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BALANCING POLYNOMIALS AND THEIR DERIVATIVES

БАЛАНСУЮЧІ ПОЛІНОМИ ТА ЇХ ПОХІДНІ

We study the generalization of balancing numbers with a new sequence of numbers called k-balancing numbers. Moreover, by using the Binet formula for k-balancing numbers, we obtain the identities including the generating function of these numbers. In addition, the properties of divisibility of these numbers are investigated. Further, balancing polynomials that are natural extensions of the k-balancing numbers are introduced and some relations for the derivatives of these polynomials in the form of convolution are also proved.

Вивчається узагальнення балансуючих чисел з новою послідовністю чисел, що називаються k-балансуючими числами. Більш того, за допомогою формули Біне для k-балансуючих чисел отримано тотожності, що включають породжуючу функцію для цих чисел. Вивчено властивості подільності цих чисел. Крім того, ми вводимо балансуючі поліноми, що є природним узагальненням k-балансуючих чисел, і доводимо деякі співвідношення для похідних цих поліномів у формі згорток.

1. Introduction. There is a huge interest of many number theorists in the study of a newly developed number sequence which is popularly known as balancing numbers. According to Behera et al., a positive integer n is called a balancing number, if it is the solution of a simple Diophantine equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r),$$

where r is the balancer corresponding to the balancing number n. The balancing number B_n is the n th term of the sequence $\{0, 1, 6, 35, 204, \ldots\}$ beginning with the values $B_0 = 0$ and $B_1 = 1$ and having recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1. \tag{1.1}$$

A number sequence closely associated with the sequence of balancing numbers is the sequence of Lucas-balancing numbers $\{C_n\}$, where the n th Lucas-balancing number C_n is given by the relation $C_n = \sqrt{8B_n^2 + 1}$ with $n \geq 0$ [6]. The recurrence relation of Lucas-balancing numbers is same as that of balancing numbers but with different initials, that is, $C_{n+1} = 6C_n - C_{n-1},\ n \geq 1$ with $C_0 = 1$ and $C_1 = 3$. The closed form of balancing numbers popularly known as Binet formula is given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the roots of the equation $\lambda^2 - 6\lambda + 1 = 0$ [1]. On the other hand, the Binet formula for Lucas-balancing numbers is the expression $C_n = \dfrac{\lambda_1^n + \lambda_2^n}{2}$ [6]. Balancing and Lucas-balancing numbers can also be extended negatively, in particular, $B_{-n} = -B_n$ and $C_{-n} = C_n$ [9]. Ray, in [9], has shown some interesting properties relating to balancing numbers using matrices. Among those properties, one important identity was the bilinear index reduction formula for balancing numbers which is given by the relation

$$B_a B_b - B_c B_d = B_{a-1} B_{b-1} - B_{c-1} B_{d-1},$$

whenever $a + b = c + d$ [9].

While searching of balancing numbers, Liptai, in [3], has found that the only balancing number in the sequence of Fibonacci numbers is 1. He has also shown that there is no Lucas-balancing number in the sequence of Fibonacci numbers [4]. In [7], Panda has established many fascinating results for balancing and Lucas-balancing numbers. For example, the identities that resemble the trigonometry identities $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ are $B_{m\pm n} = B_m C_n \pm B_n C_m$. Later, Panda et al. (see [6]) have linked balancing numbers with well known Pell and associated Pell numbers by showing that the n th balancing number B_n is the product of n th Pell number P_n and n th associated Pell number Q_n. As usual, Pell and associated Pell numbers are recursively defined as $P_{n+1} = 2P_n + P_{n-1},\ n \geq 2$ with $P_1 = 1,\ P_2 = 2$ and $Q_{n+1} = 2Q_n + Q_{n-1},\ n \geq 2$ with $Q_1 = 1,\ Q_2 = 3$ respectively.

Different forms of generalization for balancing numbers are available in literature [2, 5, 8, 12]. In [5], Liptai et al. have generalized the concept of balancing numbers in the following way. Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if

$$1^k + \ldots + (x-1)^k = (x+1)^l + \ldots + (y-1)^l.$$

Also, in [5], several effective and ineffective finiteness results were proved for (k, l)-power numerical centers using certain Baker-type Diophantine results and Bilu – Tichy theorem, respectively. For example, they proved that, for a fixed positive integer k with $k \geq 1$ and $l \in \{1, 3\}$, if $(k, l) \neq (1, 1)$, then there are only finitely many (k, l)-balancing numbers, and these balancing numbers are bounded by an effectively computable constant depending only on k. In [12], Szakács has studied a further generalization of balancing numbers by introducing multiplying balancing numbers. A positive integer n is called a multiplying balancing number if

$$1 \cdot 2 \ldots (n-1) = (n+1)(n+2) \ldots (n+r),$$

for some positive integer r which is called as multiplying balancer corresponding to the multiplying balancing number n. He has proved that the only multiplying balancing number is $n = 7$ with multiplying balancer $r = 3$. As a generalization of the notion of balancing numbers, in [2], Bérczes et al. called $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ a second order linear recurrence sequence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2}, \quad i \geq 2,$$

holds, where $A, B \neq 0,\ R_0,\ R_1$ are fixed rational integers and $|R_0| + |R_1| > 0$. They proved that any sequence $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0,\ (A, B) \neq (0, 1)$ is not a balancing sequence.

