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ALMOST MENGER PROPERTY IN BITOPOLOGICAL SPACES
МАЙЖЕ МЕНГЕРОВА ВЛАСТИВІСТЬ У БІТОПОЛОГІЧНИХ ПРОСТОРАХ

We introduce the notion of almost Menger property in bitopological spaces. We give some characterizations in terms of (i, j)-regular open sets and almost continuous surjection. We also investigate the notion of almost γ-set in the bitopological context.

Введено поняття майже менгерової властивості в бітопологічних просторах, наведено деякі характеристики в термінах (i, j)-регулярних відкритих множин і майже неперервної сюр'єкції та вивчено поняття майже γ-множини в бітопологічному контексті.

1. Introduction. In this paper we will be mostly concerned with the notion of almost Mengerness in bitopological context.

We first recall the classical definition of Menger property in topological spaces: A topological space (X, τ) is Menger if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of open covers of X, there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X [20].

Hurewicz [8] introduced the Menger property in 1925 and showed that a conjecture of Menger is equivalent to the statement that a metrizable space has the Menger property if and only if it is σ-compact. In 1927 Hurewicz [9] introduced a stronger version of the Menger property called the Hurewicz property defined as follows. A space X has the Hurewicz property if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each element of the space belongs to all but finitely many of the sets $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$. It is known that σ-compactness implies the Hurewicz property in all finite powers and that the Hurewicz property implies the Menger property.

Recently many papers related to the Menger property appeared in the literature. In [22] it is proved that there exists a non-meager, non-CDH (countable dense homogeneous) filter. In [26] Scheepers showed that if X is a separable metric space with the Hurewicz covering property, then the Banach – Mazur game played on X is determined and the implication is not true when Hurewicz covering property is replaced by Menger covering property. In [2] T. Banakh and D. Repovs defined and studied universal nowhere dense and universal meager sets in Menger manifolds.

In 1996 M. Scheepers gave general definition of selection principles and began a systematic study of selection principles in topology. For selected results see [16, 24, 25, 29].

Many topological properties are defined or characterized in terms of the following two classical selection principles.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle b_n : n \in \mathbb{N}\rangle$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $\langle A_n : n \in \mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle B_n : n \in \mathbb{N}\rangle$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

When both $\mathcal{A}$ and $\mathcal{B}$ are the collection $\mathcal{O}$ of open covers of a space X then $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ denotes the *Rothberger property* [23], while $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ is the *Menger property* [8, 10, 20].

Recently several papers on weaker forms of the Menger property have been published (see [1, 12, 21]).

Kočinac in [14, 15] introduced the notion of almost Menger property and Kocev in [12] has studied systematically this notion by giving characterizations in terms of regular open sets and almost continuous mappings. On the other hand, weakly Menger property was introduced in [3] and studied widely in [1, 21].

Both of these properties are weaker than the Menger property, and obviously every Menger space is almost Menger and every almost Menger space is weakly Menger. In [13] Kocev found conditions under which the properties Menger, almost Menger and weakly Menger are equivalent. On the other hand, there are a very few papers dealing with these concepts in bitopological context.

However, the systematic study began in [17] by studying selective versions of separability in bitopological spaces and continued in [18] by investigating selection principles in bitopological spaces and proposing possible lines of investigation in this direction.

In [18] the authors defined three versions of the Menger property in a bitopological space (X,τ_1,τ_2), which are δ_2-Menger, $(1,2)$-almost Menger and $(1,2)$-weakly Menger. In this paper we will focus on the almost Menger property in bitopolological spaces and leave the weakly Menger property for another work.

Recall that $\mathcal{O}$, Ω, Γ denote the families of open covers, ω-covers and γ-covers of a space, respectively.

An open cover $\mathfrak{U}$ of X is an ω-cover, if for each finite subset F of X there exists $U \in \mathfrak{U}$ such that $F \subseteq U$ and X is not a member of $\mathfrak{U}$.

A topological space (X,τ) is a γ-set if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of ω-covers of X there exists a sequence $\langle V_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $\mathcal{U} = \{V_n : n \in \mathbb{N}\}$ is a γ-cover of X (i.e., $\mathcal{U}$ is an open cover of X such that it is infinite and for every $x \in X$ the set $\{U \in \mathfrak{U} : x \notin U\}$ is finite).

