$$

$$= \left\{P^6 + \frac{1}{P^6}\right\} + 13\left\{P^4 + \frac{1}{P^4}\right\} + 52\left\{P^2 + \frac{1}{P^2}\right\} + 82.$$

Proof. Replace q by q^5 in Lemma 2.8, we obtain

$$\chi(q^5) = \frac{\varphi(q^5)}{f(q^5)}. \tag{3.1}$$

Dividing the equations (2.8) by (3.1), we get

$$\frac{\chi(q)}{\chi(q^5)} = \frac{\varphi(q)}{\varphi(q^5)} \frac{f(q^5)}{f(q)}. \tag{3.2}$$

Raising the power six and also multiplying q on both side of the equation (3.2), we have

$$q\frac{\chi^6(q)}{\chi^6(q^5)} = \frac{\varphi^6(q)}{\varphi^6(q^5)} \left\{ q\frac{f^6(q^5)}{f^6(q)} \right\}. \tag{3.3}$$

Using the equations (2.7) and (3.3), we obtain

$$P^4 - P^2(1-a) - 5a = 0, \tag{3.4}$$

where $P := \dfrac{\varphi(q)}{\varphi(q^5)}$, $a := q\dfrac{\chi^6(q)}{\chi^6(q^5)}$. Then the equation (3.4) can be written as

$$b^2 - b(1-a) - 5a = 0,$$

where $b = P^2$. Solve the above equation we get

$$b = \frac{1 - a + \sqrt{1 + 18a + a^2}}{2}. \tag{3.5}$$

Using the equations (3.5) and (2.1), we have

$$\big(19x^{10}y^4 - 52x^{10}y^{10} + x^{10}y^{16} + x^{16}y^{10} + x^{14}y^2 + 8x^{14}y^8 - x^{14}y^{14} + x^6 +$$
$$+ x^2y^{14} + 19x^4y^{10} - 13x^4y^4 + 8x^2y^8 - x^2y^2 + y^6 - 52x^6y^6 + 19x^6y^{12} +$$
$$+ 19x^{12}y^6 - 13x^{12}y^{12} + 8x^8y^2 - 82x^8y^8 + 8x^8y^{14}\big)\big(22x^{14}y^8 - 3x^4y^{10} -$$
$$-422x^{10}y^{10} + x^{28}y^4 + x^4y^{28} - x^{20}y^2 + x^{20}y^{32} + x^{28}y^{28} + x^{30}y^{24} + x^{24}y^{30} +$$
$$+ x^{32}y^{20} - x^{30}y^{12} - x^2y^{20} - y^{30}x^{12} - 3x^{10}y^4 - 25x^{10}y^{16} - 25x^{16}y^{10} -$$
$$-8x^{14}y^2 - 2523x^{14}y^{14} - 8x^2y^{14} + x^{12} + y^{12} + x^4y^4 + x^2y^8 - 11x^6y^6 +$$
$$+ 19x^6y^{12} + 19x^{12}y^6 + 1358x^{12}y^{12} + x^8y^2 + 101x^8y^8 + 22x^8y^{14} -$$
$$-3x^{28}y^{22} + 16x^{28}y^{10} + 43x^{28}y^{16} + 3387x^{16}y^{16} + 43x^{16}y^4 + 16x^{10}y^{28} -$$
$$-398x^{10}y^{22} - 101x^{14}y^{26} + 58x^{14}y^{20} - 25x^{22}y^{16} - 398x^{22}y^{10} + 253x^{20}y^8 +$$
$$+ 19x^{20}y^{26} + 1358x^{20}y^{20} + 58x^{20}y^{14} + 253x^8y^{20} - 19x^8y^{26} - 25x^{16}y^{22} +$$
$$+ 43x^{16}y^{28} + 16x^{22}y^4 - 3x^{22}y^{28} - 422x^{22}y^{22} + 43x^4y^{16} + 16x^4y^{22} -$$
$$-101x^{26}y^{14} - 11x^{26}y^{26} - 19x^{26}y^8 + 19x^{26}y^{20} - 8x^{30}y^{18} - 8x^{18}y^{30} -$$

$$-101x^6y^{18} - 101x^{18}y^6 + 58x^{18}y^{12} - 19x^{24}y^6 + 253x^{24}y^{12} + 58y^{18}x^{12} -$$

$$-19y^{24}x^6 + 253y^{24}x^{12} - 2523x^{18}y^{18} + 22x^{18}y^{24} + 22x^{24}y^{18} + 101x^{24}y^{24}\big) = 0,$$

where $x = q^{1/6}\dfrac{\chi(q)}{\chi(q^5)}$ and $y = q^{1/3}\dfrac{\chi(q^2)}{\chi(q^{10})}$. By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero, whereas other factor are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically.

