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INTERFERENCE OF THE WEIGHT AND BOUNDARY CONTOUR FOR ALGEBRAIC POLYNOMIALS IN THE WEIGHTED LEBESGUE SPACES. I

ПЕРЕШКОДИ, ПОВ'ЯЗАНІ З ВАГОЮ ТА ГРАНИЧНИМ КОНТУРОМ, ДЛЯ АЛГЕБРАЇЧНИХ ПОЛІНОМІВ У ЗВАЖЕНИХ ПРОСТОРАХ ЛЕБЕГА

We study the order of the height of the modulus of arbitrary algebraic polynomials with respect to the weighted Lebesgue space, where the contour and the weight functions have some singularities.

Вивчається порядок висоти модуля довільних алгебраїчних поліномів відносно зважених просторів Лебега, в яких контур та вагові функції мають деякі сингулярності.

1. Introduction. Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ be a closed Jordan curve, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = \operatorname{ext} L$. Let $\wp_n$ denotes the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$.

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we denote

$$\|P_n\|_{\mathcal{L}_p} := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z) \, |P_n(z)|^p \, |dz| \right)^{1/p}, \quad 0 < p < \infty,$$

$$\|P_n\|_{\mathcal{L}_\infty} := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \quad p = \infty.$$

Clearly, $\|\cdot\|_{\mathcal{L}_p}$ is a quasinorm (i.e., a norm for $1 \leq p \leq \infty$ and a p-norm for $0 < p < 1$).

Denoted by $w = \Phi(z)$, the univalent conformal mapping of Ω onto $\Delta := \{w\colon |w| > 1\}$ with normalization $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \dfrac{\Phi(z)}{z} > 0$ and $\Psi := \Phi^{-1}$. For $t \geq 1$, we set

$$L_t := \{z\colon \ |\Phi(z)| = t\}, \quad L_1 \equiv L, \ G_t := \operatorname{int} L_t, \quad \Omega_t := \operatorname{ext} L_t.$$

Let $\{z_j\}_{j=1}^m$ be the fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^m |z - z_j|^{\gamma_j}, \tag{1.1}$$

where $\gamma_j > -1$, for all $j = 1, 2, \ldots, m$, and h_0 is uniformly separated from zero in G_{R_0}, i.e., there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

In many problems of approximation theory, theory of polynomials and others, often need to study the following inequality:

$$\|P_n\|_{\mathcal{L}_\infty} \leq c\mu_n(L, h, p) \, \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{1.2}$$

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where $c = c(G, p) > 0$ is a constant which is independent of n and P_n, and $\mu_n(L, h, p) \to \infty$, $n \to \to \infty$, depending on the geometrical properties of curve L and weight function h in the neighborhood of the points $\{z_j\}_{j=1}^m$. In most cases, these problems can be divided in two parts. Firstly, the case where the boundary curve and weight function do not have singularities and secondly, in case where boundary curve or (and) weight function have an any singularities.

The first classical result of (1.2)-type, in case $h(z) \equiv 1$ and $L = \{z\colon |z| = 1\}$ for $0 < p < \infty$ is found by Jackson in [13]. The other classical results are similar to (1.2) belongs to Szegö and Zigmund, in [24]. The estimation of (1.2)-type for $0 < p < \infty$ and $h(z) \equiv 1$ where L is a rectifiable Jordan curve is investigated by Suetin in [25], Mamedhanov in [16, 17], Nikol'skii in [19, p. 122–133], Pritsker in [22], Andrievskii in [11] (Theorem 6), Abdullayev et al. [2–7] and etc. There are more references regarding the inequality of (1.2)-type, we can find in Milovanovic et al. [18] (Sect. 5.3).

The question arises: how can "pay off" singularity curve and weight function, so that, the estimation of (1.2) has coincided with the estimation of where the boundary curve and weight functions are not any singularities.

Let a rectifiable Jordan curve be L, has a natural parametrization $z = z(s)$, $0 \le s \le l := \operatorname{mes} L$. It is said to be $L \in C(1, \lambda)$, $0 < \lambda < 1$, if $z(s)$ is continuously differentiable and $z'(s) \in \operatorname{Lip} \lambda$. Let L belong to $C(1, \lambda)$ everywhere except for a single point $z_1 \in L$, i.e., the derivative $z'(s)$ satisfies the Lipschitz condition on the $[0, l]$ and $z(0) = z(l) = z_1$, but $z'(0) \ne z'(l)$. Assume that L has a corner at z_1 with exterior angle $\nu\pi$, $0 < \nu \le 2$, and denote the set of such curves by $C(1, \lambda, \nu)$.

Suetin, in [27], investigated this problem in case $p = 2$ for orthonormal on L polynomials $Q_n(z)$ with the weight function h defined as in (1.1) and for the curve $L \in C(1, \lambda, \nu)$. He showed that the condition of "pay off" singularity curve and weight function at the points z_1 can be given as following:

$$(1 + \gamma_1)\, \nu_1 = 1. \tag{1.3}$$

Under this conditions, for $Q_n(z)$ in case $L \in C(1, \lambda, \nu)$ Suetin [27] provided the following estimation:

$$|Q_n(z)| \le c(L)\sqrt{n}, \quad z \in L, \tag{1.4}$$

where $c(L) > 0$ is a constant independent on n.

