UDC 517.5

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ON AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

ПРО ОПЕРАТОР, ЩО ЗБЕРІГАЄ НЕРІВНОСТІ МІЖ ПОЛІНОМАМИ

Let $P(z)$ be a polynomial of degree n. We consider an operator D_α that maps $P(z)$ into $D_\alpha P(z) := nP(z)+(\alpha-z)P'(z)$ and establish some results concerning the estimates of $|D_\alpha P(z)|$ on the disk $|z| = R \geq 1$, and thereby obtain extensions and generalizations of a number of well-known polynomial inequalities.

Нехай $P(z)$ — многочлен степеня n. У роботі розглянуто оператор D_α, що відображає $P(z)$ в $D_\alpha P(z) :=$ $:= nP(z) + (\alpha - z)P'(z)$, та встановлено деякі результати щодо оцінок $|D_\alpha P(z)|$ на крузі $|z| = R \geq 1$ і, таким чином, отримано розширення та узагальнення багатьох відомих нерівностей для поліномів.

1. Introduction. Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree at most n. Let D_{k-} denote the region inside the disk $\mathbb{T}_k = \{z \in \mathbb{C}/|z| = k > 0\}$ and D_{k+} the region outside $\mathbb{T}_k$. For $P \in \mathbb{P}_n$, set

$$M(P,k) = \max_{z\in\mathbb{T}_k} |P(z)| \qquad \text{and} \qquad m(P,k) = \min_{z\in\mathbb{T}_k} |P(z)|.$$

If $P \in \mathbb{P}_n$, then concerning the estimate of $M(P',1)$ on $\mathbb{T}_1$, we have

$$M(P',1) \leq nM(P,1). \tag{1.1}$$

The above inequality is an immediate consequence of Bernstein's inequality [3] on the derivative of a trigonometric polynomial, and is best possible with equality holding for the polynomial $P(z) = \lambda z^n$, λ being a complex number.

If we restrict ourselves to the class of polynomials having no zeros in in the open unit disk, then the above inequality can be sharpened. In fact, Erdös conjectured, and later Lax [8] proved, that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1 \cup D_{1+}$, then

$$M(P',1) \leq \frac{n}{2} M(P,1). \tag{1.2}$$

The above inequality is best possible, and holds with equality for all polynomials having their zeros on $\mathbb{T}_1$.

As a refinement of (1.2), Aziz and Dawood [1] proved that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1 \cup D_{1+}$, then

$$M(P',1) \leq \frac{n}{2}\big\{M(P,1) - m(P,1)\big\}. \tag{1.3}$$

Further, as an extension of (1.3), Jain [7] (see also Dewan and Hans [5]) proved that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1 \cup D_{1+}$, then, for any β with $|\beta| \leq 1$ and $z \in \mathbb{T}_1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leq \frac{n}{2}\left\{\left(\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right)M(P,1) - \left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)m(P,1)\right\}. \tag{1.4}$$

For $P \in \mathbb{P}_n$, the polar derivative $D_\alpha P(z)$ of $P(z)$ with respect to the point α is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

It is easy to see that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, and $D_\alpha P(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\}=P'(z).$$

Corresponding to a given nth degree polynomial $P(z)$, we construct a sequence of polar derivatives as follows:

$$D_{\alpha_1}P(z)=nP(z)+(\alpha_1-z)P'(z),$$

and

$$D_{\alpha_k}D_{\alpha_{k-1}}\ldots D_{\alpha_1}P(z)=(n-k+1)D_{\alpha_{k-1}}D_{\alpha_{k-2}}\ldots D_{\alpha_1}P(z)+$$
$$+(\alpha_k-z)\big(D_{\alpha_{k-1}}D_{\alpha_{k-2}}\ldots D_{\alpha_1}P(z)\big)',\quad k=2,3,\ldots,n.$$

The points $\alpha_1,\alpha_2,\ldots\alpha_k\in\mathbb{C},\ k=1,2,3,\ldots,n,$ may or may not be distinct. Like the kth ordinary derivative $P^{(k)}(z)$ of $P(z)$, the k th polar derivative $D_{\alpha_k}D_{\alpha_{k-1}}\ldots D_{\alpha_1}P(z)$ of $P(z)$ is a polynomial of degree at most $n-k$.

