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# UNIVALENCE CRITERIA AND QUASICONFORMAL EXTENSIONS*

# КРИТЕРІЇ УНІВАЛЕНТНОСТІ ТА КВАЗІКОНФОРМНІ РОЗШИРЕННЯ

We establish more general conditions for the univalence of analytic functions in the open unit disk $\mathcal{U}$. In addition, we obtain a refinement to the criterion of quasiconformal extension for the main result.

Встановлено більш загальні умови унівалентності аналітичних функцій у відкритому одиничному колі $\mathcal{U}$. Крім того, уточнено критерій квазіконформного розширення для основного результату.

**1. Introduction.** Let $\mathcal{A}$ be the class of analytic functions $f$ in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{U}_r$ the disk $\{z \in \mathbb{C} \colon |z| < r\}$, where $0 < r \leq 1$, by $\mathcal{U} = \mathcal{U}_1$ the open unit disk of the complex plane and by $I$ the interval $[0, \infty)$.

Most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Becker [3], Nehari [18] and Ozaki, Nunokawa [19]. Some extensions of these three criteria were given by (see [15, 17, 20, 24–27, 29]). During the time, a lot of univalence criteria were obtained by different authors (see also [7, 9–11]).

In the present investigation we use the method of subordination chains to establish some sufficient conditions for the univalence of an analytic function. Also, by using Becker's method, we obtain a refinement to the criterion of quasiconformal extension for the main result.

**2. Preliminaries.** Before proving our main theorem we need a brief summary of the method of Loewner chains and quasiconformal extensions.

A function $L(z,t) \colon \mathcal{U} \times [0,\infty) \to \mathbb{C}$ is said to be *subordination chain* (*or Loewner chain*) if:

(i) $L(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t \geq 0$.

(ii) $L(z,t) \prec L(z,s)$ for all $0 \leq t \leq s < \infty$, where the symbol "$\prec$" stands for subordination.

In proving our results, we will need the following theorem due to Pommerenke [23].

**Theorem 2.1.** *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots,\ a_1(t) \neq 0$ be analytic in $\mathcal{U}_r$ for all $t \in I$, locally absolutely continuous in $I$, and locally uniform with respect to $\mathcal{U}_r$. For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \quad \forall z \in \mathcal{U}_r, \tag{2.1}$$

*where $p(z,t)$ is analytic in $\mathcal{U}$ and satisfies the condition $\Re p(z,t) > 0$ for all $z \in \mathcal{U},\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in $\mathcal{U}_r$, then, for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$.*

Let $k$ be constant in $[0,1)$. Then a homeomorphism $f$ of $G \subset \mathbb{C}$ is said to be $k$*-quasiconformal,* if $\partial_z f$ and $\partial_{\overline{z}} f$ in the distributional sense are locally integrable on $G$ and fulfill the inequality $|\partial_{\overline{z}} f| \leq k\,|\partial_z f|$ almost everywhere in $G$. If we do not need to specify $k$, we will simply call that $f$ is *quasiconformal.*

The method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [3, 4] and also [5]).

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**Theorem 2.2.** *Suppose that* $L(z,t)$ *is a Loewner chain. Consider*

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1},\qquad z\in\mathcal{U},\quad t\geq 0,$$

*where* $p(z,t)$ *is given in* (2.1). *If*

$$|w(z,t)|\leq k,\quad 0\leq k<1,$$

*for all* $z\in\mathcal{U}$ *and* $t\geq 0$, *then* $L(z,t)$ *admits a continuous extension to* $\overline{\mathcal{U}}$ *for each* $t\geq 0$ *and the function* $F(z,\overline{z})$ *defined by*

$$F(z,\overline{z})=\begin{cases}\mathcal{L}(z,0), & \text{if}\quad |z|<1,\\ \mathcal{L}\left(\dfrac{z}{|z|},\log|z|\right), & \text{if}\quad |z|\geq 1,\end{cases}$$

*is a* $k$*-quasiconformal extension of* $L(z,0)$ *to* $\mathbb{C}$.

Examples of quasiconformal extension criteria can be found in [1, 2, 6, 16, 22] and more recently in [8, 12–14, 28].