2. Definitions and examples. For undefined topological notions we refer to [6], while for undefined bitopological notions we refer to [5].

Throughout the paper (X,τ_1,τ_2) will be a bitopological space, i.e., the set X endowed with two topologies τ_1 and τ_2. For a subset A of $\mathrm{Int}_{\tau_i}(A)$ and $\mathrm{Cl}_{\tau_i}(A)$ will denote the interior and the closure of A in (X,τ_i) $(i = 1,2)$ respectively. By τ_i-open covers of X we mean that the cover of X by τ_i-open sets in X.

In [14], Kočinac introduced the following definition.

Definition 2.1 [14]. *A topological space (X,τ) is almost Menger, if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of open covers of X, there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup\{\mathcal{V}'_n : n \in \mathbb{N}\}$ is a cover of X, where $\mathcal{V}'_n = \{\mathrm{Cl}\,(V) : V \in \mathcal{V}_n\}$.*

The following definition was first introduced in [18].

Definition 2.2. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-almost Menger $(i,j = 1,2,$ $i \neq j)$ if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of τ_i-open covers of X, there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}}(\bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V))$.*

We note that if (X,τ_1) is almost Menger and $\tau_2 \leq \tau_1$, then the bitopological space (X,τ_1,τ_2) is $(1,2)$-almost Menger.

Proposition 2.1. *Let (X,τ_1,τ_2) be a bitopological space. If (X,τ_1) is Menger, then (X,τ_1,τ_2) is $(1,2)$-almost Menger.*

Proof. Obvious.

***Example* 2.1.** Let the real line $\mathbb{R}$ be endowed with the Euclidean topology τ_1 and the Sorgenfrey topology τ_2. Since $(\mathbb{R}, \tau_1)$ is Menger, the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is $(1,2)$-almost Menger by Proposition 2.1. On the other hand, $(\mathbb{R}, \tau_2)$ is not Menger.

In the following example we show that the converse of Proposition 2.1 is not true in general.

***Example* 2.2.** Let X be the Euclidean plane, τ_1 is the Sorgenfrey topology and τ_2 is the usual topology on X. It is easy to prove that the bitopological space (X, τ_1, τ_2) is $(1,2)$-almost Menger, but (X, τ_1) does not have the Menger property because X is not Lindelöff.

We can introduce the (i,j)-almost Rothberger property in a similar way.

Definition 2.3. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-almost Rothberger $(i, j = 1, 2,\ i \neq j)$ if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of τ_i-open covers of X, there is a sequence $\langle U_n : n \in \mathbb{N} \rangle$ such that for each n, $U_n \in \mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n)$.*

It is obvious that every (i,j)-almost Rothberger bitopological space is (i,j)-almost Menger.

In the following example we will see that the reverse implication is not true in general.

***Example* 2.3.** There is a (i,j)-almost Menger bitopological space which is not (i,j)-almost Rothberger.

Let the real line $\mathbb{R}$ be endowed with two topologies: one is the usual metric topology τ_1, and the other one is open-minus countable topology τ_2 ($U \in \tau_2$ if and only if $U = G \setminus C$, where G is open in the usual metric topology τ_1 on $\mathbb{R}$ and C is a countable subset of $\mathbb{R}$) (see [28]).

The bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is $(1,2)$-almost Menger. It follows from the fact that $(\mathbb{R}, \tau_1)$ is Menger and by Proposition 2.1.

To show that $(\mathbb{R}, \tau_1, \tau_2)$ is not $(1,2)$-almost Rothberger. Let $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ be a sequence of τ_1-open covers of $\mathbb{R}$ such that for each $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ $\mathrm{diam}_d(U) < 1/2^n$.