As an extension of inequality (1.3) to the polar derivative of a polynomial, Aziz and Shah [2] (see also Mir and Baba [11]) showed that if $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1\cup D_{1+}$, then, for every α with $|\alpha|\geq 1$,

$$M(D_\alpha P,1)\leq\frac{n}{2}\Big\{\big(|\alpha|+1\big)M(P,1)-\big(|\alpha|-1\big)m(P,1)\Big\}. \tag{1.5}$$

In the literature, there exist various refinements and generalization of (1.2)–(1.5) and here, we mention a few of them.

Theorem 1.1 ([9], Theorem 3). *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1\cup D_{1+}$, then, for α, $\beta\in\mathbb{C}$ with $|\alpha|\geq 1,\ |\beta|\leq 1$ and $z\in\mathbb{T}_1$,*

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|\leq$$
$$\leq\frac{n}{2}\left\{\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|+\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)M(P,1)-\right.$$
$$\left.-\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)m(P,1)\right\}. \tag{1.6}$$

Theorem 1.2 ([6], Theorem 2). *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1\cup D_{1+}$, then, for every complex number β with $|\beta|\leq 1$, $1\leq s\leq n$ and $z\in\mathbb{T}_1$,*

$$\left|z^sP^{(s)}(z)+\frac{\beta n(n-1)\ldots(n-s+1)}{2^s}P(z)\right|\leq$$
$$\leq\frac{n(n-1)\ldots(n-s+1)}{2}\left\{\left(\left|1+\frac{\beta}{2^s}\right|+\left|\frac{\beta}{2^s}\right|\right)M(P,1)-\right.$$
$$\left.-\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)m(P,1)\right\}. \tag{1.7}$$

Theorem 1.3 ([13], Theorem 1.5). *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1 \cup D_{k+}$, $k \leq 1$, then, for every complex number β with $|\beta| \leq 1$, $1 \leq s \leq n$ and $z \in \mathbb{T}_1$, we have*

$$\max_{|z|=1}\left|z^sP^{(s)}(z)+\frac{\beta n(n-1)\ldots(n-s+1)}{(1+k)^s}P(z)\right| \leq$$
$$\leq \frac{n(n-1)\ldots(n-s+1)}{2}\left\{\left(\frac{1}{k^n}\left|1+\frac{\beta}{(1+k)^s}\right|+\left|\frac{\beta}{(1+k)^s}\right|\right)M(P,k)-\right.$$
$$\left.-\left(\frac{1}{k^n}\left|1+\frac{\beta}{(1+k)^s}\right|-\left|\frac{\beta}{(1+k)^s}\right|\right)m(P,k)\right\}. \tag{1.8}$$

2. Statements of results. In this section we state our main results. Their proofs are given in the next section. From now on, we shall always assume that every $P \in \mathbb{P}_n$ is a polynomial of degree $n \geq 2$. Our main aim is to extend (1.8) to the polar derivative of a polynomial and thereby obtain a compact generalization of (1.7) as well. We start by proving the following result.

Theorem 2.1. *Let $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k \cup D_{k-}$, $k \leq 1$. Let $t \in \mathbb{N}$, $t \leq n-1$, and $(\alpha_i)_{i=1}^t$ be complex numbers satisfying $|\alpha_i| \geq k$ for $1 \leq i \leq t$. Then, for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and for every $z \in \mathbb{T}_1 \cup D_{1+}$,*

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right| \geq$$
$$\geq \frac{n(n-1)\ldots(n-t+1)}{k^n}|z|^n\left|\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right|m(P,k). \tag{2.1}$$

***Remark* 2.1.** If we take $\alpha_1 = \alpha_2 = \ldots = \alpha_t = \alpha$, divide both sides of (2.1) by $|\alpha|^t$ and let $|\alpha| \to \infty$, we obtain the following result.