In this work we study the estimations of the (1.2)-type for more general regions of the complex plane and we obtain the analog of the equality (1.3) correnponding to the general case.

2. Definitions and main results. Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and ε_0, ε_1, $\varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument; otherwise, such dependence will be explicitly stated.

For any $k \ge 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k + 1, \ldots, m$.

Before giving our new results, we need to give some definitions and the notations. Let $z = \psi(w)$ be the univalent conformal mapping of $B := \{w\colon |w| < 1\}$ onto the G normalized by $\psi(0) = 0$, $\psi'(0) > 0$. By [20, p. 286–294], we say a bounded Jordan region G is called *κ-quasidisk,* $0 \le \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \dfrac{1+\kappa}{1-\kappa}$, the homeomorphism of the plane $\overline{\mathbb{C}}$ on plane $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a *κ-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is *κ-quasidisk (κ-quasicircle)* for some $0 \leq \kappa < 1$.

We denoted the class of *κ-quasicircle* by $Q(\kappa)$, $0 \leq \kappa < 1$, and $L \in Q$, if $L \in Q(\kappa)$, for some $0 \leq \kappa < 1$. It is well-known that the quasicircle may not even be locally rectifiable in [14, p. 104].

Definition 2.1. *It is said that* $L \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, *if* $L \in Q(\kappa)$ *and* L *is rectifiable.*

Theorem 2.1. *Let* $p > 0$. *Suppose that* $L \in \widetilde{Q}(\kappa)$, *for some* $0 \leq \kappa < 1$ *and* $h(z)$ *defined in* (1.1) *for* $\gamma_j = 0$, *for all* $j = \overline{1,m}$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_1 = c_1(L,p) > 0$ *such that*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_1 n^{\frac{1+\kappa}{p}} \cdot \|P_n\|_{\mathcal{L}_p(h_0,L)}. \tag{2.1}$$

Thus, Theorem 2.1 provides an opportunity to observe the growth of $|P_n(z)|$ on the curve L. Note that, Theorem 2.1 provided for $L := \{z\colon\ |z| = 1\}$ (i.e., $\kappa = 0$) in [13], for arbitrary rectufiable curve L without weight function in [16], for polynomials in many variables in [22] (Theorem 1.1), for the special curve in [4–7] and others.

From the conditions of the theorem, we see that, it holds for k-quasidisks with $0 \leq k < 1$. But calculating the coefficient of quasiconformality κ for some curves is not an easy task. Therefore, we define a more general class of curves with another characteristic. One of them is the following:

Definition 2.2. *We say that* $L \in Q_\alpha$, $0 < \alpha \leq 1$, *if* $L \in Q$ *and* $\Phi \in \operatorname{Lip}\alpha$, $z \in \overline{\Omega}$.

We note that the class Q_α is sufficiently wide. A detailed account on it and the related topics are contained in [15, 21, 28] (see also the references cited therein). We consider only some cases:

***Remark* 2.1.** a) If $L = \partial G$ is a Dini-smooth curve [21, p. 48], then $L \in Q_1$.

b) If $L = \partial G$ is a piecewise Dini-smooth curve and largest esterior angle at L has opening $\alpha\pi$, $0 < \alpha \leq 1$ [21, p. 52], then $L \in Q_\alpha$.

c) If $L = \partial G$ is a smooth curve having continuous tangent line, then $L \in Q_\alpha$ for all $0 < \alpha < 1$.

d) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair $z_1,\ z_2 \in L$, if $s(z_1,z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L, there exists a constant $c > 1$ such that $s(z_1,z_2) \leq c\,|z_1 - z_2|$, then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{1}{2}\left(1 - \dfrac{1}{\pi}\arcsin\dfrac{1}{c}\right)^{-1}$ [28].

e) If L is "c-quasiconformal" (see, for example, [15]), then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{\pi}{2\left(\pi - \arcsin\dfrac{1}{c}\right)}$.

Also, if L is an asymptotic conformal curve, then $\Phi \in \operatorname{Lip}\alpha$ for all $0 < \alpha < 1$ [15].

Definition 2.3. *It is said that* $L \in \widetilde{Q}_\alpha$, $0 < \alpha \leq 1$, *if* $L \in Q_\alpha$ *and* L *is rectifiable.*

In this case we have the following theorem.

Theorem 2.2. *Let* $p > 0$. *Suppose that* $L \in \widetilde{Q}_\alpha$, *for some* $0 < \alpha \leq 1$ *and* $h(z)$ *defined as in* (1.1) *with* $\gamma_j = 0$, *for all* $j = \overline{1,m}$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_2 = c_2(L,p) > 0$ *such that*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_2 \|P_n\|_{\mathcal{L}_p(h_0,L)} \begin{cases} n^{\frac{1}{\alpha p}}, & \dfrac{1}{2} \leq \alpha \leq 1, \\ n^{\frac{\delta}{p}}, & 0 < \alpha < \dfrac{1}{2}, \end{cases} \tag{2.2}$$

where $\delta = \delta(L)$, $\delta \in [1,2]$, *is a certain number.*

Therefore, according to (2.1), we can calculate α in the right parts of estimation (2.2) for each case, respectively.