Recall that the usual metric $d(x,y) = |x-y|$ and $\mathrm{diam}_d(U) = \sup\{|x-y| \mid x, y \in U\}$. Now we have $\mathrm{Cl}_{\tau_1}(U) = \mathrm{Cl}_{\tau_2}(U)$ for each $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$. If for every $n \in \mathbb{N}$ we choose an element U_n of $\mathcal{U}_n$, then $\mathrm{diam}_d\left(\cup_{n \in \mathbb{N}} \mathrm{Cl}_{\tau_2}(U_n)\right) \leq \sum_{n=1}^{\infty} 1/2^n = 1$. This establishes that $(\mathbb{R}, \tau_1, \tau_2)$ is not $(1,2)$-almost Rothberger.

3. Some properties. As we mentioned in the previous section, if (X, τ_1) is Menger, then the bitopological space (X, τ_1, τ_2) is $(1,2)$-almost Menger. However the converse of this statement is not true.

Now it is natural to ask in which class of spaces the reverse implication holds.

First let us recall the following definition of (i,j)-regular bitopological space and its characterizations.

Definition 3.1 [27]. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-regular $(i, j = 1, 2,\ i \neq j)$ if for each point $x \in X$ and each τ_i-closed set F with $x \notin F$, there exist τ_i-open set U and τ_j-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.*

We may state at once:

Proposition 3.1 [27]. *A bitopological space (X, τ_1, τ_2) is (i,j)-regular if and only if for each point $x \in X$ and each τ_i-open set U with $x \in U$, there exists τ_i-open set V such that $x \in V \subseteq \subseteq \mathrm{Cl}_{\tau_j}(V) \subseteq U$.*

Now we have the following theorem.

Theorem 3.1. *If (X, τ_1, τ_2) is (i,j)-almost Menger and (i,j)-regular bitopological space, then (X, τ_i) is Menger.*

Proof. We consider only the case $i = 1,\ j = 2$. Let $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ be a sequence of τ_1-open covers of X. Since (X, τ_1, τ_2) is $(1, 2)$-regular bitopological space, for each $n \in \mathbb{N}$ there exists a τ_1-open cover $\mathcal{U}'_n$ such that $\mathrm{Cl}_{\tau_2}(\mathcal{U}'_n) = \{\mathrm{Cl}_{\tau_2}(U') \colon U' \in \mathcal{U}'_n\}$ is a refinement of $\mathcal{U}_n$.

Since (X, τ_1, τ_2) is $(1, 2)$-almost Menger bitopological space, there exists a sequence $\langle \mathcal{V}'_n : n \in \mathbb{N} \rangle$ such that for each n, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathcal{V}'_n)$ is a cover of X, where $\mathrm{Cl}_{\tau_2}(\mathcal{V}'_n) = \{\mathrm{Cl}_{\tau_2}(V') \colon V' \in \mathcal{V}'_n\}$. For every $n \in \mathbb{N}$ and $V' \in \mathcal{V}'_n$ we can choose $U_{V'} \in \mathcal{U}_n$ such that $\mathrm{Cl}_{\tau_2}(V') \subseteq U_{V'}$, since $\mathrm{Cl}_{\tau_2}(V') \in \mathrm{Cl}_{\tau_2}(\mathcal{U}'_n)$ and $\mathrm{Cl}_{\tau_2}(\mathcal{U}'_n)$ refines $\mathcal{U}_n$. Let $\mathcal{V}_n = \{U_{V'} : V' \in \mathcal{V}'_n\}$. Then $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ is a sequence with $\mathcal{V}_n$ a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a τ_1-open covers of X.

Now instead of using τ_i-open (τ_j-open) sets we will use (i, j)-regular open sets to characterize the (i, j)-almost Menger property in bitopological spaces.

First, let us recall the definition of (i, j)-regular open ((i, j)-regular closed) sets.

Definition 3.2 [11, 27]. *Let (X, τ_1, τ_2) be a bitopological space. A set $A \subseteq X$ is called (i, j)-regular open* ((i, j)*-regular closed*) $(i \neq j,\ i, j = 1, 2)$ *if $A = \mathrm{Int}_{\tau_i} \mathrm{Cl}_{\tau_j}(A)$ $(A = \mathrm{Cl}_{\tau_i} \mathrm{Int}_{\tau_j}(A))$. A is said to be pairwise regular open* (*pairwise regular closed*) *if it is both (i, j)-regular open and (j, i)-regular open* ((i, j)*-regular closed and (j, i)-regular closed*).

