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REMARK ON THE TAUTNESS MODULO AN ANALYTIC HYPERSURFACE OF HARTOGS-TYPE DOMAINS

ЗАУВАЖЕННЯ ЩОДО НАТЯГУ ОБЛАСТЕЙ ТИПУ ХАРТОГСА ЗА МОДУЛЕМ АНАЛІТИЧНОЇ ГІПЕРПОВЕРХНІ

We present sufficient conditions for the tautness modulo an analytic hypersurface of Hartogs-type domains $\Omega_H(X)$ and Hartogs – Laurent-type domains $\Sigma_{u,v}(X)$. We also propose a version of Eastwood's theorem for the tautness modulo an analytic hypersurface.

Наведено достатні умови натягу областей типу Хартогса $\Omega_H(X)$ та Хартогса – Лорана $\Sigma_{u,v}(X)$ за модулем аналітичної гіперповерхні. Сформульовано версію теореми Іствуда для натягу за модулем аналітичної гіперповерхні.

1. Introduction. Let X be a complex space and let $H\colon X \times \mathbb{C}^m \to [-\infty; +\infty)$ be an upper semicontinuous function such that $H(z, w) \geq 0$ and $H(z, \lambda w) = |\lambda| H(z, w)$ with $\lambda \in \mathbb{C},\ z \in X,$ $w \in \mathbb{C}^m$. We put

$$\Omega_H(X) := \{(z, w) \in X \times \mathbb{C}^m \colon H(z, w) < 1\},$$

and call it a Hartogs-type domain. For each $z \in X$, we denote by $\Omega_H(z) := \{w \in \mathbb{C}^m \colon H(z, w) < < 1\}$ the fiber of $\Omega_H(X)$ at z. Here, if $H(z, w) = h(w)e^{u(z)}$ for $z \in X, w \in \mathbb{C}^m$, where h, u are upper semicontinuous, $h \not\equiv 0$ and $h(\lambda w) = |\lambda| h(w)$ with $\lambda \in \mathbb{C}$, we denote $\Omega_H(X)$ by $\Omega_{u,h}(X)$ and the fiber by $\Omega_h := \{w \in \mathbb{C}^m \colon h(w) < 1\}$.

The following properties are known (see [1]):

$\Omega_h \Subset \mathbb{C}^m$ if and only if there exists a positive constant C such that $h(w) \geq C\|w\|$ for all $w \in \mathbb{C}^m$;

h is plurisubharmonic on $\mathbb{C}^m$ if and only if $\log h$ is plurisubharmonic on $\mathbb{C}^m$;

Ω_h is taut if and only if $\Omega_h \Subset \mathbb{C}^m$ and h is continuous plurisubharmonic on $\mathbb{C}^m$.

For u, v are upper semicontinuous functions on X with $u + v < 0$ on X, we put

$$\Sigma_{u,v}(X) := \{(z, \lambda) \in X \times \mathbb{C} \colon e^{v(z)} < |\lambda| < e^{-u(z)}\},$$

and call it a Hartogs – Laurent-type domain.

In the past ten years, much attention has been given to the properties of Hartogs-type domains from the viewpoint of hyperbolicity and tautness complex analysis (see [2, 3, 5, 7, 9, 11 – 14]). In [7], S. H. Park obtained necessary and sufficient conditions for the hyperbolicity and tautness of certain Hartogs-type domains and Hartogs – Laurent-type domains. In particular, in [13], D. D. Thai, M. A. Duc, P. J. Thomas and N. V. Trao considered the tautness modulo an analytic subset S of Hartogs-type domains in general situation. However, the original proof in [13] is based on Zorn's lemma, and it is not elementary. Notice that the results in [13] were proved for any analytic subset. When the analytic subset $S = \varnothing$, those results (see [13], Theorem 2.3) seem to be very different from previous results (see [7], Theorem 5.2, and [14], Theorem 1.2) by observing that given results in [13] do not need the tautness of the fibers $\Omega_H(z)$, but they do need in [7] and [14]. Moreover, in the case of an analytic hypersurface, we can show a contradiction. For instance, we consider the following example. Let

$$h(w) := h(w_1, w_2) = |w_1^2 + w_2^2|^{\frac{1}{2}}, \quad (w_1, w_2) \in \mathbb{C}^2.$$

It is easy to check that $h(\lambda w) = |\lambda| h(w) \geq 0,\ \lambda \in \mathbb{C},\ w = (w_1, w_2) \in \mathbb{C}^2$ and h is continuous on $\mathbb{C}^2$ and $\log h$ is plurisubharmonic on $\mathbb{C}^2$, but the fiber Ω_h is unbounded. Hence, Ω_h is not taut. Let $u(z) := \log |z|$, it is a continuous subharmonic function on $X := \mathbb{C} \setminus \{0, 1\}$. Put $S = \{-1\}$. It is clear to see that X is taut, hence X is taut modulo S (see Definition 2.2). Consider

$$H : X \times \mathbb{C}^2 \to [0; +\infty), \qquad H(z, w) = h(w)e^{u(z)},$$

which is continuous and log-plurisubharmonic on $X \times \mathbb{C}^2$. Thank to [7] (Theorem 5.2), we deduce that $\Omega_{u,h}(X \setminus S)$ is not taut. By Theorem 2.3 (iii) in [13] and Remark 2.1 below, we can see that $\Omega_{u,h}(X) \setminus \widetilde{S} = \Omega_{u,h}(X \setminus S)$ is taut, where $\widetilde{S} := S \times \mathbb{C}^2$. This is a contradiction.

Hence, in our opinion, for the tautness modulo an analytic subset $\widetilde{S}$ of certain Hartogs-type domains $\Omega_H(X)$, the tautness of the fibers $\Omega_H(z)$ can not be dropped.