***Corollary* 2.1.** *Let $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k \cup D_{k-}$, $k \leq 1$. Then, for every β with $|\beta| \leq 1$, $1 \leq t \leq n-1$ and for every $z \in \mathbb{T}_1 \cup D_{1+}$,*

$$\left|z^tP^{(t)}(z)+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}P(z)\right| \geq$$
$$\geq \frac{n(n-1)\ldots(n-t+1)}{k^n}|z|^n\left|1+\frac{\beta}{(1+k)^t}\right|m(P,k). \tag{2.2}$$

***Remark* 2.2.** For $k = 1$, Corollary 2.1 in particular reduces to a result of Hans and Lal ([6], Lemma 7) and for $|z| = 1$, Corollary 2.1 is exactly Theorem 2.1 recently proved by Zireh [13]. Further, for $k = 1$, Theorem 2.1 reduces to a result of Bidkham and Mezerji ([4], Corollary 3).

Next, we present the following extension of (1.8) to the polar derivative.

Theorem 2.2. *Let $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k \cup D_{k+}$, $k \leq 1$. Let $t \in \mathbb{N}$, $t \leq n-1$, and $(\alpha_i)_{i=1}^t$ be complex numbers satisfying $|\alpha_i| \geq k$ for $1 \leq i \leq t$. Then, for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and for every $z \in \mathbb{T}_1 \cup D_{1+}$,*

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right| \leq$$
$$\leq \frac{n(n-1)\ldots(n-t+1)}{2}\left\{\left(\frac{|z|^n}{k^n}\left|\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right|+\right.\right.$$

$$+\left|z^t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right|\Bigg)M(P,k)-$$

$$-\Bigg(\frac{|z|^n}{k^n}\left|\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right|-$$

$$-\left|z^t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right|\Bigg)m(P,k)\Bigg\}. \tag{2.3}$$

***Remark* 2.3.** If we take $\alpha_1=\alpha_2=\ldots=\alpha_t=\alpha$, then divide both sides of (2.3) by $|\alpha|^t$ and let $|\alpha|\to\infty$, we recover (1.8). For $t=1$, Theorem 2.2 gives the following result.

***Corollary* 2.2.** *Let $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k\cup D_{k+}$, $k\le 1$, then, for $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\ge k$, $|\beta|\le 1$ and $z\in\mathbb{T}_1\cup D_{1+}$,*

$$\left|zD_\alpha P(z)+\frac{\beta n(|\alpha|-k)}{1+k}P(z)\right|\le$$

$$\le\frac{n}{2}\Bigg\{\Bigg(\frac{|z|^n}{k^n}\left|\alpha+\frac{\beta(|\alpha|-k)}{1+k}\right|+\left|z+\frac{\beta(|\alpha|-k)}{1+k}\right|\Bigg)M(P,k)-$$

$$-\Bigg(\frac{|z|^n}{k^n}\left|\alpha+\frac{\beta(|\alpha|-k)}{1+k}\right|-\left|z+\frac{\beta(|\alpha|-k)}{1+k}\right|\Bigg)m(P,k)\Bigg\}. \tag{2.4}$$

***Remark* 2.4.** For $k=1$, the above Corollary 2.2 simplifies to inequality (1.6). For $\beta=0$, Theorem 2.2 reduces to the following result which gives a generalization of inequality (1.5).