Clearly every (i, j)-regular open set in (X, τ_1, τ_2) is τ_i-open.

Theorem 3.2. *A bitopological space (X, τ_1, τ_2) is (i, j)-almost Menger if and only if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ of covers of X by (i, j)-regular open sets, there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N} \rangle$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}}(\bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V))$.*

Proof. We consider only the case $i = 1,\ j = 2$.

$(\Rightarrow)$ Since every $(1, 2)$-regular open set in (X, τ_1, τ_2) is τ_1-open, it is obvious.

$(\Leftarrow)$ Let $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ be a sequence of τ_1-open covers of X. For each n, let $\mathcal{U}'_n = \{\mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2}(U) \colon U \in \mathcal{U}_n\}$. Then $\langle \mathcal{U}'_n : n \in \mathbb{N} \rangle$ is a sequence of covers of X by $(1, 2)$-regular open sets.

By the hypothesis there exists a sequence $\langle \mathcal{V}'_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathcal{V}'_n)$ is a cover of X, where $\mathrm{Cl}_{\tau_2}(\mathcal{V}'_n) = \{\mathrm{Cl}_{\tau_2}(V') \colon V' \in \mathcal{V}'_n\}$. For each $n \in \mathbb{N}$ and $V' \in \mathcal{V}'_n$ there exists $U_{V'} \in \mathcal{U}_n$ such that $V' = \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2}(U_{V'})$. Since $\mathrm{Cl}_{\tau_2}(U_{V'})$ is a $(2, 1)$-regular closed set we have $\mathrm{Cl}_{\tau_2}(V') = \mathrm{Cl}_{\tau_2} \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2}(U_{V'}) = \mathrm{Cl}_{\tau_2}(U_{V'})$. Thus for each $n \in \mathbb{N}$ there is a finite subset $\mathcal{V}_n = \{U_{V'} : V' \in \mathcal{V}'_n\}$ of $\mathcal{U}_n$ such that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_2}(V)$.

In [21] it was shown that almost Menger spaces are preserved by almost continuous mappings. Let us now investigate the preservation of (i, j)-almost Mengerness under the almost continuous surjective functions.

We therefore begin by recalling the concept of (i, j)-almost continuous functions.

Definition 3.3 [19]. *A function $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-almost continuous if $f^{-1}(B)$ is τ_i-open set in X for every (i, j)-regular open set B in Y. In addition, f is called p-almost continuous, if it is $(1, 2)$- and $(2, 1)$-almost continuous.*

However we give the following theorem.

Theorem 3.3. *Let (X, τ_1, τ_2) be (i, j)-almost Menger bitopological space and (Y, σ_1, σ_2) be a bitopological space. If $f\colon X \to Y$ is p-almost continuous surjection, then (Y, σ_1, σ_2) is (i, j)-almost Menger.*

Proof. We consider only the case $i = 1,\ j = 2$. Let $\langle \mathcal{U}_n : n \in \mathbb{N} \rangle$ be a sequence of covers of Y by $(1, 2)$-regular open sets. Let $\langle \mathcal{U}'_n : n \in \mathbb{N} \rangle$ be a sequence defined by $\mathcal{U}'_n = \{f^{-1}(U) \colon U \in \mathcal{U}_n\}$. Since f is $(1, 2)$-almost continuous each $\mathcal{U}'_n$ is a cover of X by τ_1-open sets. Apply the fact that X is $(1, 2)$-almost Menger; there is a sequence $\langle \mathcal{V}'_n : n \in \mathbb{N} \rangle$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n \subseteq \mathcal{U}'_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup_{V'\in\mathcal{V}'_n} \mathrm{Cl}_{\tau_2}(V')$.

For each $V' \in \mathcal{V}'_n$ we can choose $U_{V'} \in \mathcal{U}_n$ such that $V' = f^{-1}(U_{V'})$. Let $\mathcal{V}_n = \{U_{V'} : V' \in \mathcal{V}'_n\}$. Now we have a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $Y = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{Cl}_{\sigma_2}(U_{V'}) : V' \in \mathcal{V}'_n\}$.