The first purpose of this paper is to give some general versions for the tautness modulo an analytic hypersurface of Hartogs-type domains and Hartogs – Laurent-type domains. Finally, we give a version of Eastwood's theorem for the tautness modulo an analytic hypersurface of a complex space. To finish the proofs, we use the ideas and arguments in [7] to avoid using Zorn's lemma.

2. Preliminaries. Let Δ be the open unit disk in the complex plane. For a complex space X, we denote by $\mathrm{Hol}(\Delta, X)$ the set of all holomorphic maps from Δ to X and by $B_n(z, r)$ the n-dimensional Euclidean open ball with center z and radius $r > 0$ and by $\rho(a, b) := \tanh^{-1} \dfrac{|a - b|}{|1 - \overline{a}b|}$ the Poincaré distance on the open unit disk Δ.

Definition 2.1 [6, p. 68]. *Let X be a complex space and let S be an analytic subset in X. We say that X is hyperbolic modulo S if for every pair of distinct points p, q of X we have $d_X(p, q) > 0$ unless both are contained in S, where d_X is the Kobayashi pseudodistance of X.*

If $S = \varnothing$, then X is said to be hyperbolic.

Definition 2.2 [6, p. 240]. *Let X be a complex space and let S be an analytic subset in X. We say that X is taut modulo S if* $\mathrm{Hol}(\Delta, X)$ *is normal modulo S, i.e., for every sequence $\{f_n\}$ in* $\mathrm{Hol}(\Delta, X)$ *one of the following holds:*

(i) *there exists a subsequence of $\{f_n\}$ which converges uniformly on every compact subset to $f \in \mathrm{Hol}(\Delta, X)$ in* $\mathrm{Hol}(\Delta, X)$;

(ii) *the sequence $\{f_n\}$ is compactly divergent modulo S in* $\mathrm{Hol}(\Delta, X)$, *i.e., for each compact set $K \subset \Delta$ and each compact set $L \subset X \setminus S$, there exists an integer N such that $f_n(K) \cap L = \varnothing$ for all $n \geq N$.*

If $S = \varnothing$, then X is said to be taut. It is immediately from the definition that if $S \subset S' \subset X$ and X is taut modulo S, then it is taut modulo S', so in particular if X is taut, it is taut modulo S for any analytic subset S.

For z', z'' in X, we put

$$\widetilde{k}_X(z', z'') = \inf\{\rho(a, b) : a, b \in \Delta, \exists \varphi \in \mathrm{Hol}(\Delta, X), \varphi(a) = z', \varphi(b) = z''\} =$$

$$= \inf\{\rho(0, a) : a \in \Delta, \exists \varphi \in \mathrm{Hol}(\Delta, X), \varphi(0) = z', \varphi(a) = z''\},$$

which is called the Lempert function on X.

Related to the tautness of a complex space, there is a function $k_X^{(2)}$ defined as follows.

Definition 2.3 [5]. *Let X be a complex space. We define*

$$k_X^{(2)}(z', z'') = \inf\{\widetilde{k}_X(z', z_1) + \widetilde{k}_X(z_1, z'') : z_1 \in X\} =$$

$$= \inf\{\rho(0, a) + \rho(0, b) : a, b \in \Delta, \exists \varphi_1, \varphi_2 \in \mathrm{Hol}(\Delta, X),$$

$$\varphi_1(0) = z', \varphi_1(a) = \varphi_2(0), \varphi_2(b) = z''\},\ z', z'' \in X.$$

Similar to the above definition, for the tautness modulo an analytic subset S of X, there is a function $\widetilde{k}_X^{(2)}$ defined as follows.

Definition 2.4 [3]. *Let X be a complex space and let S be an analytic subset of X. We define*

$$\widetilde{k}_X^{(2)}(z', z'') = \inf\{\widetilde{k}_{X \setminus S}(z', z_1) + \widetilde{k}_X(z_1, z'') : z_1 \in X \setminus S\} =$$

$$= \inf\{\rho(0, a) + \rho(0, b) : a, b \in \Delta, \exists \varphi_1 \in \mathrm{Hol}(\Delta, X \setminus S),$$

$$\varphi_2 \in \mathrm{Hol}(\Delta, X), \varphi_1(0) = z', \varphi_1(a) = \varphi_2(0), \varphi_2(b) = z''\},$$

where $z' \in X \setminus S$ and $z'' \in X$.

Obviously, $k_X^{(2)} \leq \widetilde{k}_X^{(2)} \leq \widetilde{k}_X$.

We recall the following result, which is similar to Royden's criterion for the taut domains [8] .

Proposition 2.1 [3]. *Let X be a complex space and let S be an analytic hypersurface in X. Then X is taut modulo S if and only if*

$$B_{\widetilde{k}_G^{(2)}}(z_0, R) := \{z \in X : \widetilde{k}_X^{(2)}(z_0, z) < R\} \Subset X$$

for any $R > 0$ and $z_0 \in X \setminus S$.

Proposition 2.2 [3]. *Let X be a complex space, S be an analytic subset in X and $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. Then X is taut modulo S if and only if X_i is taut modulo $S_i := X_i \cap S$ for all $i \in I$.*

Lemma 2.1 [10]. *Let Z be a complex manifold. Let S be a hypersurface of a complex space X. If $\{\varphi_n\}_{n \geq 1} \subset \mathrm{Hol}(Z, X \setminus S)$ converging uniformly on every compact subsets of Z to a mapping $\varphi \in \mathrm{Hol}(Z, X)$, then either $\varphi(Z) \subset X \setminus S$ or $\varphi(Z) \subset S$.*

The following statement is an immediate consequence of the criterion for the tautness modulo an analytic hypersurface.