Now, we assume that the the weight function h have "singularities" at the points $\{z_i\}_{i=1}^{m}$, i.e., $\gamma_i \neq 0$ for all $i = \overline{1,m}$. In this case, we have the following "local" (at the point $z_j \in L$) estimations.

Theorem 2.3. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_{\alpha}$, for some $\frac{1}{2} \leq \alpha \leq 1$ and $h(z)$ defined as in* (1.1). *Then, for any $\gamma_i > -1$, $i = \overline{1,m}$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(L,p,\gamma_i,\alpha) > 0$ such that*

$$|P_n(z_i)| \leq c_3 n^{\frac{\gamma_i+1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)} . \tag{2.3}$$

Now, let's introduce "special" singular points on the curve L. Let us give the following definition. For $\delta > 0$ and $z \in \mathbb{C}$ let us set $B(z,\delta) := \{\zeta : \, |\zeta - z| < \delta\}$, $\Omega(z,\delta) := \Omega \cap \, B(z,\delta)$.

Definition 2.4. *We say that $L \in Q_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \leq \alpha \leq 1, i = \overline{1,m}$, if*

i) *for every sequence noncrossing in pairs circles $\{D(\zeta_i,\delta_i)\}_{i=1}^{m}$ restriction of the function Φ on $\Omega(\zeta_i,\delta_i)$ belongs to $\mathrm{Lip}\,\beta_i$ ($\Phi \,|\, \Omega(\zeta_i,\delta_i) \in \mathrm{Lip}\,\beta_i$), and restriction*

$$\Phi \left| \Omega \setminus \bigcup_{i=1}^{m} \Omega(\zeta_i,\delta_i) \in \mathrm{Lip}\ \alpha, \right.$$

ii) *there exists a sequence noncrossing in pairs circles $\{D(\zeta_i,\delta_i^*)\}_{i=1}^{m}$, such that for all $i = \overline{1,m}$, $\delta_i^* > \delta_i$ and $\xi, z \in \Omega(\zeta_i,\delta_i^*)$, $z \neq \zeta_i \neq \xi$, is fulfiled estimation*

$$|\Phi(z) - \Phi(\xi)| \leq k_i(z,\xi)\, |z - \xi|^{\alpha} , \tag{2.4}$$

where

$$k_i(z,\xi) = c_i \max\left(|\xi - \zeta_i|^{\beta_i - \alpha} ; |z - \zeta_i|^{\beta_i - \alpha} \right),$$

and c_i not depends on z and ξ.

Definition 2.5. *We say that $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, if $L \in Q_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \leq \alpha \leq 1, i = \overline{1,m}$, and $L = \partial G$ is rectifiable.*

It is clear from Definition 2.4 (2.5), that each region $L \in Q_{\alpha,\beta_1,\ldots,\beta_m}$, $0 < \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, may have "singularities" at the points $\{z_i\}_{i=1}^{m} \in L$. If a region L does not have such "singularities", i.e., if $\beta_i = \alpha$ for all $i = \overline{1,m}$, then it is written as $G \in Q_{\alpha}, 0 < \alpha \leq 1$.

Throughout this work, we will assume that the points $\{z_i\}_{i=1}^{m} \in L$ are defined in (1.1) and $\{\zeta_i\}_{i=1}^{m} \in L$ are defined in Definitions 2.4 and 2.5 coincides. Without the loss of generality, we also will assume that the points $\{z_i\}_{i=1}^{m}$ are ordered in the positive direction on the curve L.

We state our new results. Our first results is related to the simple cases. Namely, let the curve L and the weight function h has not the "singularities" at the points $\{z_i\}_{i=1}^{m}$, i.e., $\beta_i = \alpha$, and $\gamma_i = 0$ for all $i = \overline{1,m}$. In this case, we have the following "local" (at the singuler points) and "global" estimations.

Theorem 2.4. *Let $p > 0$. Suppose that $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, for some $\frac{1}{2} \leq \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, and $h(z)$ defined as in* (1.1). *Then, for any $\gamma_i > -1$, $i = \overline{1,m}$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_4 = c_4(L,p,\gamma_i,\alpha,\beta_i) > 0$ such that*

$$|P_n(z_i)| \leq c_4 n^{\frac{\gamma_i+1}{p\beta_i}} \|P_n\|_{\mathcal{L}_p(h,L)} \tag{2.5}$$

and

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_4 n^{\frac{\widetilde{\gamma}+1}{p\beta_i}} \|P_n\|_{\mathcal{L}_p(h,L)}, \tag{2.6}$$

where $\widetilde{\gamma} := \max\left\{0, \gamma_i,\ i = \overline{1,m}\right\}$.

Therefore, if contour L does not have any singular points, i.e., $\beta_i = \alpha$, for $i = \overline{1,m}$, then we have the Theorem 2.3.