The present paper is organized as follows. In Section 2, the sequence of k-balancing numbers is considered and some identities concerning these numbers are derived. Moreover, some divisibility properties of k-balancing numbers are investigated. In Section 3, balancing polynomials which are the natural extension of k-balancing numbers are introduced and it can be seen that many of their properties admit straightforward proofs. We also present the derivatives of the balancing polynomials in the form of convolution of these polynomials in Section 4.

2. k-Balancing numbers.

Definition 2.1. *If k is any positive number, the sequence of k-balancing numbers $\{B_{k,n}\}_{n=1}^{\infty}$ recursively defined as*

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}, \quad n \geq 1, \tag{2.1}$$

where $B_{k,n}$ is the nth k-balancing number with $B_{k,0} = 0$ and $B_{k,1} = 1$.

Some initial k-balancing numbers are

$$\begin{aligned}
&B_{k,0} = 0,\\
&B_{k,1} = 1,\\
&B_{k,2} = 6k,\\
&B_{k,3} = 36k^2 - 1,\\
&B_{k,4} = 216k^3 - 12k,\\
&B_{k,5} = 1296k^4 - 108k^2 + 1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Noting that, the sequence $\{B_{1,n}\}_{n=1}^{\infty}$ is the sequence of balancing numbers.

We observe that, the equation (2.1) is a second order difference equation having characteristic equation $\alpha^2 = 6k\alpha - 1$, whose roots are indeed $\alpha_1 = 3k + \sqrt{9k^2 - 1}$ and $\alpha_2 = 3k - \sqrt{9k^2 - 1}$ with $9k^2 - 1 \geq 0$. Clearly, α_2 is the conjugate root of α_1 and also notice that, the sum, the product and the difference of these roots are respectively given as $6k$, 1 and $2\sqrt{9k^2 - 1}$.

The following are some identities involving k-balancing numbers.

As $\alpha_1 = 3k + \sqrt{9k^2 - 1}$ and $\alpha_2 = 3k - \sqrt{9k^2 - 1}$ are roots of $\alpha^2 = 6k\alpha - 1$, we have $\alpha_1^2 = 6k\alpha_1 - 1$, and $\alpha_2^2 = 6k\alpha_2 - 1$. Multiplying α_1^n and α_2^n to both these equations respectively, we obtain the following result.

Lemma 2.1. *For any integer $n \geq 1$, $\alpha_1^{n+2} = 6k\alpha_1^{n+1} - \alpha_1^n$ and $\alpha_2^{n+2} = 6k\alpha_2^{n+1} - \alpha_2^n$.*

Binet formulas for certain sequences such as Fibonacci sequence, Lucas sequence, balancing sequence, etc. are useful to establish many of their identities. With the help of mathematical induction, it is easy to derive the Binet formula for k-balancing numbers and is given by the identity

$$B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}, \tag{2.2}$$

with $\alpha_1 = 3k + \sqrt{9k^2 - 1}$ and $\alpha_2 = 3k - \sqrt{9k^2 - 1}$. An application of this Binet formula gives a combinatorial identity of k-balancing numbers as follows.

Lemma 2.2. *Let $\binom{n}{j}$ denote the usual notation for combination. Then for any integer $n \geq 0$, we have*

$$\sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} 6^j k^j B_{k,j} = B_{k,2n}.$$

Proof. Using Binet formula (2.2), the left-hand side of the identity reduces to

$$\sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} 6^j k^j \left[\frac{\alpha_1^j - \alpha_2^j}{\alpha_1 - \alpha_2}\right],$$

which on simplification gives

$$\frac{1}{\alpha_1-\alpha_2}\left[\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}(6k\alpha_1)^j-\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}(6k\alpha_2)^j\right].$$

With some algebraic manipulations, the expression further simplifies to

$$\frac{1}{\alpha_1-\alpha_2}[(6k\alpha_1-1)^n-(6k\alpha_2-1)^n]=\frac{\alpha_1^{2n}-\alpha_2^{2n}}{\alpha_1-\alpha_2}=B_{k,2n},$$

which ends the proof.

The k-balancing numbers are also extended negatively like balancing numbers. Some further applications of Binet formula show the following facts.

Proposition 2.1. *For* $n\geq 1,\ B_{k,-n}=-B_{k,n}$.

Proposition 2.2. *For any natural numbers* p,q *and* $r,\ B_{k,p+q-1}=B_{k,p}B_{k,q}-B_{k,p-1}B_{k,q-1}$.

Proposition 2.3. *For any natural numbers* p, q *and* r, $B_{k,p+q-2}=\dfrac{1}{6k}\left[B_{k,p}B_{k,q}-B_{k,p-2}B_{k,q-2}\right]$.

Proposition 2.4. *For any natural numbers* p,q *and* r,

$$B_{k,p+q+r-3}=\frac{1}{6k}\left[B_{k,p}B_{k,q}B_{k,r}-6kB_{k,p-1}B_{k,q-1}B_{k,r-1}+B_{k,p-2}B_{k,q-2}B_{k,r-2}\right].$$

Since $\alpha_1+\alpha_2=6k$ and $\alpha_1\alpha_2=1$, we have $\alpha_1^2=6k\alpha_1-1=\alpha_1B_{k,2}-B_{k,1},\ \alpha_1^3= =\alpha_1(6k\alpha_1-1)=6k\alpha_1^2-\alpha_1=(36k^2-1)\alpha_1-6k=\alpha_1B_{k,3}-B_{k,2}$ and so on. Continuing this way similarly for α_2, we obtain the following result.