**3. Main results.** Making use of Theorem 2.1 we can prove now, our main results.

**Theorem 3.1.** *Consider* $f\in\mathcal{A}$ *and* $g$ *be an analytic function in* $\mathcal{U}$, $g(z)=1+b_1z+\ldots$. *Let* $\alpha$, $\beta$, $A$ *and* $B$ *complex numbers such that* $\Re(\alpha)>\dfrac{1}{2}$, $A+B\neq 0$, $|A-B|<2$, $|A|\leq 1$ *and* $|B|\leq 1$. *If the inequalities*

$$\left|\frac{1}{\alpha}\left(\frac{f'(z)}{g(z)-\beta}-1\right)\right|<\frac{|A+B|}{2-|A-B|}\tag{3.1}$$

*and*

$$\left|\left(\frac{f'(z)}{g(z)-\beta}-1\right)|z|^2+\left(1-|z|^2\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{zf'(z)}{f(z)}+\frac{zg'(z)}{g(z)-\beta}\right]-\frac{\left(\overline{A}-\overline{B}\right)(A+B)}{4-|A-B|^2}\right|\leq$$

$$\leq\frac{2\,|A+B|}{4-|A-B|^2}\tag{3.2}$$

*are satisfied for all* $z\in\mathcal{U}$, *then the function* $f$ *is univalent in* $\mathcal{U}$.

***Proof.*** We prove that there exists a real number $r\in(0,1]$ such that the function $L\colon\mathcal{U}_r\times I\rightarrow\mathbb{C}$, defined formally by

$$L(z,t)=f^{1-\alpha}(e^{-t}z)\left[f(e^{-t}z)+\left(e^{t}-e^{-t}\right)z\left(g(e^{-t}z)-\beta\right)\right]^{\alpha}\tag{3.3}$$

is analytic in $\mathcal{U}_r$ for all $t\in I$.

Since $f(z)\neq 0$ for all $z\in\mathcal{U}\setminus\{0\}$, the function

$$\varphi_1(z,t)=\frac{\left(e^{t}-e^{-t}\right)z\left(g(e^{-t}z)-\beta\right)}{f(e^{-t}z)}$$

is analytic in $\mathcal{U}$.

It follows from

$$\varphi_2(z,t) = 1 + \frac{\left(e^t - e^{-t}\right) z \left(g(e^{-t}z) - \beta\right)}{f(e^{-t}z)}$$

that there exist a $r_1$, $0 < r_1 < r$ such that $\varphi_2$ is analytic in $\mathcal{U}_{r_1}$ and $\varphi_2(0,t) = (1-\beta)\, e^{2t} + \beta$, $\varphi_2(z,t) \neq 0$ for all $z \in \mathcal{U}_{r_1}$, $t \in I$. Therefore, we choose an analytic branch in $\mathcal{U}_{r_1}$ of the function

$$\varphi_3(z,t) = \left[\varphi_2(z,t)\right]^{\alpha}.$$

From these considerations it follows that the function

$$L(z,t) = f^{1-\alpha}(e^{-t}z)\left[f(e^{-t}z) + \left(e^t - e^{-t}\right) z \left(g(e^{-t}z) - \beta\right)\right]^{\alpha} =$$
$$= f(e^{-t}z)\varphi_3(z,t) = a_1(t)z + \ldots$$

is an analytic function in $\mathcal{U}_{r_1}$ for all $t \in I$.

After simple calculation we have

$$a_1(t) = e^{(2\alpha-1)t}\left[\beta e^{-2t} + 1 - \beta\right]^{\alpha}$$

for which we consider the uniform branch equal to 1 at the origin. Because $\Re(\alpha) > \dfrac{1}{2}$, we have that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Moreover, $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $L(z,t)$ in $\mathcal{U}_{r_1}$, it follows that there exists a number $r_2$, $0 < r_2 < r_1$, and a constant $K = K(r_2)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K \quad \forall z \in \mathcal{U}_{r_2}, \quad t \in I.$$

Then, by Montel's theorem, $\left\{\dfrac{L(z,t)}{a_1(t)}\right\}_{t\in I}$ is a normal family in $\mathcal{U}_{r_2}$. From the analyticity of $\dfrac{\partial L(z,t)}{\partial t}$, we obtain that for all fixed numbers $T > 0$ and $r_3$, $0 < r_3 < r_2$, there exists a constant $K_1 > 0$ (that depends on $T$ and $r_3$) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1 \quad \forall z \in \mathcal{U}_{r_3}, \quad t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in $I$, locally uniform with respect to $\mathcal{U}_{r_3}$.