***Corollary* 2.3.** *Let $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k\cup D_{k+}$, $k\le 1$. Let $t\in\mathbb{N}$, $t\le n-1$, and $(\alpha_i)_{i=1}^t$ be complex numbers satisfying $|\alpha_i|\ge k$ for $1\le i\le t$. Then, for $z\in\mathbb{T}_1$, we have*

$$\left|D_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)\right|\le$$

$$\le\frac{n(n-1)\ldots(n-t+1)}{2}\Bigg\{\Bigg(\frac{1}{k^n}|\alpha_1\alpha_2\ldots\alpha_t|+1\Bigg)M(P,k)-$$

$$-\Bigg(\frac{1}{k^n}|\alpha_1\alpha_2\ldots\alpha_t|-1\Bigg)m(P,k)\Bigg\}. \tag{2.5}$$

***Remark* 2.5.** For $k=t=1$, (2.5) reduces to (1.5) and for $k=1$, Corollary 2.3 reduces to a result of Bidkham and Mezerji ([4], Corollary 7). Dividing the two sides of (2.4) by $|\alpha|$ and let $|\alpha|\to\infty$, we have the following generalization of the inequality (1.4).

***Corollary* 2.4.** *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k\cup D_{k+}$, $k\le 1$, then, for $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\ge k$, $|\beta|\le 1$ and $z\in\mathbb{T}_1\cup D_{1+}$,*

$$\left|zP'(z)+\frac{n\beta}{1+k}P(z)\right|\le$$

$$\le\frac{n}{2}\Bigg\{\Bigg(\frac{|z|^n}{k^n}\left|1+\frac{\beta}{1+k}\right|+\left|\frac{\beta}{1+k}\right|\Bigg)M(P,k)-$$

$$-\Bigg(\frac{|z|^n}{k^n}\left|1+\frac{\beta}{1+k}\right|-\left|\frac{\beta}{1+k}\right|\Bigg)m(P,k)\Bigg\}. \tag{2.6}$$

3. Proofs. We need the following lemmas for the proof of theorems.

Lemma 3.1 ([12], Lemma 2.3)**.** *Let $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_k \cup D_{k-}$, $k \leq 1$. Let $(\alpha_i)_{i=1}^t$, $t \leq n-1$, are complex numbers satisfying $|\alpha_i| \geq k$, $1 \leq i \leq t$. Then, for $z \in \mathbb{T}_1$, we have*

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z)\right| \geq$$

$$\geq \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)|P(z)|. \tag{3.1}$$

Lemma 3.2. *Let $P, F \in \mathbb{P}_n$ and $F(z)$ has all its zeros in $\mathbb{T}_k \cup D_{k-}$, $k \leq 1$, such that $|P(z)| \leq$ $\leq |F(z)|$ for $z \in \mathbb{T}_k$. Let $t \in \mathbb{N}$, $t \leq n-1$, and $(\alpha_i)_{i=1}^t$, are complex numbers satisfying $|\alpha_i| \geq k$, for $1 \leq i \leq t$. Then, for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $z \in \mathbb{T}_1 \cup D_{1+}$,*

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right| \leq$$

$$\leq \left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} F(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)F(z)\right|. \tag{3.2}$$

***Proof of Lemma* 3.2.** By hypothesis $|P(z)| \leq |F(z)|$ on $|z| = k$. Hence, for any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, we have $|\alpha P(z)| < |F(z)|$ on the circle $|z| = k$. Further, all the zeros of $F(z)$ lie in $|z| \leq k$, it follows by Rouche's theorem that all the zeros of $G(z) = F(z) + \alpha P(z)$ with $|\alpha| < 1$, also lie in $|z| \leq k$, $k \leq 1$. By applying Lemma 3.1 to $G(z)$, we get, for $|\alpha_i| \geq k$, $1 \leq i \leq t$, $|\alpha| < 1$ and $|z| = 1$,

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} G(z)\right| \geq$$

$$\geq \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)|G(z)|.$$

Equivalently, for $|z| = 1$,

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} F(z) + \alpha z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z)\right| \geq$$

$$\geq \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)|F(z)+\alpha P(z)|. \tag{3.3}$$