Lemma 2.3. *For any integer* $n\geq 1,\ \alpha_1^n=\alpha_1B_{k,n}-B_{k,n-1},\ \alpha_2^n=\alpha_2B_{k,n}-B_{k,n-1}$.

In [7], Panda has shown that two consecutive balancing numbers are relatively prime, that is $(B_{n-1},B_n)=1$, where (a,b) denotes the greatest common divisor of a and b. The following is a similar result concerning k-balancing numbers.

Lemma 2.4. *Any two consecutive* k*-balancing numbers are relatively prime, i.e.,* $(B_{k,n},B_{k,n-1})=1$.

Proof. The proof of this result is easy and based on Euclidean algorithm where $B_{k,n}$ is the dividend and $B_{k,n-1}$ is the divisor and we have

$$B_{k,n}=6kB_{k,n-1}-B_{k,n-2},$$

$$B_{k,n-1}=6kB_{k,n-2}-B_{k,n-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$B_{k,3}=6kB_{k,2}-B_{k,1},$$

$$B_{k,2}=6kB_{k,1}-B_{k,0},$$

consequently, $(B_{k,n},B_{k,n-1})=B_{k,1}=1$.

Panda has also shown certain divisibility properties for balancing numbers in [7]. Later, Ray, in [10], has investigated some more of these properties using congruences. The following are some results concerning divisibility properties for k-balancing numbers.

Lemma 2.5. *Let* n *and* m *be any positive integers, then* $B_{k,m}$ *divides* $B_{k,mn}$.

Proof. The proof is based on induction on n. The result is true for $n = 1$. Assume that, it is true for all $r \geq 1$, that is, $B_{k,m}$ divides $B_{k,mj}$, for every j lies between 1 and r. Therefore, by Proposition 2.2, $B_{k,m(r+1)} = B_{k,mr}B_{k,m+1} - B_{k,mr-1}B_{k,m}$. Consequently, $B_{k,m}$ divides $B_{k,m(r+1)}$ by hypothesis and the result follows.

Lemma 2.6. *For every positive integers m and t, $(B_{k,mt-1}, B_{k,m}) = 1$.*

Proof. Let $d = (B_{k,mt-1}, B_{k,m})$. It follows that, d divides $B_{k,mt-1}$ and d divides $B_{k,m}$. As $B_{k,m}$ divides $B_{k,mt}$ by Lemma 2.5, d divides $B_{k,mt}$. Further, d divides $B_{k,mt-1}$ and d divides $B_{k,mt}$ and as $B_{k,mt-1}$ and $B_{k,mt}$ are relatively prime, by division algorithm, d divides 1 which follows that $d = 1$.

Lemma 2.7. *Let m, n, s and t are positive integers. Then for $m = sn + t$, $(B_{k,m}, B_{k,n}) = (B_{k,n}, B_{k,t})$.*

Proof. Since $m = sn + t$, the greatest common divisor of $B_{k,m}$ and $B_{k,n}$ is given by $(B_{k,m}, B_{k,n}) = (B_{k,sn+t}, B_{k,n})$. Using Proposition 2.2, the right-hand side expression reduces to $(B_{k,sn}B_{k,t+1} - B_{k,sn-1}B_{k,t}, B_{k,n})$. Further simplification gives $(B_{k,m}, B_{k,n}) = (B_{k,sn-1}B_{k,t}, B_{k,n})$. Therefore, the proof of the result follows as $B_{k,sn-1}$ and $B_{k,n}$ are relatively prime by Lemma 2.6.

The greatest common divisor of any two k-balancing numbers is again a k-balancing number. The following result shows this fact.

Lemma 2.8. *For any two positive integers m and n, $(B_{k,m}, B_{k,n}) = B_{k,(m,n)}$.*

Proof. Let m and n are any two positive integers with $m \geq n$. Then by Euclidean algorithm,

$$\begin{aligned} m &= q_0 n + r_1, \qquad 0 \leq r_1 < n, \\ n &= q_1 r_1 + r_2, \qquad 0 \leq r_2 < r_1, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n, \qquad 0 \leq r_n < r_{n-1}, \\ r_{n-1} &= q_n r_n + 0. \end{aligned}$$

It follows that, $(m, n) = r_n$. By virtue of Lemma 2.7, we have

$$(B_{k,m}, B_{k,n}) = (B_{k,n}, B_{k,r_1}) = (B_{k,r_1}, B_{k,r_2}) = \ldots = (B_{k,r_{n-1}}, B_{k,r_n}).$$

On the other hand as $r_n | r_{n-1}$, by Lemma 2.5, $B_{k,r_n} | B_{k,r_{n-1}}$. It follows that $(B_{k,r_{n-1}}, B_{k,r_n}) = B_{k,r_n}$. Consequently, $(B_{k,m}, B_{k,n}) = B_{k,r_n} = B_{k,(m,n)}$.