Let $p\colon \mathcal{U}_r \times I \to \mathbb{C}$ be the analytic function in $\mathcal{U}_r$, $0 < r < r_3$, for all $t \in I$, defined by

$$p(z,t) = \frac{\partial L(z,t)}{\partial t}\Big/ z\frac{\partial L(z,t)}{\partial z}.$$

If the function

$$w(z,t) = \frac{p(z,t) - 1}{A + Bp(z,t)} = \frac{\dfrac{\partial L(z,t)}{\partial t} - \dfrac{z\partial L(z,t)}{\partial z}}{A\dfrac{z\partial L(z,t)}{\partial z} + B\dfrac{\partial L(z,t)}{\partial t}} \tag{3.4}$$

is analytic in $\mathcal{U} \times I$ and $|w(z,t)| < 1$, for all $z \in \mathcal{U}$ and $t \in I$, then $p(z,t)$ has an analytic extension with positive real part in $\mathcal{U}$, for all $t \in I$. From equality (3.4) we obtain

$$w(z,t)=\frac{-2\phi(z,t)}{(A-B)\,\phi(z,t)+A+B} \tag{3.5}$$

for $z\in\mathcal{U}$ and $t\in I$, where

$$\phi(z,t)=\left(\frac{1}{\alpha}\frac{f'(e^{-t}z)}{g(e^{-t}z)-\beta}-1\right)e^{-2t}+\left(1-e^{-2t}\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}+\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)-\beta}\right]. \tag{3.6}$$

From (3.1), (3.5), (3.6) and $\Re(\alpha)>\dfrac{1}{2}$ we have

$$|w(z,0)|=\left|\frac{1}{\alpha}\left(\frac{f'(z)}{g(z)-\beta}-1\right)\right|<\frac{|A+B|}{2-|A-B|}$$

and

$$|w(0,t)|=\left|\left(\frac{1}{\alpha\,(1-\beta)}-1\right)e^{-2t}\right|<\frac{|A+B|}{2-|A-B|},$$

where $A+B\neq 0,\ |A-B|<2,\ |A|\leq 1$ and $|B|\leq 1$.

Since $\left|e^{-t}z\right|\leq\left|e^{-t}\right|=e^{-t}<1$ for all $z\in\overline{\mathcal{U}}=\{z\in\mathbb{C}\colon\ |z|\leq 1\}$ and $t>0$, we find that $w(z,t)$ is an analytic function in $\overline{\mathcal{U}}$. Using the maximum modulus principle it follows that for all $z\in\mathcal{U}-\{0\}$ and each $t>0$ arbitrarily fixed there exists $\theta=\theta(t)\in\mathbb{R}$ such that

$$|w(z,t)|<\max_{|z|=1}|w(z,t)|=\left|w(e^{i\theta},t)\right|$$

for all $z\in\mathcal{U}$ and $t\in I$.

Denote $u=e^{-t}e^{i\theta}$. Then $|u|=e^{-t}$ and from (3.5) we get

$$\left|w(e^{i\theta},t)\right|=\left|\frac{2\phi(e^{i\theta},t)}{(A-B)\,\phi(e^{i\theta},t)+A+B}\right|,$$

where

$$\phi(e^{i\theta},t)=\left(\frac{1}{\alpha}\frac{f'(u)}{g(u)-\beta}-1\right)|u|^2+\left(1-|u|^2\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{uf'(u)}{f(u)}+\frac{ug'(u)}{g(u)-\beta}\right].$$

Because $u\in\mathcal{U}$, the inequality (3.2) implies that

$$\left|w(e^{i\theta},t)\right|\leq 1$$

for all $z\in\mathcal{U}$ and $t\in I$. Therefore $|w(z,t)|<1$ for all $z\in\mathcal{U}$ and $t\in I$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $L(z,t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$, for all $t\in I$. For $t=0$ we have $L(z,0)=f(z)$, for $z\in\mathcal{U}$ and therefore the function $f$ is analytic and univalent in $\mathcal{U}$.