Therefore, for any β with $|\beta| < 1$, we have by Rouche's theorem, the polynomial

$$T(z) = \left(z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} F(z) + \alpha z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z)\right) +$$

$$+ \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)(F(z)+\alpha P(z)) =$$

$$= \left(z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} F(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)F(z)\right) +$$

$$+ \alpha\left(z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right) \neq$$

$$\neq 0 \quad \text{for} \quad |z| \geq k. \tag{3.4}$$

Since $k \le 1$, we have $T(z) \ne 0$ for $|z| \ge 1$ as well. Now choosing the argument of α in (3.4) suitably and letting $|\alpha| \to 1$, we get, for $|z| \ge 1$ and $|\beta| < 1$,

$$\left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)P(z) \right| \le$$

$$\le \left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} F(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)F(z) \right|.$$

For β with $|\beta| = 1$, the above inequality holds by continuity.

Lemma 3.2 is proved.

Lemma 3.3. *Let $P \in \mathbb{P}_n$, $t \in \mathbb{N}$, $t \le n-1$, and $(\alpha_i)_{i=1}^t$ are complex numbers satisfying $|\alpha_i| \ge k$, for $1 \le i \le t$. Then, for any complex β with $|\beta| \le 1$ and $|z| \ge 1$,*

$$\left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)P(z) \right| +$$

$$+ k^n \left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} Q\Big(\frac{z}{k^2}\Big) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)Q\Big(\frac{z}{k^2}\Big) \right| \le$$

$$\le n(n-1)\ldots(n-t+1)\left\{ \frac{|z|^n}{k^n}\left| \alpha_1\alpha_2\ldots\alpha_t + \frac{\beta(|\alpha_1| - k)\ldots(|\alpha_t| - k)}{(1+k)^t} \right| + \right.$$

$$\left. + \left| z^t + \frac{\beta(|\alpha_1| - k)\ldots(|\alpha_t| - k)}{(1+k)^t} \right| \right\} M(P,k), \tag{3.5}$$

where $Q(z) = z^n \overline{P\left(\dfrac{1}{\overline{z}}\right)}$.

Proof. Since $M(P,k) = \max_{|z|=k} |P(z)|$. It follows by Rouche's theorem, that for any γ with $|\gamma| > 1$, the polynomial $T(z) = P(z) + \dfrac{\gamma M(P,k) z^n}{k^n}$ has all zeros in $|z| < k$. If we set

$$S(z) = z^n \overline{T\left(\frac{1}{\overline{z}}\right)} = Q(z) + \frac{\overline{\gamma} M(P,k)}{k^n},$$

then $\left| k^n S\Big(\dfrac{z}{k^2}\Big) \right| = |T(z)|$ for $|z| = k$. Hence, for every complex η with $|\eta| > 1$, the polynomial $W(z) = k^n S\Big(\dfrac{z}{k^2}\Big) + \eta T(z)$ has all its zeros in $|z| < k$. Therefore, by applying Lemma 3.1 to $W(z)$, we obtain for $|\alpha_i| \ge k$, $1 \le i \le t$, $t \le n-1$,

$$\left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} W(z) \right| \ge$$

$$\ge \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)|W(z)| \quad \text{for} \quad |z| = 1.$$

This implies, for any β with $|\beta| < 1$ and $|z| = 1$,

$$\left| z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} W(z) \right| >$$

$$> |\beta| \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1| - k)\ldots(|\alpha_t| - k)|W(z)| \quad \text{for} \quad |z| = 1. \tag{3.6}$$

Since by Laguerre's theorem [10, p. 52], the polynomial $D_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}W(z)$ has all its zeros in $|z|<k,\ k\leq 1,$ for every α_i with $|\alpha_i|\geq k,\ 1\leq i\leq t,\ t\leq n-1.$ Rouche's theorem together with (3.6) implies that the polynomial

$$G(z)=z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}W(z)+$$
$$+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)W(z)=$$
$$=k^n\left(z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}S\left(\frac{z}{k^2}\right)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)S\left(\frac{z}{k^2}\right)\right)+$$
$$+\eta\left(z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}T(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)T(z)\right)\neq$$
$$\neq 0\quad\text{for}\quad |z|\geq k. \tag{3.7}$$