***Corollary* 2.1.** *For any two positive integers m and n, if $(m, n) = 1$, then $B_{k,m}B_{k,n}$ divides $B_{k,mn}$.*

Proof. By virtue of Lemma 2.5, $B_{k,m}$ divides $B_{k,mn}$ and $B_{k,n}$ divides $B_{k,mn}$. Therefore, their least common multiple $[B_{k,m}, B_{k,n}]$ divides $B_{k,mn}$. Indeed, $(B_{k,m}, B_{k,n}) = B_{k,mn} = B_{k,1} = 1$, and using the fact that $ab = (a, b)[a, b]$, we get $[B_{k,m}, B_{k,n}] = B_{k,m}B_{k,n}$, and the result follows.

The matrix used to represent the recurrence relation (2.1) is $M = \begin{pmatrix} 6k & -1 \\ 1 & 0 \end{pmatrix}$, which for $k = 1$ reduces to balancing Q_B matrix studied in [9]. It is easy to notice that $M^n = \begin{pmatrix} B_{k,n+1} & -B_{k,n} \\ B_{k,n} & B_{k,n-1} \end{pmatrix}$. Consider the matrix $I - sM = \begin{pmatrix} 1-6ks & -s \\ s & 1 \end{pmatrix}$, where I be the identity matrix same order as M. Since the determinant of the matrix $I - sM$ is $1 - 6ks + s^2$, its inverse will be

$$(I-sM)^{-1}=\frac{1}{1-6ks+s^2}\begin{pmatrix} 1 & -s \\ s & 1-6ks \end{pmatrix}.$$

Let $g(s)=\sum_{n=0}^{\infty}s^nM^n=(I-sM)^{-1}$ be the generating function for the sequence of k-balancing numbers. Then

$$s^0M^0+s^1M^1+s^2M^2+\ldots=\begin{pmatrix} \dfrac{1}{1-6ks+s^2} & \dfrac{-s}{1-6ks+s^2} \\ \dfrac{s}{1-6ks+s^2} & \dfrac{1-6s}{1-6ks+s^2} \end{pmatrix}.$$

By equating (2.1) entry from both sides, we get

$$B_{k,0}+sB_{k,1}+s^2B_{k,2}+\ldots=g(s)=\frac{s}{1-6ks+s^2},$$

and we have the following result.

Lemma 2.9. *The generating function of k-balancing numbers is given by*

$$\sum_{n=0}^{\infty}B_{k,n}s^n=\frac{s}{1-6ks+s^2}.$$

The following combinatorial identity straightforwardly obtain from the generating function for k-balancing numbers.

Lemma 2.10. *If $B_{k,n}$ denotes the $n\,th$ k-balancing number, then*

$$B_{k,n}=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^j\begin{pmatrix} n-j-1 \\ j \end{pmatrix}(6k)^{n-2j-1}.$$

Proof. The defining generating function of the Chebyshev polynomials of the second kind $U_n(t)$ is

$$\sum_{n=0}^{\infty}U_n(t)z^n=\frac{1}{1-2tz+z^2}=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^j\begin{pmatrix} n-j \\ j \end{pmatrix}(2t)^{n-2j}\right]z^n.$$

Setting $t=3k$ and $z=s$, where $i^2=-1$, we get

$$\frac{s}{1-6ks+s^2}=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^j\begin{pmatrix} n-j \\ j \end{pmatrix}(6k)^{n-2j}\right]s^{n+1}. \tag{2.3}$$

From (2.3) and Lemma 2.9, it is observed that

$$B_{k,n+1}=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^j\begin{pmatrix} n-j \\ j \end{pmatrix}(6k)^{n-2j}=$$

$$=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^j\begin{pmatrix} n-j \\ j \end{pmatrix}(6k)^{n-2j},$$

which ends the proof.

3. Balancing polynomials. In this section, the sequence of k-balancing numbers $\{B_{k,n}\}$ is extended to the sequence of balancing polynomials $\{B_{x,n}\}$ by replacing k with a real variable x. It is also observed that many of the properties of balancing polynomials admit straightforward proofs.

Definition 3.1. *The sequence of balancing polynomials is recursively defined as follows:*

$$B_{n+1}(x) = \begin{cases} 1, & \text{if } n = 0, \\ 6x, & \text{if } n = 1, \\ 6xB_n(x) - B_{n-1}(x), & \text{if } n > 1. \end{cases} \tag{3.1}$$

Some initial balancing polynomials are

$$B_0(x) = 0,$$
$$B_1(x) = 1,$$
$$B_2(x) = 6x,$$
$$B_3(x) = 36x^2 - 1,$$
$$B_4(x) = 216x^3 - 12x,$$
$$B_5(x) = 1296x^4 - 108x^2 + 1.$$

It is observed that, $B_j(1) = B_j$ for each j.

Looking into the identity from Lemma 2.10, one can easily obtain the general term of balancing polynomials as, for $n \geq 0$,

$$B_{n+1}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (6x)^{n-2j}. \tag{3.2}$$

Solving the auxiliary equation of (3.1), the roots are $\lambda_1(x) = 3x + \sqrt{9x^2 - 1}$ and its conjugate $\lambda_2(x) = 3x - \sqrt{9x^2 - 1}$, where $9x^2 - 1 \geq 0$. Notice that, $\lambda_1(1) = 3 + \sqrt{8}$ and $\lambda_2(1) = 3 - \sqrt{8}$ are balancing constant and its conjugate for the balancing numbers.