Theorem 3.1 is proved.

***Remark* 3.1.** Here by using the definition of (1.3), then above Theorem 3.1 is also can be written as $P = I_{5,n}I_{5,4n}$ and $Q = \dfrac{I_{5,n}}{I_{5,4n}}$.

Theorem 3.2. *If* $P := q^{1/3}\dfrac{\chi(q)}{\chi(q^5)}$ *and* $Q := q^{-1/6}\dfrac{\chi(q^3)}{\chi(q^{15})}$, *then*

$$\frac{P^6}{Q^6} + \frac{Q^6}{P^6} + 18 = 9\left\{\frac{P^3}{Q^3} + \frac{Q^3}{P^3}\right\} + \left\{P^3Q^3 + \frac{1}{P^3Q^3}\right\}.$$

Proof. Employing the equations (3.5) and (2.2), we obtain

$$\big(-9Q^9P^3 - Q^3P^3 - Q^9P^9 - 9Q^3P^9 + 18P^6Q^6 + Q^{12} + P^{12}\big)\times$$

$$\times\big(9Q^9P^3 + Q^3P^3 + Q^9P^9 + 9Q^3P^9 + 18P^6Q^6 + Q^{12} + P^{12}\big) = 0.$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factor are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically.

Theorem 3.2 is proved.

***Remark* 3.2.** Here by using the definition of (1.3), then above Theorem 3.2 is also can be written as $P = I_{5,n}$ and $Q = I_{5,9n}$.

***Remark* 3.3.** $I_{m,n}$ has positive real value less than 1 and that the values of $I_{m,n}$ decrease as n increases when $m > 1$. Thus, by Lemma 2.9, $I_{m,n} < 1$ for all $n > 1$ if $m > 1$.

***Corollary* 3.1.** *We have*

$$I_{5,3} = \left[\frac{7 - 3\sqrt{5}}{2}\right]^{1/6}, \tag{3.6}$$

$$I_{5,1/3} = \left[\frac{7 + 3\sqrt{5}}{2}\right]^{1/6}, \tag{3.7}$$

$$I_{5,9} = \left[4 - \sqrt{15}\right]^{1/3}, \tag{3.8}$$

$$I_{5,1/9} = \left[4 + \sqrt{15}\right]^{1/3}. \tag{3.9}$$

Proof. Setting $n = 1/3$ in Theorem 3.2 and using the Lemma 2.10, we obtain

$$I_{5,3}^{12} + I_{5,3}^{-12} - 9\big(I_{5,3}^{6} + I_{5,3}^{-6}\big) + 16 = 0.$$

Equivalently,

$$C^2 - 9C + 14 = 0, \tag{3.10}$$

where $C = I_{5,3}^{6} + I_{5,3}^{-6}$.

Solving (3.10) and using the fact in Remark 3.3, we get

$$C = 7. \tag{3.11}$$

Employing (3.10) and (3.11), solving the resulting equation for $I_{5,3}$ and nothing that $I_{5,3} < 1$, we arrive (3.6).

Again setting $n = 1$ in Theorem 3.2 and using Lemma 2.9, we obtain

$$I_{5,9}^{6} + I_{5,9}^{-6} - 10\big(I_{5,9}^{3} + I_{5,9}^{-3}\big) + 18 = 0.$$

Equivalently,

$$D^2 - 10D + 16 = 0, \tag{3.12}$$

where $D = I_{5,9}^{3} + I_{5,9}^{-3}$.

Solving (3.12) and using the fact in Remark 3.3, we have

$$D = 8. \tag{3.13}$$

Employing (3.12) and (3.13), solving the resulting equation for $I_{5,9}$ and nothing that $I_{5,9} < 1$, we arrive (3.8).

Theorem 3.3. *If* $P := q^{5/6}\dfrac{\chi(q)\chi(q^4)}{\chi(q^5)\chi(q^{20})}$ *and* $Q := q^{-1/2}\dfrac{\chi(q)\chi(q^{20})}{\chi(q^5)\chi(q^4)}$, *then*

$$\left\{Q^{14} + \frac{1}{Q^{14}}\right\} - 56\left\{Q^{12} + \frac{1}{Q^{12}}\right\} + 861\left\{Q^{10} + \frac{1}{Q^{10}}\right\} - 5824\left\{Q^{8} + \frac{1}{Q^{8}}\right\} +$$

$$+22524\left\{Q^{6} + \frac{1}{Q^{6}}\right\} - 59015\left\{Q^{4} + \frac{1}{Q^{4}}\right\} + 102884\left\{Q^{2} + \frac{1}{Q^{2}}\right\} -$$