As $k\leq 1,$ we have $G(z)\neq 0$ for $|z|\geq 1$ as well. Hence on choosing the arguement of η suitably in (3.7) and letting $|\eta|\to 1,$ we get, for $|z|\geq 1$ and $|\beta|<1,$

$$k^n\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}S\left(\frac{z}{k^2}\right)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)S\left(\frac{z}{k^2}\right)\right|\leq$$
$$\leq\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}T(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)T(z)\right|. \tag{3.8}$$

Replacing $T(z)$ by $P(z)+\dfrac{\gamma M(P,k)z^n}{k^n}$ and $S(z)$ by $Q(z)+\dfrac{\overline{\gamma}M(P,k)}{k^n}$ in (3.8), we get, for $|z|\geq 1,$

$$k^n\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}Q\left(\frac{z}{k^2}\right)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)+\right.$$
$$\left.+\frac{\overline{\gamma}}{k^n}n(n-1)\ldots(n-t+1)\left\{z^t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right\}M(P,k)\right|\leq$$
$$\leq\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)+\right.$$
$$\left.+\frac{\gamma}{k^n}\,n(n-1)\ldots(n-t+1)\left\{\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right\}M(P,k)z^n\right|. \tag{3.9}$$

Applying Lemma 3.2 to the right-hand side of (3.9) and choosing the argument of γ so that

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)+\right.$$
$$\left.+\frac{\gamma}{k^n}n(n-1)\ldots(n-t+1)\left\{\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right\}M(P,k)z^n\right|=$$
$$=\left|\frac{\gamma}{k^n}\,n(n-1)\ldots(n-t+1)\left\{\alpha_1\alpha_2\ldots\alpha_t+\frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t}\right\}M(P,k)z^n\right|-$$
$$-\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right|,$$

we get

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \beta \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right| +$$

$$+k^n \left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} Q\left(\frac{z}{k^2}\right) + \beta \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)\right| \leq$$

$$\leq n(n-1)\ldots(n-t+1)|\gamma| \left\{ \frac{|z|^n}{k^n} \left| \alpha_1\alpha_2\ldots\alpha_t + \frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t} \right| + \right.$$

$$\left. + \left| z^t + \frac{\beta(|\alpha_1|-k)\ldots(|\alpha_t|-k)}{(1+k)^t} \right| \right\} M(P,k). \qquad (3.10)$$

Making $|\gamma| \to 1$ and using the continuity for $|\beta| = 1$ in (3.10), we get the desired result.

***Proof of Theorem* 2.1.** If $P(z)$ has a zeros on $|z| = k$, then the theorem is trivial. Therefore, assume that $P(z)$ has all its zeros in $|z| < k,\ k \leq 1$, so that $m(P,k) > 0$ and hence for every γ with $|\gamma| < 1$, we have

$$\left| \frac{\gamma m(P,k) z^n}{k^n} \right| < |P(z)| \quad \text{for} \quad |z| = k.$$

It follows by Rouche's theorem, that the polynomial $G(z) = P(z) - \dfrac{\gamma m(P,k) z^n}{k^n}$ of degree n has all its zeros in $|z| < k,\ k \leq 1$. On applying Lemma 3.1 to $G(z)$, we have, for $|\alpha_i| \geq k,\ 1 \leq i \leq t$, and $|z| = 1$,

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} G(z)\right| \geq \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)|G(z)|,$$

i.e.,

$$\left|z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) - \frac{\gamma m(P,k)}{k^n} n(n-1)\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t z^n\right| \geq$$

$$\geq \frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)\left|P(z) - \frac{\gamma m(P,k) z^n}{k^n}\right| \quad \text{for} \quad |z| = 1. \qquad (3.11)$$