Theorem 3.1 is proved.

***Remark* 3.1.** Some particular cases of Theorem 3.1 are the following:

(i) When $\alpha=1,\ \beta=0,\ A=B=1$ and $g(z)=f'(z)$ inequality (3.2) becomes

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right|\leq 1,\quad z\in\mathcal{U}, \tag{3.7}$$

which is Becker's condition of univalence [3].

(ii) A result due to N. N. Pascu [21] is obtained when $\alpha = 1,\ A = B = 1$ and $g(z) = f'(z)$.

***Remark* 3.2.** It is worth to notice that the condition (3.2) assures the univalence of an analytic function in more general case than that of condition (3.7).

***Remark* 3.3.** If we put $g(z) = \dfrac{f(z)}{z}$, $\alpha = 1$ and $\beta = 0$ into Theorem 3.1, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{|A+B|}{2 - |A-B|}, \quad z \in \mathcal{U},$$

the class of functions starlike with respect to origin.

**4. Quasiconformal extension.** In this section we will obtain the univalence condition given in Theorem 3.1 to a quasiconformal extension criterion.

**Theorem 4.1.** *Consider $f \in \mathcal{A}$, $g$ be an analytic function in $\mathcal{U}$, $g(z) = 1 + b_1 z + \ldots$ and $k \in [0,1)$. Let $\alpha$, $\beta$, $A$ and $B$ complex numbers such that $\Re(\alpha) > \dfrac{1}{2}$, $A + B \neq 0$, $k\,|A-B| < 2$, $|A| \le 1$ and $|B| \le 1$. If the inequalities*

$$\left|\frac{1}{\alpha}\left(\frac{f'(z)}{g(z)-\beta} - 1\right)\right| < \frac{k\,|A+B|}{2 - k\,|A-B|}$$

*and*

$$\left|\left(\frac{f'(z)}{g(z)-\beta} - 1\right)|z|^2 + \left(1 - |z|^2\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z)-\beta}\right] - \frac{k^2\left(\overline{A}-\overline{B}\right)(A+B)}{4 - k^2\,|A-B|^2}\right| \le$$
$$\le \frac{2k\,|A+B|}{4 - k^2\,|A-B|^2}$$

*are satisfied for all $z \in \mathcal{U}$, then the function $f$ has a $k$-quasiconformal extension to $\mathbb{C}$.*

***Proof.*** In the proof of Theorem 3.1 has been proved that the function $L(z,t)$ given by (3.3) is a subordination chain in $\mathcal{U}$. Applying Theorem 2.2 to the function $w(z,t)$ given by (3.5), we obtain that the assumption

$$|w(z,t)| = \left|\frac{-2\phi(z,t)}{(A-B)\,\phi(z,t) + A + B}\right| \le k, \qquad z \in \mathcal{U}, \quad t \ge 0, \quad k \in [0,1), \tag{4.1}$$

where $\phi(z,t)$ is defined by (3.6).

Lenghty but elementary calculation shows that the last inequality (4.1) is equivalent to

$$\left|\left(\frac{1}{\alpha}\frac{f'(e^{-t}z)}{g(e^{-t}z)-\beta} - 1\right)e^{-2t} + \left(1 - e^{-2t}\right)\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)-\beta}\right] - \right.$$
$$\left. - \frac{k^2\left(\overline{A}-\overline{B}\right)(A+B)}{4 - k^2\,|A-B|^2}\right| \le \frac{2k\,|A+B|}{4 - k^2\,|A-B|^2}. \tag{4.2}$$

Inequality (4.2) implies $k$-quasiconformal extensibility of $f$.

Theorem 4.1 is proved.

***Remark* 4.1.** For $A = B = 1$ $\alpha = 1$, $\beta = 0$, $g = f'$ in Theorem 4.1, we have result of Becker [3].

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