$$-224\left\{P^6Q^6 + \frac{1}{P^6Q^6}\right\} + 1800\left\{P^3Q^3 + \frac{1}{P^3Q^3}\right\} -$$

$$-224\left\{\frac{P^6}{Q^6} + \frac{Q^6}{P^6}\right\} + 1800\left\{\frac{P^3}{Q^3} + \frac{Q^3}{P^3}\right\} =$$

$$= \left\{P^{12} + \frac{1}{P^{12}}\right\}\left\{Q^{2} + \frac{1}{Q^{2}}\right\} - \left\{P^{9} + \frac{1}{P^{9}}\right\}\left[\left\{Q^{7} + \frac{1}{Q^{7}}\right\} -\right.$$

$$\left. - 8\left\{Q^{5} + \frac{1}{Q^{5}}\right\} - 3\left\{Q^{3} + \frac{1}{Q^{3}}\right\} - 16\left\{Q + \frac{1}{Q}\right\}\right] - \left\{P^{6} + \frac{1}{P^{6}}\right\} \times$$

$$\times\left[24\left\{Q^8+\frac{1}{Q^8}\right\}+628\left\{Q^4+\frac{1}{Q^4}\right\}-1357\left\{Q^2+\frac{1}{Q^2}\right\}1196\right]+$$

$$+\left\{P^3+\frac{1}{P^3}\right\}\left[16\left\{Q^{11}+\frac{1}{Q^{11}}\right\}-109\left\{Q^9+\frac{1}{Q^9}\right\}+336\left\{Q^7+\frac{1}{Q^7}\right\}-\right.$$

$$\left.-1261\left\{Q^5+\frac{1}{Q^5}\right\}-2228\left\{Q+\frac{1}{Q}\right\}\right]+127634. \tag{3.14}$$

Proof. Employing the equations (3.5) and (2.3), we obtain (3.14).

***Remark* 3.4.** Here by using the definition of (1.3), then above Theorem 3.3 is also can be written as $P=I_{5,n}I_{5,16n}$ and $Q=\dfrac{I_{5,n}}{I_{5,16n}}$.

***Corollary* 3.2.** *We have*

$$I_{5,4}=\frac{(11+5\sqrt{5})^{1/4}-\sqrt{\sqrt{11+5\sqrt{5}}-4}}{2}, \tag{3.15}$$

$$I_{5,1/4}=\frac{(11+5\sqrt{5})^{1/4}+\sqrt{\sqrt{11+5\sqrt{5}}-4}}{2}. \tag{3.16}$$

Proof. Employing Theorem 3.3 and Lemma 2.10, solving the resulting equation for $I_{5,4}$ and nothing that $I_{5,4}<1$, we arrive (3.15).

Theorem 3.4. *If* $P:=q\dfrac{\chi(q)\chi(q^5)}{\chi(q^5)\chi(q^{25})}$ *and* $Q:=q^{-2/3}\dfrac{\chi(q)\chi(q^{25})}{\chi^2(q^5)}$, *then*

$$\left\{Q^3+\frac{1}{Q^3}\right\}\left[\left\{P^2+\frac{1}{P^2}\right\}+\left\{P+\frac{1}{P}\right\}+1\right]=$$

$$=\left\{P^4+\frac{1}{P^4}\right\}+6\left\{P^3+\frac{1}{P^3}\right\}+11\left\{P^2+\frac{1}{P^2}\right\}+16\left\{P+\frac{1}{P}\right\}+22. \tag{3.17}$$

Proof. Employing the equations (3.5) and (2.4), we obtain (3.17).

***Remark* 3.5.** Here by using the definition of (1.3), then above Theorem 3.4 is also can be written as $P=I_{5,n}I_{5,25n}$ and $Q=\dfrac{I_{5,n}}{I_{5,25n}}$.

***Corollary* 3.3.** *We have*

$$\begin{aligned}
I_{5,5}&=\left[9-4\sqrt{5}\right]^{1/3},\\
I_{5,1/5}&=\left[9+4\sqrt{5}\right]^{1/3},\\
I_{5,25}&=\frac{a-\sqrt{a^2-4}}{2},\\
I_{5,1/25}&=\frac{a+\sqrt{a^2-4}}{2},
\end{aligned} \tag{3.18}$$

where

$$a = \frac{b^4 + xb^3 + yb^2 + z}{5b^3}, \qquad b = \left\{ \frac{125\left[250 + 110\sqrt{5} - (5 + 3\sqrt{5})\sqrt{10 + 2\sqrt{5}}\right]}{4} \right\}^{1/5},$$

$$x = 5(5 + 2\sqrt{5}), \qquad y = \frac{25\left[5 + \sqrt{5} + \sqrt{10 + 2\sqrt{5}}\right]}{2},$$

and

$$z = \frac{125\left[12 + 4\sqrt{5} - (\sqrt{5} + 1)\sqrt{10 + 2\sqrt{5}}\right]}{4}.$$

Proof. Employing Theorem 3.4, Lemmas 2.9 and 2.10, solving the resulting equation for $I_{5,5}$, $I_{5,25}$ and nothing that $I_{5,5} < 1$ and $I_{5,25} < 1$, we arrive (3.18).