Applying Laguerre's theorem [10, p. 52] repeatedly, we deduce that for $|\alpha_i| \geq k,\ 1 \leq i \leq t$, and $|\gamma| < 1$, the polynomial $D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} G(z)$ has all its zeros in $|z| < k,\ k \leq 1$, and therefore for every complex β with $|\beta| < 1$, the polynomial

$$T(z) = \left(z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) - \frac{\gamma m(P,k)}{k^n} n(n-1)\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t z^n\right) +$$

$$+ \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)\left(P(z) - \frac{\gamma m(P,k) z^n}{k^n}\right) =$$

$$= \left(z^t D_{\alpha_t} D_{\alpha_{t-1}} \ldots D_{\alpha_1} P(z) + \frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right) -$$

$$- \frac{\gamma m(P,k) z^n}{k^n} \left\{ n(n-1)\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t + \right.$$

$$+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)\Bigg\}\neq 0 \quad \text{for} \quad |z|\geq k. \tag{3.12}$$

Since $k\leq 1$, we have $T(z)\neq 0$ for $|z|\geq 1$ as well. Now choosing the argument of γ in (3.12) suitably and letting $|\gamma|\to 1$, we get, for $|z|\geq 1$ and $|\beta|<1$,

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right|\geq$$

$$\geq\left|\frac{m(P,k)z^n}{k^n}\Bigg\{n(n-1)\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t+\right.$$

$$\left.+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)\Bigg\}\right|$$

or

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)\right|\geq$$

$$\geq\frac{|z|^n}{k^n}\left|n(n-1)\ldots(n-t+1)\alpha_1\alpha_2\ldots\alpha_t+\right.$$

$$\left.+\frac{\beta n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)\right|m(P,k).$$

For β with $|\beta|=1$, the above inequality holds by continuity.

***Proof of Theorem* 2.2.** Since $m(P,k)=\min_{z\in\mathbb{T}_k}|P(z)|$. Also $P(z)$ has all its zeros in $|z|\geq k$, $k\leq 1$, therefore

$$m(P,k)\leq|P(z)| \quad \text{for} \quad |z|=k.$$

Hence, it follows by Rouche's theorem that for $m(P,k)>0$ and for any complex λ with $|\lambda|<1$, the polynimial $h(z)=P(z)-\lambda m(P,k)$ does not vanish in $|z|<k,\ k\leq 1$. Let

$$g(z)=z^n\overline{h\left(\frac{1}{\overline{z}}\right)}=z^n\overline{P\left(\frac{1}{\overline{z}}\right)}-\overline{\lambda}m(P,k)z^n=Q(z)-\overline{\lambda}m(P,k)z^n,$$

then the polynomial $g\Big(\dfrac{z}{k^2}\Big)$ has all its zeros in $|z|\leq k$. Also $\left|k^ng\Big(\dfrac{z}{k^2}\Big)\right|=|h(z)|$ for $|z|=k$. By applying Lemma 3.2 to $k^ng\Big(\dfrac{z}{k^2}\Big)$, we get, for $|\alpha_i|\geq k,\ 1\leq i\leq t,\ t\leq n-1,\ |\beta|\leq 1$ and $|z|\geq 1$,

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}h(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)h(z)\right|\leq$$

$$\leq k^n\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}g\Big(\frac{z}{k^2}\Big)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)g\Big(\frac{z}{k^2}\Big)\right|.$$

Equivalently for $|z|\geq 1$, we obtain

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\ldots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\ldots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\ldots(|\alpha_t|-k)P(z)-\right.$$

$$-\lambda n(n-1)\dots(n-t+1)\left(z^t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right)m(P,k)\Bigg|\leq$$

$$\leq k^n\Bigg|z^tD_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\left(\frac{z}{k^2}\right)+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)-$$
$$-\overline{\lambda}\frac{n(n-1)\dots(n-t+1)}{k^{2n}}\left(\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right)m(P,k)z^n\Bigg|. \tag{3.13}$$