***Corollary* 2.1** [3]. *Let X be a complex space and let S be an analytic hypersurface of X. If X is not taut modulo S, then there exist a number $R > 0$, sequences $\{z_n\}_{n \geq 0} \subset X$, $\{f_n\}_{n \geq 1} \subset$ $\subset \mathrm{Hol}(\Delta, X \setminus S)$, $\{g_n\}_{n \geq 1} \subset \mathrm{Hol}(\Delta, X)$ and $\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1} \subset [0; 1)$ such that for $n \geq 1$, we have*

(i) $\widetilde{k}_X^{(2)}(z_0, z_n) < R$,

(ii) $f_n(0) = z_0 \in X \setminus S$,

(iii) $f_n(\alpha_n) = g_n(0)$,

(iv) $g_n(\beta_n) = z_n,\ z_n \to w \in \partial X$ *or* $|z_n| \to \infty$,

(v) $\alpha_n \to \alpha_0,\ \beta_n \to \beta_0$.

***Remark* 2.1.** We could have $X \setminus S$ being taut without X being taut modulo S. For instance, $\mathbb{C} \setminus \{0,1\}$ is taut, but $\mathbb{C}$ is not taut modulo $\{0,1\}$. On the other hand, there are examples of domains taut modulo S such that $X \setminus S$ is not taut. Just take X a taut domain and S such that the codimension of S is at least 2. Then $X \setminus S$ is not pseudoconvex, therefore not taut.

However, when S is an analytic hypersurface, using Lemma 2.1, we can show that if X is taut modulo S then $X \setminus S$ is also taut. Indeed, we take any sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, X \setminus S)$. Suppose that $\{f_n\}$ is not compactly divergent in $\mathrm{Hol}(\Delta, X \setminus S)$, we deduce that $\{f_n\}$ is not compactly divergent modulo S in $\mathrm{Hol}(\Delta, X)$ either. By the tautness modulo hypersurface S of X, it implies that $\{f_n\}$ converges uniformly on every compact subset of Δ to a mapping $f \in \mathrm{Hol}(\Delta, X)$. By Lemma 2.1, we have either $f \in \mathrm{Hol}(\Delta, X \setminus S)$ or $f(\Delta) \subset S$. By the assumption, $\{f_n\}$ is normally convergent in $\mathrm{Hol}(\Delta, X \setminus S)$.

3. The tautness of Hartogs-type domains and of Hartogs – Laurent-type domains. Firstly, we give a theorem for the tautness modulo an analytic hypersurface of Hartogs-type domains as follows.

Theorem 3.1. *Let X be a complex space and let S be an analytic hypersurface in X. If X is taut modulo S, the fiber $\Omega_H(z)$ is taut for each $z \in X$, H is continuous on $X \times \mathbb{C}^m$ and H is plurisubharmonic on $(X \setminus S) \times \mathbb{C}^m$, then $\Omega_H X$ is taut modulo $\widetilde{S} := S \times C^m$.*

Proof. Suppose that $\Omega_H(X)$ is not taut modulo $\widetilde{S}$. By Proposition 2.2, we can assume that X is an irreducible complex space. By Corollary 2.1, we can choose a number $R > 0$, sequences $\{z_n\}_{n \geq 0} \subset \Omega_H(X)$ and

$$\{f_n\}_{n \geq 1} \subset \mathrm{Hol}(\Delta, \Omega_H(X) \setminus \widetilde{S}), \qquad \{g_n\}_{n \geq 1} \subset \mathrm{Hol}(\Delta, \Omega_H(X))$$

and $\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1} \subset [0;1)$ satisfying the properties (i) to (v). By the definition of $\widetilde{k}^{(2)}$, we have

$$\widetilde{k}^{(2)}_{\Omega_H(X)}(z_0, z_n) \geq \widetilde{k}^{(2)}_X(z_0^1, z_n^1),$$

where $z_n = (z_n^1, z_n^2) \in X \times \mathbb{C}^m$. Since the property (i), it implies that $\{z_n^1\}_{n \geq 1} \subset B_{\widetilde{k}^{(2)}_X}(z_0^1, R)$. By Proposition 2.1, we may see that $\{z_n^1\}_{n \geq 1} \to a_0^1 \in X$ as $n \to \infty$. For each $n \geq 1$, we denote

$$f_n := (f_n^1, f_n^2) \in \mathrm{Hol}(\Delta, X \setminus S) \times \mathrm{Hol}(\Delta, \mathbb{C}^m)$$

and

$$g_n := (g_n^1, g_n^2) \in \mathrm{Hol}(\Delta, X) \times \mathrm{Hol}(\Delta, \mathbb{C}^m).$$

By the property (ii), we have $f_{n_k}^1(0) = z_0^1 \in X \setminus S$. Then, since the tautness modulo S of X, we may choose a subsequence $\{f_{n_k}^1\} \subset \{f_n^1\}$ such that

$$f_{n_k}^1 \overset{K}{\rightrightarrows} f_0^1 \in \mathrm{Hol}(\Delta, X),$$

i.e., $\{f_{n_k}^1\}$ is converging uniformly on every compact subset of Δ to $f_0^1 \in \mathrm{Hol}(\Delta, X)$. It is clear to see that $f_0^1(0) = z_0^1 \in X \setminus S$. Since $f_n^1 \in \mathrm{Hol}(\Delta, X \setminus S)$, applying Lemma 2.1, we have $f_0^1 \in \mathrm{Hol}(\Delta, X \setminus S)$. Then the properties (iii) and (v) yield that

$$\lim_{k \to \infty} g_{n_k}^1(0) = \lim_{k \to \infty} f_{n_k}^1(\alpha_{n_k}) = f_0^1(\alpha_0) \in X \setminus S.$$