In order to find the Binet formula for balancing polynomials, once again method of induction is used to get

$$B_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}, \tag{3.3}$$

where $\lambda_1(x) = 3x + \sqrt{9x^2 - 1}$ and $\lambda_2(x) = 3x - \sqrt{9x^2 - 1}$ with $9x^2 - 1 \geq 0$.

The following result demonstrates the limit of the quotient of two consecutive terms of balancing polynomials.

Lemma 3.1. *For* $x > \dfrac{1}{3}$, $\lim_{n\to\infty} \dfrac{B_{n+1}(x)}{B_n(x)} = \lambda_1(x)$.

Proof. Using the Binet formula (3.3), we have

$$\lim_{n\to\infty} \frac{B_{n+1}(x)}{B_n(x)} = \lim_{n\to\infty} \frac{\lambda_1^{n+1}(x) - \lambda_2^{n+1}(x)}{\lambda_1^n(x) - \lambda_2^n(x)} =$$

$$= \lim_{n\to\infty} \frac{\lambda_1(x) - \left(\dfrac{\lambda_2(x)}{\lambda_1(x)}\right)^n \lambda_2(x)}{1 - \left(\dfrac{\lambda_2(x)}{\lambda_1(x)}\right)^n}.$$

Since $\lambda_2(x) < \lambda_1(x)$ for every $x > \dfrac{1}{3}$, $\lim_{n\to\infty} \left(\dfrac{\lambda_2(x)}{\lambda_1(x)}\right)^n = 0$ and the desired result is obtained.

Observation 3.1. It can be observed that, for $x = -\dfrac{1}{3}$ or $\dfrac{1}{3}$, $\lim_{n\to\infty} \left|\dfrac{B_{n+1}}{B_n}\right| = 1$ and for $x < -\dfrac{1}{3}$, $\lim_{n\to\infty} \left|\dfrac{B_{n+1}}{B_n}\right| = \lambda_1(x)$. However, this limit does not exist for x lies between $-\dfrac{1}{3}$ and $\dfrac{1}{3}$.

The following result straightforwardly deduce by iterating recurrence relation for balancing polynomials.

Proposition 3.1. *For an integer r lying between 1 and $n-1$,*

$$B_{n+1}(x) = B_r(x)B_{n-(r-2)}(x) - B_{r-1}(x)B_{n-(r-1)}(x).$$

The following result is obtained on replacing $n-r-2$ by n and r by $m+1$ in the expression given in Proposition 3.1.

Proposition 3.2. *Let m and n are any two integers, then*

$$B_{m+n}(x) = B_{m+1}(x)B_n(x) - B_m(x)B_{n-1}(x).$$

In particular, for $m = n-1$ and $m = n$ in the above identity respectively, an expression for the polynomial of even degree $B_{2n-1}(x) = B_n^2(x) - B_{n-1}^2(x)$ and the expression $B_{2n}(x) = B_n(x)[B_{n+1}(x) - B_{n-1}(x)]$ which is equivalently, $B_{2n}(x) = \dfrac{B_{n+1}^2(x) - B_{n-1}^2(x)}{6x}$ are obtained.

The matrix corresponds to balancing polynomials is represented by a second order matrix N whose entries are the first three balancing polynomials, that is $N = \begin{pmatrix} 6x & -1 \\ 1 & 0 \end{pmatrix}$. It can be easily verified by induction, that the matrix N when raised to the n th power is given by

$$N^n = \begin{pmatrix} B_{n+1}(x) & -B_n(x) \\ B_n(x) & -B_{n-1}(x) \end{pmatrix},$$

for any integer n. In fact, N^0 represent the identity matrix and N^{-n} is the matrix inverse of N. Since

$$\det N = 1,\ \det N^n = (\det N)^n = 1.$$

It follows that

$$B_n^2(x) - B_{n-1}(x)B_{n+1}(x) = 1,$$

which we call as Cassini formula for balancing polynomials. It is also observed that

$$\begin{pmatrix} B_{-n+1}(x) & -B_{-n}(x) \\ B_{-n}(x) & -B_{-n-1}(x) \end{pmatrix} = N^{-k} = (N^k)^{-1} = \begin{pmatrix} -B_{n-1}(x) & B_n(x) \\ -B_n(x) & B_{n+1}(x) \end{pmatrix}.$$

Equating (2.1) entry from above matrices, we get $B_{-n}(x) = -B_n(x)$, that shows balancing polynomials are also extended negatively. Further, as $N^m N^n = N^{m+n}$ for all $m, n \in \mathbb{Z}$, the left expression is

$$N^m N^n = \begin{pmatrix} B_{m+1}(x)B_{n+1}(x) - B_m(x)B_n(x) & B_m(x)B_{n-1}(x) - B_{m+1}(x)B_n(x) \\ B_m(x)B_{n+1}(x) - B_{m-1}(x)B_n(x) & B_{m-1}(x)B_{n-1}(x) - B_m(x)B_n(x) \end{pmatrix},$$

while the right-hand side expression reduces to

$$N^{m+n} = \begin{pmatrix} B_{m+n+1}(x) & -B_{m+n}(x) \\ B_{m+n}(x) & -B_{m+n-1}(x) \end{pmatrix}.$$

Equating (2.1) entry from both the matrices, we obtain the identity given in Proposition 3.2.