Now let us prove that $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\sigma_2}(\mathcal{V}_n)$, where $\mathrm{Cl}_{\sigma_2}(\mathcal{V}_n) = \{\mathrm{Cl}_{\sigma_2}(U_{V'}) : V' \in \mathcal{V}'_n\}$ is a cover of Y. Indeed, if $y = f(x) \in Y$, then there exists $n \in \mathbb{N}$ and $U_{V'} \in \mathcal{V}_n$ such that $x \in \mathrm{Cl}_{\tau_2}(f^{-1}(U_{V'}))$. Since $U_{V'} \subseteq Y$ is $(1, 2)$-regular open we have $\mathrm{Cl}_{\sigma_2}(U_{V'}) = \mathrm{Cl}_{\sigma_2} \mathrm{Int}_{\sigma_1} \mathrm{Cl}_{\sigma_2}(U_{V'})$. Since $Y \setminus \mathrm{Cl}_{\sigma_2}(U_{V'})$ is $(2, 1)$-regular open, f is $(2, 1)$-almost continuous $f^{-1}(Y \setminus \mathrm{Cl}_{\sigma_2}(U_{V'})) = X \setminus f^{-1}(\mathrm{Cl}_{\sigma_2}(U_{V'}))$ is τ_2-open and $f^{-1}(\mathrm{Cl}_{\sigma_2}(U_{V'}))$ is τ_2-closed. Then $\mathrm{Cl}_{\tau_2}(f^{-1}(U_{V'})) \subseteq f^{-1}(\mathrm{Cl}_{\sigma_2}(U_{V'}))$. This means $y \in \mathrm{Cl}_{\sigma_2}(U_{V'})$. Hence (Y, σ_1, σ_2) is $(1, 2)$-almost Menger.

For $k \in \mathbb{N}$, the power bitopological space X^k of a bitopological space (X, τ_1, τ_2) is defined as $(X^k, {\tau_1}^k, {\tau_2}^k)$ in [4].

Theorem 3.4. *Let (X, τ_1, τ_2) be a bitopological space. For each $n \in \mathbb{N}$, $(X^n, {\tau_1}^n, {\tau_2}^n)$ is $(1, 2)$-almost Menger if and only if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of τ_1-ω-covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for every finite set $F \subseteq X$, there exists $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $F \subseteq \mathrm{Cl}_{\tau_2}(V)$.*

Proof. ($\Longrightarrow$) Let $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ be a sequence of τ_1-ω-covers of X. Let $\{N_m : m \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ such that for each $m \in \mathbb{N}$, N_m is infinite. For each $m \in \mathbb{N}$ and each $i \in N_m$, let ${\mathcal{U}_i}^m = \{U^m : U \in \mathcal{U}_i\}$. Then $\langle {\mathcal{U}_i}^m : i \in N_m\rangle$ is a sequence of ${\tau_1}^m$-open covers of X^m. Since $(X^m, {\tau_1}^m, {\tau_2}^m)$ is $(1, 2)$-almost Menger there exists a sequence $\langle {\mathcal{V}_i}^m : i \in N_m\rangle$ such that for every $i \in N_m$, ${\mathcal{V}_i}^m$ is a finite subset of ${\mathcal{U}_i}^m$ and $\bigcup_{i\in N_m} \bigcup \mathrm{Cl}_{\tau_2^m}({\mathcal{V}_i}^m) = X^m$ where $\mathrm{Cl}_{\tau_2^m}({\mathcal{V}_i}^m) = \{\mathrm{Cl}_{\tau_2^m}(V) : V \in {\mathcal{V}_i}^m\}$.

On the other hand, for every $V \in {\mathcal{V}_i}^m$, $m \in \mathbb{N}$, $i \in N_m$, pick an element $U_V \in \mathcal{U}_i$ such that $V = {U_V}^m$. Then $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ is the desired sequence, where $\mathcal{V}_i = \{U_V : V \in {\mathcal{V}_i}^m\}$ is a finite subfamily of $\mathcal{U}_i$.