Hence, there exists a subsequence of $\{g_{n_k}^1\}$, without loss of generality, we assume that $g_{n_k}^1 \overset{K}{\rightrightarrows} g_0^1 \in$ $\in \mathrm{Hol}(\Delta, X)$. Then we have

$$g_0^1(0) = f_0^1(\alpha_0) \in X \setminus S. \tag{1}$$

In particular,

$$g_0^1(\beta_0) = \lim_{k\to\infty} g_{n_k}^1(\beta_{n_k}) = \lim_{k\to\infty} z_{n_k}^1 := a_0^1 \in X. \tag{2}$$

Assume that $\lim_{n\to\infty} \|z_n^2\| = \infty$. Put $z_n^2 = r_n w_n$ with $\|w_n\| = 1$ and $r_n \in \mathbb{R},\ n \geq 1$. It implies that $\lim_{n\to\infty} r_n = \infty$ and $\lim_{k\to\infty} w_{n_k} = w_0 \neq 0$ with some subsequence $\{w_{n_k}\}$ of $\{w_n\}$. By $H(z_n^1, r_n w_n) = r_n H(z_n^1, w_n) < 1$ and since continuity of H on $X \times \mathbb{C}^m$, we have

$$\lim_{k\to\infty} H\big(z_{nk}^1, w_{n_k}\big) = H\big(a_0^1, w_0\big) = 0.$$

Thus, $H(a_0^1, L) = 0 < 1$ with a complex line $L = tw_0$ in $\mathbb{C}^m$ $(t \in \mathbb{C})$. It implies that the fiber $\Omega_H(a_0^1) = L$. But L is not taut, so we have a contradiction to the assumption. Therefore, since the property (iv), we obtain

$$\lim_{n\to\infty} z_n = (a_0^1, a_0^2) = a_0 \in \partial\Omega_{H(X)}. \tag{3}$$

Step 1: Choose $c_2 \in (0,1)$ such that $\beta_n \in c_2^2\Delta,\ n \geq 1$. Since (1), we have $(g_0^1)(\Delta) \not\subset S$. It implies that $(g_0^1)^{-1}(S)$ is an analytic subset in the open unit disc Δ, so it is a discrete set. Then $(g_0^1)^{-1}(S)$ does not have any accumulation point in Δ. Therefore, we can assume that

$$c_2\Delta \cap (g_0^1)^{-1}(S) = \varnothing. \tag{4}$$

We put $E_2 := {c_2}^{-1}\Delta$. For each $n \geq 1$, we define a map $\widetilde{g}_n : E_2 \to X \times C^m$ by

$$\widetilde{g}_n(\lambda) = \big(\widetilde{g}_n^1(\lambda), \widetilde{g}_n^2(\lambda)\big) := g_n(\beta_n\lambda).$$

Clearly,

$$\{\widetilde{g}_n\}_{n\geq 1} \subset \mathrm{Hol}(E_2, \Omega_H(X)). \tag{5}$$

Put $F_2 := \bigcup_{n\geq 1}(\beta_n E_2)$. Using (v), it is easy to check that $F_2 \Subset \Delta$. Let $M := g_0^1(\overline{F_2})$. Since $\beta_n \in c_2^2\Delta,\ n \geq 1$, we have $\beta_n < c_2^2$. It implies that

$$\beta_n E_2 = \beta_n {c_2}^{-1}\Delta \subset c_2\Delta,\ n \geq 1.$$

Then $g_0^1(\overline{F_2}) \subset g_0^1(\overline{c_2\Delta})$. This and (4) imply that $M \Subset X \setminus S$. Notice that X is hyperbolic modulo S and $d = d_X$ is the Kobayashi pseudodistance, then $\delta := \mathrm{dist}(M, \partial(X \setminus S))/3 > 0$. Since $\{g_{n_k}^1\}$ converges uniformly on $\overline{F}_2$, we may take $n_0 \in \mathbb{N}$ such that $d\big(g_{n_k}^1(\lambda), g_0^1(\lambda)\big) < \delta,\ \lambda \in \overline{F}_2$ and $n_k > n_0$. Hence, for $v_0 \in \partial(X \setminus S),\ \lambda \in \overline{F}_2$, we obtain

$$d\big(g_{n_k}^1(\lambda), v_0\big) \geq d\big(g_0^1(\lambda), v_0\big) - d\big(g_{n_k}^1(\lambda), g_0^1(\lambda)\big) \geq \mathrm{dist}(M, \partial X) - \delta \geq 2\delta.$$

Then we get $d\big(g_{n_k}^1(\overline{F}_2), \partial(X \setminus S)\big) \geq 2\delta > 0$, which implies that

$$K := g_0^1(\overline{F}_2) \cup \left(\bigcup_{n_k\geq n_0} g_{n_k}^1(\overline{F}_2)\right) \Subset X \setminus S. \tag{6}$$

Particularly,

$$K' := \{g^1_{n_k}(\beta_{n_k}\lambda), g^1_0(\beta_0\lambda) : \lambda \in E_2, n_k \geq n_0\} \subset K. \tag{7}$$

We now assume that the family $\{\widetilde{g}^2_n\}$ is not uniformly bounded in E_2. Then there exist a subsequence $\{\widetilde{g}^2_{n_k}\}$ and a sequence $\{\lambda_k\} \subset E_2$ such that

$$\lim_{k\to\infty} \|\widetilde{g}^2_{n_k}(\lambda_k)\| = \infty.$$

Put $\widetilde{g}^2_{n_k}(\lambda_k) = r_k w_k$ with $\|w_k\| = 1$ and $r_k \in \mathbb{R}$. Similar to the above argument, we have $\lim_{k\to\infty} r_k = \infty$ and $\lim_{k\to\infty} w_k = w_0 \neq 0$. Since (6) and (7), we deduce that

$$\lim_{k\to\infty} \widetilde{g}^1_{n_k}(\lambda_k) = b_0 \in \overline{K} \subset X \setminus S.$$