Proposition 3.3 (Catalan identity)**.** *For integers* n, r *with* $n > r$, $B_{n-r}(x)B_{n+r}(x) - B_n^2(x) = -B_r^2(x)$.

Proof. Using Binet formula (3.3), the left-hand side expression reduces to

$$-\left(\frac{\lambda_1^{2r}(x) + \lambda_2^{2r}(x) - 2}{[\lambda_1(x) - \lambda_2(x)]^2}\right),$$

and we obtain the desired result.

It can be seen that, for $r = 1$, the Catalan identity reduces to the Cassini formula for balancing polynomials. Further, replacing n by $4n$ and $r = 2n$, and then $n = 2n + r$ in the Catalan identity, we get the following corollary.

***Corollary* 3.1.** $B_{2n}(x)[B_{2n}(x)+B_{6n}(x)] = B_{4n}^2(x)$ *and* $B_{2n}(x)B_{2n+2r}(x)+B_r^2(x) = B_{2n+r}^2(x)$.

Proposition 3.4 (General bilinear index reduction formula)**.** *For integers* n, r, a, b, c, d *with* $a + b = c + d$,

$$B_a(x)B_b(x) - B_c(x)B_d(x) = B_{a-r}(x)B_{b-r}(x) - B_{c-r}(x)B_{d-r}(x).$$

Proof. Using Binet formula (3.3), for all integers n, h and k, the following identity is easily verified:

$$B_{n+h}(x)B_{n+k}(x) - B_n(x)B_{n+h+k}(x) = B_h(x)B_k(x).$$

On setting $n = c$, $h = a - c$ and $k = b - c$ in the above identity gives

$$B_a(x)B_b(x) - B_c(x)B_{a+b-c}(x) = B_{a-c}(x)B_{b-c}(x).$$

Again putting $n = c - r$, $h = a - c$ and $k = b - c$ in the same identity, we obtain

$$B_{a-r}(x)B_{b-r}(x) - B_{c-r}(x)B_{a+b-c-r}(x) = B_{a-c}(x)B_{b-c}(x).$$

Comparing these two identities and using $a + b - c = d$ follows the proof of the formula.

Setting $a = n + 1$, $b = m$, $c = n$, $d = m + 1$ and $r = n - 1$ in the general bilinear index reduction formula, we have the following result.

***Corollary* 3.2.** *For* n, m *integers with* $n \leq m$, $B_{m-n}(x) = B_{n+1}(x)B_m(x) - B_n(x)B_{m+1}(x)$.

We end this section with the derivation of a product formula for balancing polynomials.

Theorem 3.1. *If $B_n(x)$ be the $n\,th$ balancing polynomial, then*

$$B_n(x) = \prod_{1 \le l \le n-1} \left[6x - 2\cos\frac{l\pi}{n} \right].$$

Proof. Clearly, the degree of the balancing polynomial $B_n(x)$ is $n-1$ for $n \ge 1$. To find out a formula for the zeros of $B_n(x)$, we express $B_n(x)$ in terms of hyperbolic functions using Binet formula. Setting $3x = \cosh z$, we have

$$\lambda_1(x) = \cosh z + \sinh z = e^z$$

and

$$\lambda_2(x) = \cosh z - \sinh z = e^{-z}.$$

It follows that

$$B_n(x) = \frac{e^{nz} - e^{-nz}}{e^z - e^{-z}} = \frac{\sinh(nz)}{\sinh z}.$$

Let $z = u + iv$ where the imaginary number $i = \sqrt{-1}$. As $\sinh z \neq 0$, $B_n(x)$ will be zero only when $\sinh(nz) = 0$, that is $e^{2nz} = 1$ which is equivalent to $e^{2nu}(\cos 2nv + i \sin 2nv) = 1$. Equating the real part, $u = 0$ and therefore $\sinh(inv) = i \sin nv = 0$. It follows that, $v = \dfrac{l\pi}{n}$ for any integer l, and hence $z = i\dfrac{l\pi}{n}$. Consequently, for any integer l, we have

$$3x = \cosh\left(i\frac{l\pi}{n}\right) = \cos\left(\frac{l\pi}{n}\right).$$

As a result, the theorem follows when $1 \le l \le n-1$.

Observation 3.2. It is observed that, the number of zeros of the balancing polynomial $B_n(x)$ is the range of l, i.e., the number of roots of the balancing polynomial $B_n(x)$ are in between 1 to $(n-1)$. For instance, if the balancing polynomial $B_4(x) = 216x^3 - 12x$ is considered, then it can be observed that the number of roots of $B_4(x)$ are 3 and they are Indeed, $x = \dfrac{1}{3\sqrt{2}}, \dfrac{1}{3}, -\dfrac{1}{3\sqrt{2}}$.

Observation 3.3. It is also observed that, for $x = 1$, the product formula mentioned in Theorem 3.1 gives an alternating expression for B_n. For example

$$B_4 = \prod_{1 \le k \le 3} \left(6 - 2\cos\frac{k\pi}{4}\right) = \left(6 - 2\cdot\frac{1}{\sqrt{2}}\right)(6 - 2\cdot 0)\left(6 + 2\cdot\frac{1}{\sqrt{2}}\right) = 204.$$

4. Derivative sequences of balancing polynomials. In this section, the sequences that are obtained by differentiating the balancing polynomials are studied. Several properties of these sequences and some relations between the balancing polynomials and their derivatives are also shown.