Indeed, let $F = \{x_1, \ldots, x_k\}$ be a finite subset of X. Since $\bigcup_{i\in N_k} \mathrm{Cl}_{\tau_2^k}({\mathcal{V}_i}^k)$ is a cover of X^k there exists an $i \in N_k$ and $V \in {\mathcal{V}_i}^k$ such that $(x_1, \ldots, x_k) \in \mathrm{Cl}_{\tau_2^k}(V)$. On the other side, we have $\mathrm{Cl}_{\tau_2^k}(V) = \mathrm{Cl}_{\tau_2^k}({U_V}^k) = (\mathrm{Cl}_{\tau_2}(U_V))^k$ for some $U_V \in \mathcal{V}_i$. Thus $F \subseteq \mathrm{Cl}_{\tau_2}(U_V)$ for $U_V \in \mathcal{V}_i$.

($\Longleftarrow$) Let $k \in \mathbb{N}$ be fixed and $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ be a sequence of τ_1^k-open covers of X^k, where each $\mathcal{U}_n = \{U_{nm} : m \in I_n\}$.

Let F be a finite subset of X. Since F^k is a finite subset of X^k, for each $n \in \mathbb{N}$, there exists a finite subset $I_n^F \subseteq I_n$ such that $F^k \subseteq \bigcup_{m\in I_n^F} U_{nm}$. In this case, there exists a τ_1-open set V_F such that $F \subseteq V_F$ and $V_F^k \subseteq \bigcup_{m\in I_n^F} U_{nm}$.

Thus $\mathcal{V}_n = \{V_F : F \subseteq X \text{ is finite}\}$ is a τ_1-ω-open cover of X. By assumption there exists a sequence $\langle \mathcal{W}_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and each finite set $T \subseteq X$ there exists $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$ such that $T \subseteq \mathrm{Cl}_{\tau_2}(W)$.

Let for each $n \in \mathbb{N}$, $\mathcal{W}_n = \{V_{F_j} : j \in J_n\}$, where J_n is a finite index set and let $\mathcal{K}_n = \{U_{nm} : m \in H_n\}$, where $H_n = \{m \in I_n : m \in I_n^{F_j},\ j \in J_n\}$. Then we have $\mathcal{K}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2^k}(\mathcal{K}_n)$ is a cover of X^k, where $\mathrm{Cl}_{\tau_2^k}(\mathcal{K}_n) = \{\mathrm{Cl}_{\tau_2^k}(U_{nm}) : m \in H_n\}$.

To see, $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2^k}(\mathcal{K}_n)$ is indeed a cover of X^k, now let $x = (x_1, \ldots, x_k) \in X^k$. Then $T = \{x_1, \ldots, x_k\}$ is a finite subset of X. So there exists $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$ such that $T \subseteq \mathrm{Cl}_{\tau_2}(W)$. Let $W = V_{F_j}$ for some $j \in J_n$. Then we have

$$T^k \subseteq (\mathrm{Cl}_{\tau_2}(V_{F_j}))^k \subseteq \mathrm{Cl}_{\tau_2^k}(V_{F_j}^k) \subseteq \bigcup_{m\in I_n^{F_j}} \mathrm{Cl}_{\tau_2^k}(U_{nm}).$$

So, there exists $m \in I_n^{F_j} \subseteq H_n$ such that $x \in \mathrm{Cl}_{\tau_2^k}(U_{nm})$. Thus $(X^k, {\tau_1}^k, {\tau_2}^k)$ is $(1,2)$-almost Menger.

4. Almost γ-set in bitopological spaces.

Definition 4.1. *Let (X, τ_1, τ_2) be a bitopological space and $\mathcal{U}$ be an infinite τ_i-open cover of X. If for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin \mathrm{Cl}_{\tau_j}(U)\}$ is finite, then we call $\mathcal{U}$ is (i,j)-almost γ-cover for X.*

Gerlits and Nagy [7] defined the notion of γ-set and Kocev in [12] defined the notion of almost γ-set and its characterizations. Now let us give the definition of almost γ-set for bitopological spaces.

Definition 4.2. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-almost γ-set if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of τ_i-ω-covers of X, there exists a sequence $\langle V_n : n \in \mathbb{N}\rangle$ such that for all $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and the set $\{V_n : n \in \mathbb{N}\}$ is an (i,j)-almost γ-cover for X.*

Theorem 4.1. *(X, τ_1, τ_2) is (i,j)-almost γ-set if and only if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of τ_i-ω-covers of X by (i,j)-regular open sets, there exists a sequence $\langle V_n : n \in \mathbb{N}\rangle$ such that for all $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and the set $\{V_n : n \in \mathbb{N}\}$ is an (i,j)-almost γ-cover for X.*

Proof. We consider only the case $i = 1,\ j = 2$.