Since $P(z)\neq 0$ in $|z|<k,\ k\leq 1$, we have $Q\left(\dfrac{z}{k^2}\right)$ has all its zeros in $|z|\leq k$ and

$$k^n\min_{|z|=k}\left|Q\left(\frac{z}{k^2}\right)\right|=\min_{|z|=k}|P(z)|=m(P,k).$$

Hence by inequality (2.1) of Theorem 2.1 applied to $Q\left(\dfrac{z}{k^2}\right)$, we get, for $|z|\geq 1$,

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\left(\frac{z}{k^2}\right)+\frac{\beta n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)\right|\geq$$
$$\geq\frac{n(n-1)\dots(n-t+1)}{k^n}\left|\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|\min_{|z|=k}\left|Q\left(\frac{z}{k^2}\right)\right|=$$
$$=\frac{n(n-1)\dots(n-t+1)}{k^{2n}}\left|\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|m(P,k). \tag{3.14}$$

Now choosing the arguement of λ on the right-hand side of (3.13), such that

$$k^n\Bigg|z^tD_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\left(\frac{z}{k^2}\right)+$$
$$+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)-$$
$$-\overline{\lambda}\ \frac{n(n-1)\dots(n-t+1)}{k^n}\left(\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right)m(P,k)z^n\Bigg|=$$
$$=k^n\Bigg|z^tD_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\left(\frac{z}{k^2}\right)+$$
$$+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\left(\frac{z}{k^2}\right)\Bigg|-$$
$$-|\lambda|\frac{n(n-1)\dots(n-t+1)}{k^n}\left|\left(\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right)m(P,k)z^n\right|,$$

which is possible by (3.14), we have from (3.13), for $|z|\geq 1$,

$$\left|z^tD_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)P(z)\right|-$$
$$-|\lambda|\ n(n-1)\dots(n-t+1)\left|z^t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|m(P,k)\leq$$

$$\leq k^n\left|z^t D_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\Big(\frac{z}{k^2}\Big)+\right.$$

$$\left.+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\Big(\frac{z}{k^2}\Big)\right|-$$

$$-\frac{|\lambda||z|^n n(n-1)\dots(n-t+1)}{k^n}\left|\alpha_1\alpha_2\dots\alpha_t+\right.$$

$$\left.+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|m(P,k). \tag{3.15}$$

Letting $|\lambda|\to 1$, we obtain from (3.15), for $|z|\geq 1$,

$$\left|z^t D_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}P(z)+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)P(z)\right|-$$

$$-k^n\left|z^t D_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}Q\Big(\frac{z}{k^2}\Big)+\right.$$

$$\left.+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)Q\Big(\frac{z}{k^2}\Big)\right|\leq$$

$$\leq n(n-1)\dots(n-t+1)\left\{\left|z^t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|-\right.$$

$$\left.-\frac{|z|^n}{k^n}\left|\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|\right\}m(P,k). \tag{3.16}$$

Combining (3.16) with Lemma 3.3, we get, for $|z|\geq 1$,

$$2\left|z^t D_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}P(z)+\right.$$

$$\left.+\beta\frac{n(n-1)\dots(n-t+1)}{(1+k)^t}(|\alpha_1|-k)\dots(|\alpha_t|-k)P(z)\right|\leq$$

$$\leq n(n-1)\dots(n-t+1)\left[\left\{\frac{|z|^n}{k^n}\left|\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|+\right.\right.$$

$$\left.+\left|z^t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|\right\}M(P,k)-$$

$$-\left\{\frac{|z|^n}{k^n}\left|\alpha_1\alpha_2\dots\alpha_t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|-\right.$$

$$\left.\left.-\left|z^t+\frac{\beta(|\alpha_1|-k)\dots(|\alpha_t|-k)}{(1+k)^t}\right|\right\}m(P,k)\right],$$

which is equivalent to (2.3).

Theorem 2.2 is proved.

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Received 09.12.15,
after revision — 26.09.16