Since (5), we obtain

$$r_k H\big(\widetilde{g}^1_{n_k}(\lambda_k), w_k\big) = H\big(\widetilde{g}^1_{n_k}(\lambda_k), \widetilde{g}^2_{n_k}(\lambda_k)\big) < 1.$$

By continuity of H, we have

$$\lim_{k\to\infty} H\big(\widetilde{g}^1_{n_k}(\lambda_k), w_k\big) = H(b_0, w_0) = 0.$$

Repeat the same argument in the above, we will get a contradiction to the tautness of the fiber $\Omega_H(z)$ again. Therefore, $\widetilde{g}^2_{n_k}$ is uniformly bounded in E_2. Applying to Montel's theorem, without loss of generality, we can assume

$$\widetilde{g}^2_{n_k} \overset{K}{\rightrightarrows} \widetilde{g}^2_0 \in \mathrm{Hol}(E_2, \mathbb{C}^m)$$

as $k \to \infty$. In particular,

$$\widetilde{g}^2_0(1) = \lim_{k\to\infty} \widetilde{g}^2_{n_k}(1) = \lim_{k\to\infty} g^2_{n_k}(\beta_{n_k}) = \lim_{k\to\infty} z^2_{n_k} = a^2_0. \tag{8}$$

Put $\varphi_{n_k} := H \circ \widetilde{g}_{n_k}$ on E_2. Since $\varphi_{n_k} < 1$ on E_2 for any $n_k \geq n_0$, we have $\varphi_0 := H \circ \widetilde{g}_0 \leq 1$ on E_2, where $\widetilde{g}_0 := \big(\widetilde{g}^1_0, \widetilde{g}^2_0\big)$ and $\widetilde{g}^1_0(\lambda) := g^1_0(\beta_0\lambda)$, $\lambda \in E_2$. It follows from (2) that

$$\widetilde{g}^1_0(1) = g^1_0(\beta_0) = a^1_0. \tag{9}$$

Since (3), (8) and (9), we get $\varphi_0(1) = H(a_0) = 1$. By H is plurisubharmonic on $(X \setminus S) \times \mathbb{C}^m$, φ_0 is subharmonic on E_2. Thus, the maximum principle for subharmonic functions implies that $\varphi_0 \equiv 1$ on E_2, and hence

$$\widetilde{g}_0(0) = \big(\widetilde{g}^1_0(0), \widetilde{g}^2_0(0)\big) \in \partial\Omega_H(X). \tag{10}$$

Step 2: We are going to apply the same argument as in Step 1 to $\{f_n\}_{n\geq 1}$ and $\{\alpha_n\}_{n\geq 0}$. Choose $c_1 \in (0,1)$ such that $\alpha_n \in E_1 := {c_1}^{-1}\Delta$ for $n \geq 1$. We define a holomorphic function $\widetilde{f}_n : E_1 \to \Omega_H(X)$ by

$$\widetilde{f}_n(\lambda) = (\widetilde{f}^1_n(\lambda), \widetilde{f}^2_n(\lambda)) := f_n(\alpha_n\lambda), \quad \lambda \in E_1.$$

Then we also have

$$\widetilde{f}^2_{n_k} \overset{K}{\rightrightarrows} \widetilde{f}^2_0 \in \mathrm{Hol}(E_1, \mathbb{C}^m)$$

as $k \to \infty$. Since condition (iii), we observe that

$$\widetilde{g}_0(0) = \lim_{k\to\infty} \widetilde{g}_{n_k}(0) = \lim_{k\to\infty} g_{n_k}(0) = \lim_{k\to\infty} f_{n_k}(\alpha_{n_k}) = \lim_{k\to\infty} \widetilde{f}_{n_k}(1) = \widetilde{f}_0(1), \tag{11}$$

where $\widetilde{f}_0 := (\widetilde{f}_0^1, \widetilde{f}_0^2)$ and $\widetilde{f}_0^1(\lambda) := f_0^1(\alpha_0\lambda),\ \lambda \in E_1$. Put

$$\psi_0(\lambda) = H \circ (\widetilde{f}_0)(\lambda) \leq 1, \quad \lambda \in E_1.$$

Obviously, since (10) and (11), we obtain $\psi_0(1) = H(\widetilde{f}_0(1)) = 1$. It follows from the maximum principle for ψ_0 that $\psi_0 \equiv 1$ on E_1. This implies that

$$\widetilde{f}_0(0) = \lim_{k\to\infty} \widetilde{f}_{n_k}(0) = \lim_{k\to\infty} f_{n_k}(0) = z_0 \in \partial\Omega_H(X),$$

which contradicts the condition (ii).

Theorem 3.1 is proved.

We know that the tautness of $X \setminus S$ does not imply the tautness modulo S of X in general. But, using Theorem 1.2 in [14], we have the following assertion in special situations of Hartogs-type domains.

***Corollary* 3.1.** *Let X be a complex space being taut modulo an analytic hypersurface S. Assume that H is continuous on $\widetilde{S} := S \times \mathbb{C}^m$ and the fiber $\Omega_H(z)$ is taut for each $z \in S$. If $\Omega_H(X) \setminus \widetilde{S}$ is taut, then $\Omega_H(X)$ is taut modulo $\widetilde{S}$.*

Due to Barth [1], Ω_h is taut if and only if $\Omega_h \Subset \mathbb{C}^m$ and h is continuous plurisubharmonic on $\mathbb{C}^m$. In addition, if u is plurisubharmonic then e^u is. We immediately have the following corollary.