Theorem 3.5. *If* $P := q^{4/3}\dfrac{\chi(q)}{\chi(q^5)}$ *and* $Q := q^{-1}\dfrac{\chi(q^7)}{\chi(q^{35})}$, *then*

$$\frac{P^4}{Q^4} + \frac{Q^4}{P^4} - 7\left\{\frac{P^3}{Q^3} + \frac{Q^3}{P^3}\right\} + 21\left\{\frac{P^2}{Q^2} + \frac{Q^2}{P^2}\right\} - 42\left\{\frac{P}{Q} + \frac{Q}{P}\right\} + 56 =$$

$$= P^3Q^3 + \frac{1}{P^3Q^3}. \tag{3.19}$$

Proof. Employing the equations (3.5) and (2.5), we obtain (3.19).

***Remark* 3.6.** Here by using the definition of (1.3), then above Theorem 3.5 is also can be written as $P = I_{5,n}$ and $Q = I_{5,49n}$.

***Corollary* 3.4.** *We have*

$$I_{5,7} = \frac{a - \sqrt{a^2 - 4}}{2}, \qquad I_{5,1/7} = \frac{a + \sqrt{a^2 - 4}}{2}, \tag{3.20}$$

where $a = \dfrac{(71 - 3\sqrt{105})\left[16x^2 + (x - 1)(71 + 3\sqrt{105})\right]}{12288}$ *and* $x = [71 + 3\sqrt{105}]^{1/3}$,

$$I_{5,49} = \frac{b - \sqrt{b^2 - 4}}{2}, \qquad I_{5,1/49} = \frac{b + \sqrt{b^2 - 4}}{2}, \tag{3.21}$$

where $b = \dfrac{(9 - 2\sqrt{15})\left[y^2 + (y + 6)(9 + 2\sqrt{15})\right]}{16}$ *and* $y = [189 + 42\sqrt{15}]^{1/3}$.

Proof. Employing Theorem 3.5, Lemmas 2.9 and 2.10, solving the resulting equation for $I_{5,7}$, $I_{5,49}$ and nothing that $I_{5,7} < 1$ and $I_{5,49} < 1$, we arrive (3.20), (3.21).

Theorem 3.6. *If* $P := q^2 \dfrac{\chi(q)\chi(q^{11})}{\chi(q^5)\chi(q^{55})}$ *and* $Q := q^{-5/3} \dfrac{\chi(q)\chi(q^{55})}{\chi(q^5)\chi(q^{11})}$, *then*

$$\left\{P^5 + \frac{1}{P^5}\right\} - 11\left\{P^4 + \frac{1}{P^4}\right\} + 11\left\{P^3 + \frac{1}{P^3}\right\} +$$

$$+88\left\{P^2 + \frac{1}{P^2}\right\} + 33\left\{P + \frac{1}{P}\right\} =$$

$$= \left\{Q^6 + \frac{1}{Q^6}\right\} - 11\left\{Q^3 + \frac{1}{Q^3}\right\}\left[\left\{P^2 + \frac{1}{P^2}\right\} + 2\left\{P + \frac{1}{P}\right\} + 3\right] + 165. \tag{3.22}$$

Proof. Employing the equations (3.5) and (2.6), we obtain (3.22).

Remark **3.7.** Here by using the definition of (1.3), then above Theorem 3.6 is also can be written as $P = I_{5,n} I_{5,121n}$ and $Q = \dfrac{I_{5,n}}{I_{5,121n}}$.

Corollary **3.5.** *We have*

$$I_{5,11} = \left[\frac{9(11 + 5\sqrt{5}) - \sqrt{110(181 + 81\sqrt{5})}}{4}\right]^{1/6}, \tag{3.23}$$

$$I_{5,1/11} = \left[\frac{9(11 + 5\sqrt{5}) + \sqrt{110(181 + 81\sqrt{5})}}{4}\right]^{1/6}. \tag{3.24}$$

Proof. Employing Theorem 3.6 and Lemma 2.10, solving the resulting equation for $I_{5,11}$ and nothing that $I_{5,11} < 1$, we arrive (3.23).

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