$(\Longrightarrow)$ Since every $(1,2)$-regular open set in (X, τ_1, τ_2) is τ_1-open, it is obvious.

$(\Longleftarrow)$ Let $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ be a sequence of τ_1-ω-covers of X. So the set $\mathcal{U}'_n = \{\mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2}(U) : U \in \mathcal{U}_n\}$ is a τ_1-ω-cover of X, by $(1,2)$-regular open sets. By the hypothesis there exists a sequence $\langle V'_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $V'_n \in \mathcal{U}'_n$ and $\mathcal{V}' = \{V'_n : n \in \mathbb{N}\}$ is a $(1,2)$-almost γ-cover of X.

For each $n \in \mathbb{N}$, there exists $V_n \in \mathcal{U}_n$ such that $V'_n = \mathrm{Int}_{\tau_1} \mathrm{Cl}_{\tau_2}(V_n)$. It is easy to check that the sequence $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ witnesses for (X, τ_1, τ_2) is $(1,2)$-almost γ-set.

Theorem 4.2. *Let (X, τ_1, τ_2) be an (i,j)-almost γ-set and (Y, σ_1, σ_2) be a bitopological space. If $f : X \to Y$ is an (i,j)-almost continuous surjection, then (Y, σ_1, σ_2) is an (i,j)-almost γ-set.*

Proof. We consider only the case $i = 1,\ j = 2$. Let $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ be a sequence of σ_1-ω-covers of Y by $(1,2)$-regular open set. Since f is $(1,2)$-almost continuous surjection, the set $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is a τ_1-ω-cover of X.

Since (X, τ_1, τ_2) is $(1,2)$-almost γ-set there exists a sequence $\langle U'_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $U'_n \in \mathcal{U}'_n$ and $\mathcal{U}' = \{U'_n : n \in \mathbb{N}\}$ is an $(1,2)$-almost γ-cover for X. On the other hand, for all $n \in \mathbb{N}$ there exists $U_n \in \mathcal{U}_n$ such that $U'_n = f^{-1}(U_n)$. We claim that $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an $(1,2)$-almost γ-cover for Y.

Since $\mathcal{U}'$ is infinite, so is $\mathcal{U}$. If $y \in Y$ there exists $x \in X$ such that $f(x) = y$ and $\{U'_n \in \mathcal{U}' : x \notin \mathrm{Cl}_{\tau_2}(U'_n)\}$ is finite since $\mathcal{U}'$ is $(1,2)$-almost γ-cover for X. That means there exists $M \in \mathbb{N}$ such that $\forall n \in \mathbb{N}\ n > M$, we have $x \in \mathrm{Cl}_{\tau_2}(U'_n)$. For all $n > M$, $y = f(x) \in f(\mathrm{Cl}_{\tau_2}(U'_n))$. Since $\mathrm{Cl}_{\tau_2}(U'_n) \subseteq f^{-1}(\mathrm{Cl}_{\sigma_2}(U_n))$ we have for each $n > M$, $y \in \mathrm{Cl}_{\sigma_2}(U_n)$ so $\{U_n \in \mathcal{U} : y \notin \mathrm{Cl}_{\sigma_2}(U_n)\}$ is finite, hence (Y, σ_1, σ_2) is $(1,2)$-almost γ-set.

5. Concluding remarks. The following definition of (i,j)-weakly Menger bitopological space was first introduced in [18].

Definition 5.1. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-weakly Menger, $i, j = 1, 2$, $i \neq j$, if for each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of τ_i-open covers of X, there exists a sequence $\langle \mathcal{V}_n : n \in \mathbb{N}\rangle$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathcal{V}_n)$.*

It would be interesting to investigate the properties of (i,j)-weakly Mengerness in bitopological context and the relations between (i,j)-almost Menger and (i,j)-weakly Menger bitopological spaces.

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