***Corollary* 3.2.** *Let X be a complex space and S be an analytic hypersurface in X. If X is taut modulo S, the fiber Ω_h is taut, u is continuous on X and u is plurisubharmonic on $X \setminus S$, then $\Omega_{u,h}(X)$ is taut modulo $\widetilde{S} := S \times C^m$.*

We recall the Example 2.4 in [13]. Let $X = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$ and $S = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$. We can check that X is taut modulo S. We put $u(z) = u(z_1, z_2) := \log|z_2|$ and $h(w) = |w|$ with $w \in \mathbb{C}$. Obviously, u is plurisubharmonic on $X \setminus S$ and continuous on X. It is easy to see that the fiber Ω_h is taut, $H(z, w) := h(w)e^{u(z)}$ is continuous on $X \times \mathbb{C}$ and $\log H$ is plurisubharmonic on $(X \setminus S) \times \mathbb{C}$. Applying Corollary 3.2, we deduce that $\Omega_{u,h}$ is taut modulo $\widetilde{S} := S \times \mathbb{C}$.

Notice that we also obtain this conclusion by direct proof as in [13]. However, we can not get one from Theorem 2.3 (iii) in [13]. Because $\log H$ is not plurisubharmonic on $X \times \mathbb{C}$, since u is not plurisubharmonic on X.

Now, we give a necessary condition for the tautness modulo of Hartogs – Laurent-type domains.

Proposition 3.1. *If $\Sigma_{u,v}(X)$ is taut modulo $\widetilde{S} := S \times \mathbb{C}$, then u and v are continuous on $X \setminus S$, where S is an analytic subset of X.*

Proof. Suppose the contrary. Without loss of generality, we can assume that u is not continuous at $z_0 \in X \setminus S$. By upper semicontinuity of u, we can choose a number $R \in \mathbb{R}$ and a sequence $\{z_n\} \subset X \setminus S$ such that $z_n \to z_0$ as $n \to \infty$ and $-u(z_0) < -R < -u(z_n)$ for any $n \in \mathbb{N}$. Since $u(z_0) \neq -\infty$ and $u(z_0) + v(z_0) < 0$, we may take an $\alpha \in \mathbb{R}$ such that $v(z_0) < -\alpha < -u(z_0)$. Since upper semicontinuity of v, we can assume that $v(z_n) < -\alpha$ for $n > 1$. Put

$$C := \frac{1}{2} \min\big\{ -u(z_0) + \alpha, -R + u(z_0)\big\} > 0$$

and

$$\hat{u} := u - u(z_0) - \frac{C}{2}, \qquad \hat{v} := v + u(z_0) + \frac{C}{2}.$$

Obviously, the mapping

$$(z, w) \in \Sigma_{u,v}(X) \mapsto \left(z, we^{u(z_0)+\frac{C}{2}}\right) \in \Sigma_{\hat{u},\hat{v}}(X)$$

is biholomorphic, so $\Sigma_{\hat{u},\hat{v}}(X)$ is taut modulo $\widetilde{S}$. We put

$$\hat{R} := -u(z_0) + R - \frac{C}{2}, \qquad \hat{\alpha} := -u(z_0) + \alpha - \frac{C}{2}.$$

It is easy to show that $\hat{v}(z_n) < -\hat{\alpha}$ for any $n \in \mathbb{N}$. Hence, for any $n \geq 1$,

$$\max\{\hat{v}(z_0), \hat{v}(z_n)\} < -\hat{\alpha} < -C < 0 < -\hat{u}(z_0) < C < -\hat{R} < -\hat{u}(z_n). \tag{12}$$

We define $f_n(\lambda) := (z_n, e^{C\lambda})$, $\lambda \in \Delta$ for $n \geq 1$. Observe that

$$e^{\hat{v}(z_n)} < e^{-C} < |e^{C\lambda}| < e^{C} < e^{-\hat{u}(z_n)}, \qquad n \geq 1, \quad \lambda \in \Delta.$$

It implies that $\{f_n\} \subset \mathrm{Hol}(\Delta, \Sigma_{\hat{u},\hat{v}} \setminus \widetilde{S})$ by $z_n \in X \setminus S$, $n \geq 1$. Because, $e^{\hat{v}(z_0)} < e^{-C} < e^0 <$ $< e^{-\hat{u}(z_0)}$ and $z_0 \in X \setminus S$, we have

$$f_n(0) = (z_n, 1) \to (z_0, 1) \in \Sigma_{\hat{u},\hat{v}} \setminus \widetilde{S}.$$

By the tautness modulo $\widetilde{S}$ of $\Sigma_{\hat{u},\hat{v}}$, we get

$$f_n(\lambda) \overset{K}{\Rightarrow} f(\lambda) = (z_0, e^{C\lambda}) \in \mathrm{Hol}(\Delta, \Sigma_{\hat{u},\hat{v}})$$

as $n \to \infty$. It implies that $e^{\hat{v}(z_0)} < e^{CRe\lambda} < e^{-\hat{u}(z_0)}$ for any $\lambda \in \Delta$. By letting $\lambda \to 1$, we have a contradiction to (12).

Hence, Proposition 3.1 is proved.

The following proposition gives a sufficient condition for the tautness modulo of Hartogs–Laurent-type domains.