Now, we conside the general case: assume that the curve L have "singularity" on the boundary points $\{z_i\}_{i=1}^m$, i.e., $\beta_i \neq \alpha$, for all $i = \overline{1,m}$, and the weight function h have "singularity" at the same points, i.e., $\gamma_i \neq 0$ for some $i = \overline{1,m}$. For simplesity, let us suppose $\gamma_i \neq 0$ for all $i = \overline{1,m}$. In this case, we have the following "global" estimations.

Theorem 2.5. *Let* $p > 0$. *Suppose that* $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, *for some* $\dfrac{1}{2} \leq \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, *and* $h(z)$ *defined as in* (1.1) *and*

$$\gamma_i + 1 = \frac{\beta_i}{\alpha}, \tag{2.7}$$

for each points $\{z_i\}_{i=1}^m$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_5 = c_5(L, p, \gamma_i, \alpha) > 0$ *such that*

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_5 n^{\frac{1}{\alpha p}} \|P_n\|_{\mathcal{L}_p(h,L)}. \tag{2.8}$$

Comparing Theorem 2.5 with Theorem 2.2, it is seen that, if the equality (2.7) is satisfied, then the growth rate of the polynomials $P_n(z)$ on L does not depend on whether the weight function $h(z)$ and the boundary contour L have singularity or not. The condition (2.7) is called the condition of "interference of singularities" of weight function h and contour L at the "singular" points $\{z_i\}_{i=1}^m$.

***Corollary* 2.1.** *If* $L \in C(1, \lambda, \nu)$, *then* $L \in \widetilde{Q}_{\alpha,\beta_1}$ *for* $\alpha = 1$ (2.1) *and* $\beta_1 = \dfrac{1}{\nu}$ [15]. *In this case, for* $p = 2$ *from* (2.7) *and* (2.8), *we have*

$$(\gamma_1 + 1)\,\nu_1 = 1,$$

$$\|P_n\|_{\mathcal{L}_\infty} \leq c_5 \sqrt{n}\, \|P_n\|_{\mathcal{L}_2(h,L)}. \tag{2.9}$$

For $P_n \equiv Q_n$, *the estimation* (2.9) *coincides from* (1.4). *Therefore, Theorem* 2.5 *is generalised the result in* [27] (Theorem 1).

2.1. Sharpness of estimates. The sharpness of the estimations (2.1), (2.2) and for some special cases can be discussed by comparing them with the following results:

***Remark* 2.2.** For any $n \in \mathbb{N}$, there exists polynomials $P_n^* \in \wp_n$, weight functions h^* and the constants $c_6 = c_6(L) > 0$ such that, for $L := \{z\colon\ |z| = 1\}$ we have

$$\|P_n^*\|_{C(L)} \geq c_6 n^{\frac{1}{p}} \|P_n^*\|_{\mathcal{L}_p(h^*,\ L)}.$$

3. Some auxiliary results. For $a > 0$ and $b > 0$, we shall use the notations "$a \preceq b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

The following definitions of the K-quasiconformal curves are well-known (see, for example, [9; 14, p. 97; 23]):

Definition 3.1. *The Jordan arc (or curve) L is called K-quasiconformal $(K \geq 1)$, if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).*

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and lets define

$$K_L := \inf \{K(f) \colon f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a K-quasiconformal curve, if $K_L \leq K$.

***Remark* 3.1.** It is well-known that, if we are not interested with the coefficients of quasiconformality of the curve, then the definitions of "quasicircle" and "quasiconformal curve" are identical. However, if we are also interested with the coefficients of quasiconformality of the given curve, then we will consider that if the curve L is K-quasiconformal, then it is κ-quasicircle with $\kappa = \dfrac{K^2-1}{K^2+1}$.

By Remark 3.1, for simplicity, we will use both terms, depending on the situation.

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$, we set

$$d(z, M) = \operatorname{dist}(z, M) := \inf \{|z - \zeta| : \ \zeta \in M\}.$$

Lemma 3.1 [1]. *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z \colon |z - z_1| \preceq \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent and similarly so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon < 1$, $c > 1$, $0 < r_0 < 1$ are constants, depending on G and $L_{r_0} := \{z = \psi(w) \colon |w| = r_0\}$.

Lemma 3.2. *Let $G \in Q(\kappa)$ for some $0 \leq \kappa < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa}$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from [20, p. 287] (Lemma 9.9) and the estimation for the Ψ' (see, for example, [10], Theorem 2.8)

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{3.1}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed the system of the points on L and the weight function $h(z)$ defined as (1.1).

Lemma 3.3 [6]. *Let L be a rectifiable Jordan curve, $h(z)$ defined as in* (1.1). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n + \frac{1+\gamma*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \ p > 0. \tag{3.2}$$