The identity (3.2) can be rewritten as

$$B_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} (6x)^{n-1-2j}. \tag{4.1}$$

Differentiation of (4.1) with respect to x yields

$$B_n'(x) = \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^j (n-1-2i) \binom{n-1-j}{j} (6x)^{n-2j-2}, \tag{4.2}$$

with $B_0'(x) = 0$ and $B_1'(x) = 0$. Few first derivative sequences of balancing polynomials with respect to x are given below

$$\begin{aligned}
&B_1'(x) = 0, \qquad B_2'(x) = 6,\\
&B_3'(x) = 72x, \qquad B_4'(x) = 648x^2 - 12,\\
&B_5'(x) = 5184x^3 - 216x,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Different types of numerical sequences can be generated on substitution of integers in the variable x of the balancing polynomials such as

$$\begin{aligned}
&\{B_n'(1)\} = \{0, 6, 72, 636, 4968, \ldots\},\\
&\{B_n'(2)\} = \{0, 6, 144, 2580, 41040, \ldots\},\\
&\{B_n'(3)\} = \{0, 6, 216, 5820, 139320, \ldots\},\\
&\{B_n'(4)\} = \{0, 6, 288, 10356, 323612, \ldots\}.
\end{aligned}$$

The following results are some interesting connections between balancing polynomials with their first derivatives.

Proposition 4.1. *Let $B_n'(x)$ denote the first derivative of $B_n(x)$. Then*

$$B_n'(x) = \frac{3nB_{n+1}(x) - 18xB_n(x) - 3nB_{n-1}(x)}{2(9x^2-1)} \quad \textit{for} \quad x \neq \pm\frac{1}{3}. \tag{4.3}$$

Proof. Recall that $\lambda_1(x) = 3x + \sqrt{9x^2-1}$ and $\lambda_2(x) = 3x - \sqrt{9x^2-1}$ with $\lambda_1(x)\lambda_2(x) = 1$. Their first derivatives are respectively

$$\lambda_1'(x) = \frac{6\lambda_1(x)}{\triangle} \quad \text{and} \quad \lambda_2'(x) = -\frac{6\lambda_2(x)}{\triangle},$$

where $\triangle = 2\sqrt{9x^2-1}$. Therefore, use of Binet formula (3.3) yields

$$B_n'(x) = \frac{d}{dx}\left[\frac{\lambda_1^n(x) - \lambda_2^n(x)}{\triangle}\right] =$$

$$= \frac{6n\triangle\left[\dfrac{\lambda_1^n(x) + \lambda_2^n(x)}{\triangle}\right] - 36x\left[\dfrac{\lambda_1^n(x) - \lambda_2^n(x)}{\triangle}\right]}{\triangle^2} =$$

$$= \frac{6n\left[\dfrac{\lambda_1^n(x)\{\lambda_1(x) - \lambda_2(x)\} + \lambda_2^n(x)\{\lambda_1(x) - \lambda_2(x)\}}{\triangle}\right] - 36xB_n(x)}{\triangle^2} =$$

$$= \frac{6n\left[\dfrac{\lambda_1^{n+1}(x)-\lambda_2^{n+1}(x)}{\triangle} - \dfrac{\lambda_1^{n-1}(x)-\lambda_2^{n-1}(x)}{\triangle}\right] - 36xB_n(x)}{\triangle^2} =$$

$$= \frac{3nB_{n+1}(x) - 18xB_n(x) - 3nB_{n-1}(x)}{2(9x^2-1)},$$

which completes the proof.

In particular, for $x = 1$, the identity (4.3) reduces to

$$B_n'(1) = \frac{3n(B_{n+1} - B_{n-1}) - 18B_n}{16}.$$

Using the well known identity $B_{n+1} - B_{n-1} = 2C_n$ [9], the above identity further reduces to

$$B_n'(1) = \frac{3nC_n - 9B_n}{8}.$$

Using the recurrence relation for balancing polynomials, (4.3) may also be rewritten as

$$B_n'(x) = \frac{9(n-1)xB_n(x) - 3nB_{n-1}(x)}{9x^2-1} \quad \text{for} \quad x \neq \pm\frac{1}{3}. \tag{4.4}$$

The derivative of balancing polynomials may be obtained by the self-convolution of balancing polynomials. The following proposition establishes the fact.