Proposition 3.2. *If X is taut modulo an analytic hypersurface S, u is continuous on X, plurisubharmonic on $X \setminus S$ and v is continuous plurisubharmonic on X, then $\Sigma_{u,v}(X)$ is taut modulo $\widetilde{S} := S \times \mathbb{C}$.*

Proof. Let a sequence $\{\varphi_n\} \subset \mathrm{Hol}(\Delta, \Sigma_{u,v}(X))$. We have $\Sigma_{u,v}(X) \subset \Omega_{u,|.|}(X)$, where $|.|$ is the norm on $\mathbb{C}$. By Corollary 3.2, $\Omega_{u,|.|}(X)$ is taut modulo $\widetilde{S}$. It implies that there exists a subsequence $\{\varphi_{n_k}\} \subset \{\varphi_n\}$ which is either normally convergent or compactly divergent modulo $\widetilde{S}$ in $\mathrm{Hol}(\Delta, \Omega_{u,|.|}(X))$. In the latter case, the sequence $\{\varphi_{n_k}\}$ as a subfamily of $\mathrm{Hol}(\Delta, \Sigma_{u,v}(X))$, diverges compactly modulo $\widetilde{S}$. Then, we only suppose that $\{\varphi_{n_k}\}$ is normally convergent in $\mathrm{Hol}(\Delta, \Omega_{u,|.|}(X))$. Put $\varphi_{n_k} := (f_{n_k}, g_{n_k})$, where $\{f_{n_k}\} \subset \mathrm{Hol}(\Delta, X)$ and $\{g_{n_k}\} \subset \mathrm{Hol}(\Delta, \mathbb{C})$. We denote

$$\varphi := (f, g) \in \mathrm{Hol}(\Delta, \Omega_{u,|.|}(X)),$$

where $f \in \mathrm{Hol}(\Delta, X)$ and $g \in \mathrm{Hol}(\Delta, \mathbb{C})$, such that $f_{n_k} \overset{K}{\Rightarrow} f$ and $g_{n_k} \overset{K}{\Rightarrow} g$ as $k \to \infty$. We have

$$e^{(v \circ f_{n_k})(\lambda)} < |g_{n_k}(\lambda)| < e^{-(u \circ f_{n_k})(\lambda)}$$

and

$$|g(\lambda)| < e^{-(u\circ f)(\lambda)}, \quad \lambda \in \Delta.$$

Since $(g_{n_k})^{-1}(0) = \varnothing$ for any $k \geq 1$, it follows from Hurwitz's theorem that either $g \equiv 0$ or g never vanishes. If $g \equiv 0$ then $\varphi(\Delta) \subset \partial\Sigma_{u,v}(X)$, which implies that $\{\varphi_{n_k}\}$ as a subfamily of $\mathrm{Hol}(\Delta, \Sigma_{u,v}(X))$ diverges compactly. Now, we suppose that $g \not\equiv 0$ and define

$$\hat{v} := \frac{1}{|g(\lambda)|} e^{(v\circ f)(\lambda)}, \quad \lambda \in \Delta.$$

It implies that $\hat{v}$ is continuous subharmonic on Δ. By continuity of v, we have $\hat{v}(\lambda) \leq 1$ for any $\lambda \in \Delta$. It follows from the maximum principle for subharmonic that either $\hat{v} \equiv 1$ on Δ or $\hat{v} < 1$ on Δ. Therefore, it is either $\varphi(\Delta) \subset \partial\Sigma_{u,v}(X)$ or $\varphi(\Delta) \subset \Sigma_{u,v}(X)$. Then $\{\varphi_n\}$ is either normally convergent in $\mathrm{Hol}(\Delta, \Sigma_{u,v}(X))$ or compactly divergent. Thus, since the above arguments, $\{f_n\}$ is either compactly divergent modulo $\widetilde{S}$ or normally convergent in $\mathrm{Hol}(\Delta, \Sigma_{u,v}(X))$. It implies that $\Sigma_{u,v}(X)$ is taut modulo $\widetilde{S}$.

4. Eastwood's theorem for the tautness modulo. Similar to Eastwood's theorem for the hyperbolicity and tautness of a complex space (see [4, 9, 11]), we give a version of Eastwood's theorem for the tautness modulo an analytic hypersurface of a complex space.

Theorem 4.1. *Let $\widetilde{X}$ and X be two complex spaces. Let $\pi\colon \widetilde{X} \to X$ be a holomorphic mapping and let S be an analytic hypersurface in X. Suppose that for each $p \in X$, there exists an open neighborhood $U := U(p)$ in X such that $\pi^{-1}(U)$ is taut modulo $\widetilde{S} := \pi^{-1}(S)$. If X is taut modulo S, then $\widetilde{X}$ is also taut modulo $\widetilde{S}$.*

Proof. As in the proof of Theorem 3.1, we can consider X as an irreducible complex space. Suppose that $\widetilde{X}$ is not taut modulo $\widetilde{S}$. Then by Corollary 2.1, we can take sequences $\{z_n\}_{n\geq 0} \subset \subset \Omega_H(X)$, $\{f_n\} \subset \mathrm{Hol}(\Delta, \widetilde{X} \setminus \widetilde{S}), \{g_n\} \subset \mathrm{Hol}(\Delta, \widetilde{X})$ and sequences $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1} \subset [0; 1)$ satisfying the properties (ii) to (v). We put

$$\widetilde{f}_n = \pi \circ f_n \in \mathrm{Hol}(\Delta, X \setminus S) \tag{13}$$

and

$$\widetilde{g}_n = \pi \circ g_n \in \mathrm{Hol}(\Delta, X), \quad n \geq 1.$$

By the property (ii), $\lim_{n\to\infty} \widetilde{f}_n(0) = \lim_{n\to\infty} \pi(z_0) \in X \setminus S$. Since X is taut modulo S, there exists a sequence $\{\widetilde{f}_{n_k}\} \subset \{\widetilde{f}_n\}$ such that

$$\widetilde{f}_{n_k} \overset{K}{\Rightarrow} \varphi_1 \in \mathrm{Hol}(\Delta, X)$$

as $k \to \infty$. By (13) and applying Lemma 2.1, it implies that $\varphi_1 \in \mathrm{Hol}(\Delta, X \setminus S)$. By the property (iii), we get

$$\lim_{k\to\infty} \widetilde{g}_{n_k}(0) = \lim_{k\to\infty} \widetilde{f}_{n_k}(\alpha_{n_k}) = \varphi_1(\alpha_0) \in X \setminus S.$$

Then $\{\widetilde{g}_n\}$ contains a subsequence $\{\widetilde{g}_{n_k}\}$ converging uniformly on compact subsets to a map $\varphi_2 \in \in \mathrm{Hol}(\Delta, X)$ as $k \to \infty$. Therefore, for any $\lambda \in \Delta$, there exists an open neighborhood $V_\lambda \Subset \Delta$ of λ, $U_{\varphi_2(\lambda)}$ and $k_\lambda \in \mathbb{N}$, such that $\pi^{-1}(U_{\varphi_2(\lambda)})$ is taut modulo $\widetilde{S}$ and $g_{n_k}(V_\lambda) \Subset \pi^{-1}(U_{\varphi_2(\lambda)}) \subset \widetilde{X}$, for any $k \geq k_\lambda$.