***Remark* 3.2.** In case of $h(z) \equiv 1$, the estimation (3.2) has been proved in [12].

4. Proof of theorems. ***4.1. Proof of Theorem 2.4.*** Suppose that $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, for some $\frac{1}{2} \le \beta_i \le \alpha \le 1$, $i = \overline{1,m}$, be given and $h(z)$ defined as in (1.1). Let $w = \varphi_R(z)$ be the univalent conformal mapping of G_R, $R > 1$, onto the B normalized by $\varphi_R(0) = 0$, $\varphi_R'(0) > 0$, and let $\{\zeta_j\}$, $1 \le j \le m \le n$, be a zeros of $P_n(z)$ lying on G_R. Let

$$B_{m,R}(z) := \prod_{j=1}^{m} B_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \overline{\varphi_R(\zeta_j)}\varphi_R(z)}, \tag{4.1}$$

denotes a Blashke function with respect to zeros $\{\zeta_j\}$, $1 \le j \le m \le n$, of $P_n(z)$ [29]. Clearly,

$$|B_{m,R}(z)| \equiv 1, \quad z \in L_R, \tag{4.2}$$

and

$$|B_{m,R}(z)| < 1, \quad z \in G_R. \tag{4.3}$$

For any $p > 0$ and $z \in G_R$ let us set

$$T_n(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2} . \tag{4.4}$$

The function $T_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_n(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i} \int_{L_R} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in G_R, \tag{4.5}$$

or

$$T_n(z_j) = \frac{1}{2\pi i} \int_{L_R} T_n(\zeta) \frac{d\zeta}{\zeta - z_j}.$$

Now, let $z \in L$. Multiplying the numerator and the determinator of the integrand by $h^{1/2}(\zeta)$, according to the Hölder inequality, from (4.2) and (4.3), we obtain

$$|P_n(z_j)| \le \left(\frac{1}{2\pi}\right)^{2/p} \left(\int_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta|\right)^{1/p} \times$$

$$\times \left(\int_{L_R} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{2+\gamma_j}}\right)^{1/p} =: \left(\frac{1}{2\pi}\right)^{2/p} I_{n,1} \times I_{n,2}, \tag{4.6}$$

where

$$I_{n,1} := \|P_n\|_{\mathcal{L}_p(h,L_R)}, \quad I_{n,2} := \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{2+\gamma_j}} \right)^{1/p}.$$

Then, by Lemma 3.3, for any points $\{z_j\}_{j=1}^m \in L$, we have

$$|P_n(z_j)| \preceq \|P_n\|_{\mathcal{L}_p} (I_{n,2})^{1/p}. \tag{4.7}$$

To estimate the integral $I_{n,2}$, we introduce

$$w_j := \Phi(z_j),\ \varphi_j := \arg w_j,\ L^j := L \cap \overline{\Omega}^j, \quad L_R^j := L_R \cap \overline{\Omega}^j, \quad j = \overline{1,m}, \tag{4.8}$$

where $\Omega^j := \Psi(\Delta_j^{'})$,

$$\Delta_1^{'} := \left\{ t = \mathrm{Re}^{i\theta} : R > 1,\ \frac{\varphi_m + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$
$$\Delta_m^{'} := \left\{ t = \mathrm{Re}^{i\theta} : R > 1,\ \frac{\varphi_{m-1} + \varphi_m}{2} \le \theta < \frac{\varphi_m + \varphi_1}{2} \right\}, \tag{4.9}$$

and, for $j = \overline{2, m-1}$,

$$\Delta_j^{'} := \left\{ t = \mathrm{Re}^{i\theta} : R > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\}. \tag{4.10}$$

Then, we get

$$(I_{n,2})^p = \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{2+\gamma_j}} \asymp \sum_{i=1}^{m} \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}} =: \sum_{i=1}^{m} I_{n,2}^i, \tag{4.11}$$

where

$$I_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{2+\gamma_i}}, \quad i = \overline{1,m}, \tag{4.12}$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. It remains to estimate the integrals $I_{n,2}^i$ for each $i = \overline{1,m}$. For simplicity of our next calculations, we assume that

$$m = 1, \quad R = 1 + \frac{\varepsilon_0}{n}. \tag{4.13}$$

Let the numbers $\delta_{1,}\ \delta_1^*,\ 0 < \delta_1 < \delta_1^* < \delta_0 < \mathrm{diam}\overline{G},$ are choosen from Definition 2.4. By denoted

$$l_{R,1}^1 := L_R^1 \cap \Omega(z_1, \delta_1), \quad l_{R,2}^1 := L_R^1 \backslash l_{R,1}^1, \quad F_{R,i}^1 := \Phi(l_{R,i}^1), \quad i = 1, 2,$$

we get

$$I_{n,2}^1 := \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{l_{R,1}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} + \int\limits_{l_{R,2}^1} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}}. \tag{4.14}$$

By applying the Lemma 3.1, we have

$$\int\limits_{l^1_{R,1}} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} = \int\limits_{\Phi(l^1_{R,1})} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{2+\gamma_1}\,(|\tau| - 1)} \preceq$$

$$\preceq \int\limits_{\Phi(l^1_{R,1})} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{1+\gamma_1}\,(|\tau| - 1)} \preceq$$

$$\preceq n \int\limits_{\Phi(l^1_{R,1})} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma_1+1}{\beta_1}}} \preceq n^{\frac{\gamma_1+1}{\beta_1}}, \tag{4.15}$$

$$\int\limits_{l^1_{R,2}} \frac{|d\zeta|}{|\zeta - z_1|^{2+\gamma_1}} \preceq (\delta_1)^{2+\gamma_1} \operatorname{mes} l^1_{R,1} \preceq 1. \tag{4.16}$$

Then, from (4.14), we get

$$I^1_{n,2} \preceq n^{\frac{\gamma_1+1}{\beta_1}}. \tag{4.17}$$

By combining the relations (4.7) – (4.17), we obtain

$$|P_n(z_1)| \preceq n^{\frac{\gamma_1+1}{p\beta_1}} \, \|P_n\|_{\mathcal{L}_p},$$

and, according to (4.13), we completed the proof.