Proposition 4.2 (the derivative of balancing polynomials and the convolved balancing polynomials).

$$B_1'(x) = 0 \quad and \quad B_n'(x) = \sum_{j=1}^{n-1} 6B_j(x)B_{n-j}(x) \quad for \quad n > 1. \tag{4.5}$$

Proof. The proof of this result is based on induction on n. Clearly the result holds for $n = 2$. As an inductive hypothesis, let the formula is true for every polynomial $B_m'(x)$ with $m \leq n$. For inductive step, differentiate the recurrence relation for balancing polynomials w.r.t. x and use the inductive hypothesis to get

$$B_{n+1}'(x) = 6B_n(x) + 6xB_n'(x) - B_{n-1}'(x) =$$

$$= 6B_n(x) + 6x\sum_{j=1}^{n-1} 6B_j(x)B_{n-j}(x) - \sum_{j=1}^{n-2} 6B_j(x)B_{n-1-j}(x) =$$

$$= 6B_n(x) + 36xB_{n-1}(x)B_1(x) + 6x\sum_{j=1}^{n-2} 6B_j(x)B_{n-j}(x) - \sum_{j=1}^{n-2} 6B_j(x)B_{n-1-j}(x) =$$

$$= 6B_n(x) + 36xB_{n-1}(x) + \sum_{j=1}^{n-2} 6B_j(x)[6B_{n-j}(x) - B_{n-1-j}(x)] =$$

$$= 6B_n(x) + 36xB_{n-1}(x) + \sum_{j=1}^{n-2} 6B_j(x)B_{n+1-j}(x) =$$

$$= 6B_n(x)B_1(x) + 6B_{n-1}(x)B_2(x) + \sum_{j=1}^{n-2} 6B_j(x)B_{n+1-j}(x) =$$

$$= \sum_{j=1}^{n} B_j(x)B_{n+1-j}(x),$$

which ends the proof.

Observation 4.1. It is observed that, the identities (4.4) and (4.5) together yields the following formula:

$$\sum_{j=1}^{n-1} B_j(x)B_{n-j}(x) = \frac{3(n-1)xB_n(x) - nB_{n-1}(x)}{2(9x^2-1)} \quad for\ n > 1 \quad and \quad x \neq \pm\frac{1}{3}.$$

Setting $x = 1$, the corresponding formula for balancing numbers will obtain.

Proposition 4.3. *For* $n \geq 1,\ B'_{n+1}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 6(n-2j)B_{n-2j}$.

Proof. Method of induction is used to prove this result. For $n = 1$, the result is trivial, since

$$B'_2(x) = \sum_{j=0}^{0} 6(1-j)B_{1-j} = 6.$$

We assume that the formula is true for $k \leq n$. Then

$$B'_n(x) = \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} 6(n-1-2j)B_{n-1-2j}$$

and

$$B'_{n+1}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 6(n-2j)B_{n-2j}.$$

Differentiating the recurrence relation for balancing polynomials $B_{n+2}(x) = 6xB_{n+1}(x) - B_n(x)$, using the hypothesis and after some algebra, the result is easily verified for $n+1$.

Proposition 4.4. *For* $n \geq 1,\ B_n(x) = \frac{1}{6n}\Big[B'_{n+1}(x) - B'_{n-1}(x)\Big]$.

Proof. Using identity (4.2), the right-hand side expression reduces to

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (6x)^{n-2j} - \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^j \binom{n-2-j}{j} (6x)^{n-2-2j}.$$

Further simplification yields

$$(6x)^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \left[\binom{n-j}{j} + \binom{n-1-j}{j-1}\right] (6x)^{n-2j}.$$

Therefore, using usual properties for combinations, we obtain

$$B_{n+1}(x) - B_{n-1}(x) = (6x)^n + n\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^j \binom{n-1-j}{j-1} \frac{1}{j}(6x)^{n-2j}.$$

Differentiating the above equation with respect to x, we get

$$B'_{n+1}(x) - B'_{n-1}(x) = 6n(6x)^{n-1} + 6n\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor}(-1)^j \binom{n-1-j}{j-1} \frac{n-2i}{j}(6x)^{n-1-2j}.$$

It follows that

$$\frac{B'_{n+1}(x) - B'_{n-1}(x)}{6n} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{j-1} \binom{n-1-j}{j-1} (6x)^{n-1-2j} = B_n(x),$$

and the result follows.

Explicit formulas for any derivative can be obtained similarly. For example, differentiating (4.2) with respect to x, the second derivative of balancing polynomials becomes

$$B''_n(x) = \sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} (-1)^j 6^{n-1-2j}(n-1-2j)(n-2j-2) \binom{n-1-j}{j} x^{n-2j-3}, \tag{4.6}$$

with $B''_1(x) = 0, B''_2(x) = 0$ and for $n \geq 2$.

Some second derivative sequences of balancing polynomials with respect to x are given below

$$\begin{aligned}
&B''_1(x) = 0, B''_2(x) = 0,\\
&B''_3(x) = 72, B''_4(x) = 1296x,\\
&B''_5(x) = 15552x^2 - 216,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

For the second derivative of balancing polynomials, the numerical sequences by replacing x with an integer are given by

$$\begin{aligned}
\{B''_n(1)\} &= \{0, 0, 72, 1296, 15336, \ldots\},\\
\{B''_n(2)\} &= \{0, 0, 72, 2592, 61992, \ldots\},\\
\{B''_n(3)\} &= \{0, 0, 72, 3888, 139752, \ldots\},\\
\{B''_n(4)\} &= \{0, 0, 72, 5184, 248832, \ldots\}.
\end{aligned}$$

Differentiating r times the identity (4.6) yields the recurrence relation for derivative sequences as follows:

$$B^{(r)}_{n+1}(x) = \begin{cases} 0, & \text{if } n < r,\\ 6^r r!, & \text{if } n = r,\\ \dfrac{1}{n-r}\left[6nxB^{(r)}_n(x) - (n+r)B^{(r)}_{n-1}(x)\right], & \text{if } n > r. \end{cases}$$

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