Now, we take a point $0 < s < 1$. By the compactness of $[-s, \beta_0] \subset \Delta$, we can choose a finite set $\{x_\mu : \mu = 1, \ldots, q\} \subset [-s, \beta_0]$ such that $[-s, \beta_0] \subset \bigcup_{\mu=1}^{q} V_{x_\mu}$ and for all $\mu \in \{1, \ldots, q\}$ exists

$\nu \in \{1, \ldots, q\} \setminus \{\mu\}$ such that $V_{x_\mu} \cap V_{x_\nu} \neq \varnothing$. After a rearrangement, we can assume that $\beta_0 \in V_q$ and $V_{x_\mu} \cap V_{x_{\mu+1}} \neq \varnothing$, $\mu \in \{1, \ldots, q-1\}$. For $\lambda = \beta_0$, we consider $g_{n_k} \in \mathrm{Hol}(V_{\beta_0}, \pi^{-1}(U_{\varphi_2(\beta_0)}))$.

Assume that there exists a subsequence $\{g_{n_{k_1}}\} \subset \{g_{n_k}\}$ converging uniformly on compact subset to map $g_{\beta_0} \in \mathrm{Hol}(V_{\beta_0}, \pi^{-1}(U_{\varphi_2(\beta_0)}))$ as $k_1 \to \infty$. By the property (iv), we have

$$\lim_{k_1 \to \infty} z_{n_{k_1}} = \lim_{k_1 \to \infty} g_{n_{k_1}}(\beta_{n_{k_1}}) = g_{\beta_0}(\beta_0) \in \widetilde{X}.$$

That is a contradiction to the condition (iv). Hence, by the tautness modulo $\widetilde{S}$ of $\pi^{-1}(U_{\varphi_2(\beta_0)})$, it implies that g_{n_k} diverges compactly modulo on V_{β_0}. But, since $\beta_0 \in V_{x_q} \cap V_{\beta_0} \neq \varnothing$, we can choose a sequence $\{g_{n_{k_2}}\} \subset \{g_{n_{k_1}}\}$ which diverges compactly modulo on V_{x_q}. Because $V_{x_q} \cap V_{x_q - 1} \neq \varnothing$, we also choose a subsequence $\{g_{n_{k_3}}\} \subset \{g_{n_{k_2}}\}$ which diverges compactly modulo on $V_{x_{q-1}}$. And, we can proceed to $q-2$, in this manner, we can choose $\mu_0 \in \{1, \ldots, q\}$ with $0 \in V_{x_{\mu_0}}$ and a subsequence $\{g_{n_{k_4}}\} \subset \{g_{n_{k_3}}\}$ diverges compactly modulo on $V_{x_{\mu_0}}$. Thus, in view of (iii), we have either

$$\lim_{k_4 \to \infty} f_{n_{k_4}}(\alpha_{n_{k4}}) = \lim_{k_4 \to \infty} g_{n_{k_4}}(0) = \hat{a_0} \in \partial \widetilde{X}, \tag{14}$$

or

$$\lim_{k_4 \to \infty} f_{n_{k_4}}(\alpha_{n_{k4}}) \in \widetilde{S}. \tag{15}$$

Applying the above argument for the sequence $f_{n_{k_4}} \subset \mathrm{Hol}(\Delta, \widetilde{X} \setminus \widetilde{S})$, we can choose a subsequence $\{f_{n_{k_5}}\}$ of $\{f_{n_{k_4}}\}$ that diverges compactly modulo on V_{α_0}. Because if $f_{n_{k_5}} \overset{K}{\Rightarrow} f_{\alpha_0} \in$ $\in \mathrm{Hol}(V_{\alpha_0}, \pi^{-1}(U_{\varphi_1(\alpha_0)}))$, by Lemma 2.1, we have $f_{\alpha_0} \in \mathrm{Hol}(V_{\alpha_0}, \pi^{-1}(U_{\varphi_1(\alpha_0)}) \setminus \widetilde{S})$. It implies that

$$\lim_{k_5 \to \infty} f_{n_{k_5}}(\alpha_{n_{k_5}}) = f_{\alpha_0}(\alpha_0) \in \pi^{-1}(U_{\varphi_1(\alpha_0)}) \setminus \widetilde{S} \subset \widetilde{X} \setminus \widetilde{S}.$$

This is a contradiction to (14) and (15). So, we can take a subsequence $\{f_{n_{k_6}}\} \subset \{f_{n_{k_5}}\}$ such that $\{f_{n_{k_6}}(0)\}$ converges to a point in $\partial \widetilde{X}$ or in $\widetilde{S}$. Obviously, this is a contradiction to the property (ii). Therefore, $\widetilde{X}$ is taut modulo $\widetilde{S}$.

Immediately, we get the following corollary.

***Corollary* 4.1.** *If $\pi : \widetilde{X} \to X$ is a holomorphic covering between complex spaces, then $\widetilde{X}$ is taut modulo $\widetilde{S}$ if and only if X is taut modulo an analytic hypersurface S in X, where $\widetilde{S} := \pi^{-1}(S)$.*

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