4.2. Proof of Theorem 2.5. Suppose that $L \in \widetilde{Q}_{\alpha,\beta_1,\ldots,\beta_m}$, for some $\frac{1}{2} \leq \beta_i \leq \alpha \leq 1$, $i = \overline{1,m}$, be given and $h(z)$ defined as in (1.1). Let $w = \varphi_R(z)$, $B_{m,R}(z)$ and $T_n(z)$ be defined as in begining to proof of the Theorem 2.4 by (4.1) and (4.4). Then Cauchy integral representation for the $T_n(z)$ in G_R gives

$$T_n(z) = \frac{1}{2\pi i} \int\limits_{L_R} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in G_R, \tag{4.18}$$

or

$$\left| \left[\frac{P_n(z)}{B_{m,R}(z)} \right]^{p/2} \right| \leq \frac{1}{2\pi} \int\limits_{L_R} \left| \frac{P_n(\zeta)}{B_{m,R}(\zeta)} \right|^{p/2} \frac{|d\zeta|}{|\zeta - z|} \leq \int\limits_{L_R} |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z|},$$

since $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L_R$. Lets now $z \in L$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, by the Hölder inequality, we obtain

$$\left| \frac{P_n(z)}{B_{m,R}(z)} \right|^{p/2} \leq \frac{1}{2\pi} \left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta| \right)^{1/2} \times$$

$$\times \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod_{j=1}^m |\zeta - z_j|^{\gamma_j}\,|\zeta - z|^2} \right)^{1/2} =: \frac{1}{2\pi} J_{n,1} \times J_{n,2}, \tag{4.19}$$

where

$$J_{n,1}:=\left(\int\limits_{L_R} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/2},\quad J_{n,2}:=\left(\int\limits_{L_R}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\right)^{1/2}.$$

Then, since $|B_{m,R}(z)|<1,$ for $z\in L,$ from Lemma 3.3, we have

$$|P_n(z)|\preceq (J_{n,1}J_{n,2})^{2/p}\preceq \|P_n\|_p\,(J_{n,2})^{2/p},\quad z\in L. \tag{4.20}$$

The integral $J_{n,2}$ we estimate analogous to the integral $I_{n,2}$. By using designations (4.8) – (4.10), we obtain

$$(J_{n,2})^2=\sum_{i=1}^m\int\limits_{L_R^i}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}\,|\zeta-z|^2}\asymp\sum_{i=1}^m\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2}=:\sum_{i=1}^m J_{n,2}^i, \tag{4.21}$$

where

$$J_{n,2}^i:=\int\limits_{L_R^i}\frac{|d\zeta|}{|\zeta-z_i|^{\gamma_i}\,|\zeta-z|^2},\quad i=\overline{1,m}, \tag{4.22}$$

since the points $\{z_j\}_{j=1}^m\in L$ are distinct. It remains to estimate the integrals $J_{n,2}^i$ for each $i=\overline{1,m}.$ As we have assumed in (4.13) for simplicity of calculations, here, we also assume that

$$m=1,\quad R=1+\frac{\varepsilon_0}{n}. \tag{4.23}$$

Let the numbers $\delta_1,\ \delta_1^*,\ 0<\delta_1<\delta_1^*<\delta_0<\operatorname{diam}\overline{G},$ are choosen from Definition 2.4. We denote

$$L_{R,1}^1:=L_R^1\cap\Omega(z_1,\delta_1),$$
$$L_{R,2}^1:=L_R^1\cap(\Omega(z_1,\delta_1^*)\backslash\Omega(z_1,\delta_1)),$$
$$L_{R,3}^1:=L_R\backslash(L_{R,1}^1\cup L_{R,2}^1),\qquad F_{R,i}^1:=\Phi(L_{R,i}^1),$$
$$L_1^1:=L^1\cap D(z_1,\delta_1),$$
$$L_2^1:=L^1\cap(D(z_1,\delta_1^*)\backslash D(z_1,\delta_1)),$$
$$L_3^1:=L\backslash(L_1^1\cup L_2^1),\quad F_i^1:=\Phi(L_i^1),\quad i=1,2,3.$$

By taking into consideration these designations and by replacing the variable $\tau=\Phi(\zeta),$ from (3.1) and (4.12), we have

$$J_{n,2}^1\asymp\sum_{i=1}^3\int\limits_{F_{R,i}^1}\frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2}\asymp$$

$$\asymp\sum_{i=1}^3\int\limits_{F_{R,i}^1}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)}=:\sum_{i=1}^3 J(F_{R,i}^1). \tag{4.24}$$

So, we need to evaluate the integrals $J(F^1_{R,i})$ for each $i=1,2,3$. Therefore, we will continue in the following manner. Let

$$\|P_n\|_\infty =: \left|P_n\left(z'\right)\right|, \; z'\in L, \tag{4.25}$$

and $w' = \Phi(z')$. There are two possible cases: the point z' may lie on L^1 or L^2.

1) Suppose first that $z'\in L^1$. If $z'\in L^1_i$, then $w'\in F^1_i$ for $i=1,2,3$. Let's $F^{1,1}_{R,j} := \left\{\tau\in F^1_{R,j}\colon |\tau-w_1|\ge|\tau-w'|\right\}$, $F^{1,2}_{R,j} := F^1_{R,j}\backslash F^{1,1}_{R,j}$, $j=1,2$. Consider the individual cases.

1.1) Let $z'\in L^1_1$. Applying Lemma 3.1, we have

$$J(F^1_{R,1}) = \int\limits_{F^1_{R,1}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \preceq$$

$$\preceq n\int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{\gamma_1+1}} + n\int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1+1}} \preceq$$

$$\preceq n\int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau-w'|^{\frac{\gamma_1+1}{\beta}-\frac{1}{\alpha}}\,|\tau-w'|^{\frac{1}{\alpha}}} + n\int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau-w_1|^{\frac{\gamma_1+1}{\beta}-\frac{1}{\alpha}}\,|\tau-w_1|^{\frac{1}{\alpha}}} \preceq$$

$$\preceq n\int\limits_{F^{1,1}_{R,1}} \frac{|d\tau|}{|\tau-w'|^{\frac{1}{\alpha}}} + n\int\limits_{F^{1,2}_{R,1}} \frac{|d\tau|}{|\tau-w_1|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}, \tag{4.26}$$

$$J(F^1_{R,2}) = \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \preceq$$

$$\preceq \int\limits_{F^1_{R,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1\ge 0,\\ \left(2\mathrm{diam}\overline{G}\right)^{-\gamma_1}, & -1<\gamma_1<0, \end{cases} \preceq$$

$$\preceq \int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|\,(|\tau|-1)}. \tag{4.27}$$

Setting in (2.4) $z:=z'$, $\xi\colon \zeta=\Psi(\tau)$ and according to $|\zeta-z_1|>|z'-z_1|$, we obtain

$$\left|\zeta-z'\right|^\alpha \succeq \max\left\{|\zeta-z_1|^{\alpha-\beta_1};\; \left|z'-z_1\right|^{\alpha-\beta_1}\right\}\left|w'-\tau\right| =$$

$$= |\zeta-z_1|^{\alpha-\beta_1}\left|w'-\tau\right| \ge \delta_1^{\alpha-\beta_1}\left|w'-\tau\right| \succeq \left|w'-\tau\right|. \tag{4.28}$$

Then, from (4.27), we get

$$\begin{aligned} J(F^1_{R,2}) &\preceq n\int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|} \preceq n\int\limits_{F^1_{R,2}} \frac{|d\tau|}{|\tau-w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}},\\ J(F^1_{R,3}) &= \int\limits_{F^1_{R,3}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1}\,|\Psi(\tau)-\Psi(w')|^2\,(|\tau|-1)} \preceq \end{aligned} \tag{4.29}$$

$$\preceq \frac{n}{\delta_1^* - \delta_1} \int\limits_{F_{R,3}^1} |d\tau| \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2\mathrm{diam}\overline{G}\right)^{-\gamma_1}, & -1 < \gamma_1 < 0, \end{cases} \preceq n.$$

1.2) Let $z' \in L_2^1$. Analogously to case 1.1, we have

$$J(F_{R,1}^1) = \int\limits_{F_{R,1}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1} |\Psi(\tau) - \Psi(w')|^2 (|\tau| - 1)} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^{1,1}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{\gamma_1 + 1}} + n \int\limits_{F_{R,2}^{1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1 + 1}} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^{1,1}} \frac{|d\tau|}{|\tau - w'|^{\frac{\gamma_1+1}{\beta} - \frac{1}{\alpha}} |\tau - w'|^{\frac{1}{\alpha}}} + n \int\limits_{F_{R,2}^{1,2}} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma_1+1}{\beta} - \frac{1}{\alpha}} |\tau - w_1|^{\frac{1}{\alpha}}} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^{1,1}} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} + n \int\limits_{F_{R,2}^{1,2}} \frac{|d\tau|}{|\tau - w_1|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}. \tag{4.30}$$

For $z' \in L_2^1$, applying (2.4), we see that

$$\left|\zeta - z'\right|^{\alpha} \succeq \max\left\{|\zeta - z_1|^{\alpha - \beta_1};\ \left|z' - z_1\right|^{\alpha - \beta_1}\right\} \left|w' - \tau\right| \geq$$

$$\geq \delta_1^{\alpha - \beta_1} \left|w' - \tau\right| \succeq \left|w' - \tau\right|,$$

and, consequently, we obtain

$$J(F_{R,2}^1) = \int\limits_{F_{R,2}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1} |\Psi(\tau) - \Psi(w')|^2 (|\tau| - 1)} \preceq$$

$$\preceq \int\limits_{F_{R,2}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w')|^2 (|\tau| - 1)} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2\mathrm{diam}\overline{G}\right)^{-\gamma_1}, & -1 < \gamma_1 < 0, \end{cases} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \preceq n \int\limits_{F_{R,2}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}, \tag{4.31}$$

$$J(F_{R,3}^1) = \int\limits_{F_{R,3}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1} |\Psi(\tau) - \Psi(w')|^2 (|\tau| - 1)} \preceq$$

$$\preceq \frac{n}{(\delta_1^*)^{\alpha - \beta_1}} \int\limits_{F_{R,3}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2\mathrm{diam}\overline{G}\right)^{-\gamma_1}, & -1 < \gamma_1 < 0, \end{cases} \preceq$$

$$\preceq n \int\limits_{F_{R,3}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}. \tag{4.32}$$

1.3) Let $z' \in L_3^1$. In this situation for the integral $J(F_{R,1}^1)$, we get

$$J(F_{R,1}^1) = \int\limits_{F_{R,1}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}\, |\Psi(\tau) - \Psi(w')|^2\, (|\tau| - 1)} \preceq$$

$$\preceq n \int\limits_{F_{R,1}^1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1 + 1}} \preceq n \int\limits_{F_{R,1}^1} \frac{|d\tau|}{|\tau - w_1|^{\frac{\gamma_1+1}{\beta_1}}} \preceq$$

$$\preceq n.n^{\frac{\gamma_1+1}{\beta_1} - 1} = n^{\frac{1}{\alpha}}. \tag{4.33}$$

Applying (2.4), we see that

$$\left|\zeta - z'\right|^{\alpha} \succeq \max\left\{|\zeta - z_1|^{\alpha - \beta_1};\; \left|z' - z_1\right|^{\alpha-\beta_1}\right\} \left|w' - \tau\right| \geq$$

$$\geq \delta_1^{\alpha - \beta_1} \left|w' - \tau\right| \succeq \left|w' - \tau\right|,$$

so, we get

$$J(F_{R,2}^1) = \int\limits_{F_{R,2}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}\, |\Psi(\tau) - \Psi(w')|^2\, (|\tau| - 1)} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^1} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|} \begin{cases} (\delta_1)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2\,\mathrm{diam}\overline{G}\right)^{-\gamma_1}, & -1 < \gamma_1 < 0, \end{cases} \preceq$$

$$\preceq n \int\limits_{F_{R,2}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}} \tag{4.34}$$

and

$$J(F_{R,3}^1) = \int\limits_{F_{R,3}^1} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1}\, |\Psi(\tau) - \Psi(w')|^2\, (|\tau| - 1)} \preceq$$

$$\preceq \frac{n}{(\delta_1^*)^{\alpha - \beta_1}} \int\limits_{F_{R,3}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \begin{cases} (\delta_1^*)^{-\gamma_1}, & \gamma_1 \geq 0, \\ \left(2\,\mathrm{diam}\,\overline{G}\right)^{-\gamma_1}, & -1 < \gamma_1 < 0, \end{cases} \preceq$$

$$\preceq n \int\limits_{F_{R,3}^1} \frac{|d\tau|}{|\tau - w'|^{\frac{1}{\alpha}}} \preceq n^{\frac{1}{\alpha}}. \tag{4.35}$$

By the relations (4.24) – (4.35), we obtain

$$J_{n,2}^1 \preceq n^{\frac{1}{\alpha}}. \tag{4.36}$$

Therefore, in case of $z' \in L^1$ for each $\gamma_1 > -1$ and for all $z \in L$, from (4.7), (4.11) and (4.36) we get

$$|P_n(z)| \preceq n^{\frac{1}{p\alpha}} \|P_n\|_p . \tag{4.37}$$

Theorem 2.5 is proved.

4.3. ***Proof of Remark*** **2.2.** a) Let $L := \{z\colon |z| = 1\}$, $h^*(z) \equiv 1$ and $P_n^*(z) = \sum_{j=1}^n z^j$. Then $L \in \widetilde{Q}_1$, $|P_n^*(z)| \le \sum_{j=1}^n |z|^j = n$, $|z| = 1$. On the other hand, $|P_n^*(1)| = n$. Therefore, $\|P_n^*\|_{\mathcal{L}_\infty} = n$. $\|P_n^*\|_{\mathcal{L}_2(1,L)} = \sqrt{2\pi n}$. Then

$$\|P_n^*\|_{\mathcal{L}_\infty} = n = \frac{\sqrt{n}}{\sqrt{2\pi}} \|P_n^*\|_{\mathcal{L}_2(1,L)} \ge \frac{1}{\sqrt{2\pi}} \sqrt{n} \|P_n^*\|_{\mathcal{L}_2(1,L)} .$$

Theorem 2.1 follows from [8] (Theorem 2.2). Theorem 2.3 is obtained from Theorem 2.4 for the case $\beta_i = \alpha$, $i = \overline{1,m}$. For $\frac{1}{2} < \alpha \le 1$, the Theorem 2.2 we get from the Theorem 2.3 in the case $\gamma_i = 0$, $i = \overline{1,m}$. Case of $\alpha = \frac{1}{2}$ follows by using the estimation $d(z, L_R) \succeq n^{-2}$, where is true for arbitrary continuum with connected complement (see, for example, [10], Corollary 2.7).

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