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HALF INTEGER VALUES OF ORDER-TWO HARMONIC NUMBERS SUMS

НАПІВЦІЛІ ЗНАЧЕННЯ СУМ ГАРМОНІЧНИХ ЧИСЕЛ ДРУГОГО ПОРЯДКУ

Half integer values of harmonic numbers and reciprocal binomial coefficients sums are investigated in this paper. Closed-form representations and integral expressions are developed for the infinite series.

Вивчаються напівцілі значення сум гармонічних чисел та обернених біноміальних коефіцієнтів. Для нескінченних рядів отримано зображення в замкненій формі та інтегральні вирази.

1. Introduction and preliminaries. It is known that the harmonic number H_n has the usual definition

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \sum_{j=1}^{\infty} \frac{n}{j(j+n)} = \int_0^1 \frac{1-x^n}{1-x}\, dx \quad (H_0 := 0) \tag{1.1}$$

for $n \in \mathbb{N}$ where $\mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An unusual, but intriguing representation has recently been given by Ciaurri et al. [5], as

$$H_n = \pi \int_0^1 \left(x - \frac{1}{2}\right) \left(\frac{\cos\left(\dfrac{(4n+1)\,\pi x}{2}\right) - \cos\left(\dfrac{\pi x}{2}\right)}{\sin\left(\dfrac{\pi x}{2}\right)} \right) dx.$$

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers. We define harmonic numbers at half integer values as $H_{n-\frac{1}{2}}$, which may be expressed in terms of the digamma (or Psi) function $\psi(z),\ z \in \mathbb{R}$, and the Euler–Mascheroni constant, γ as $H_{n-\frac{1}{2}} = \gamma + \psi\left(n + \frac{1}{2}\right)$. The digamma function is defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\, dt.$$

The Lerch transcendent $\Phi(z, t, a) = \sum_{m=0}^{\infty} \dfrac{z^m}{(m+a)^t}$ is defined for $|z| < 1$ and $\Re(a) > 0$ and satisfies the recurrence

$$\Phi(z, t, a) = z\ \Phi(z, t, a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function, $\zeta(t, a)$ at $z = 1$,

$$\Phi(1, t, a) = \zeta(t, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the polylogarithm, or de Jonquière’s function, when $a = 1$,

$$Li_t\left(z\right) := \sum_{m=1}^{\infty} \frac{z^m}{m^t}, \quad t \in \mathbb{C}, \qquad \text{when} \quad |z| < 1; \qquad \Re\left(t\right) > 1 \quad \text{when} \quad |z| = 1.$$

Moreover

$$\int_0^1 \frac{Li_t\left(px\right)}{x} dx = \begin{cases} \zeta(1+t) & \text{for} \quad p = 1, \\ \left(2^{-r} - 1\right)\zeta(1+t) & \text{for} \quad p = -1, \end{cases}$$

where $\zeta(s)$ denotes the Riemann zeta function defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \Re(s) > 1.$$

A generalized binomial coefficient $\binom{\lambda}{\mu}$, $\lambda, \mu \in \mathbb{C}$, is defined, in terms of the familiar (Euler's) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\mu+1\right)\Gamma\left(\lambda-\mu+1\right)}, \quad \lambda, \mu \in \mathbb{C},$$

which, in the special case when $\mu = n$, $n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \qquad \text{and} \qquad \binom{\lambda}{n} := \frac{\lambda\left(\lambda-1\right)\ldots\left(\lambda-n+1\right)}{n!} = \frac{(-1)^n\left(-\lambda\right)_n}{n!}, \quad n \in \mathbb{N},$$

where $(\lambda)_\nu$, $\lambda, \nu \in \mathbb{C}$, is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0, \quad \lambda \in \mathbb{C} \setminus \{0\}, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \nu = n \in \mathbb{N}, \quad \lambda \in \mathbb{C}, \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed that the Γ-quotient exists. A *generalized harmonic number* $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m}, \quad m, n \in \mathbb{N}, \qquad \text{and} \qquad H_0^{(m)} := 0, \quad m \in \mathbb{N},$$

and

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\}, \quad n \in \mathbb{N}_0.$$

In the case of non integer values of the argument $z = \dfrac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta\left(\alpha+1\right) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{q}+1\right), \qquad \frac{r}{q} \neq \{-1, -2, -3, \ldots\},$$

where $\zeta\left(z\right)$ is the zeta function. We also define

$$H_{\frac{r}{q}} = H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q}+1\right).$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [7], or Choi and Cvijovic [2] in terms of the polylogarithmic or other special functions. Some specific values are given as

$$H^{(2)}_{\frac{3}{2}} = \frac{40}{9} - 2\zeta(2), \qquad H^{(2)}_{\frac{1}{2}} = 4 - 2\zeta(2), \qquad H^{(3)}_{\frac{3}{2}} = \frac{224}{27} - 6\zeta(3).$$

Many others are listed in the books [13, 19, 20]. In this paper we will develop identities, closed form representations of alternating half integer harmonic numbers and reciprocal binomial coefficients of the form

$$W_k(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n^p \binom{n+k}{k}} \tag{1.2}$$

for $p = 0$ and 1. While there are many results for sums of harmonic numbers with positive terms, see, for example, [1, 3, 4, 6, 8–12, 14, 16–18, 21–23] and references therein. There are fewer results for sums of the type (1.2).

The following lemma will be useful in the development of the main theorems.

Lemma 1.1. *Let r be a positive integer. Then for $p \in \mathbb{N}$*

$$\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p}\left(H^{(p)}_{\left[\frac{r}{2}\right]} + H^{(p)}_{\left[\frac{r-1}{2}\right]}\right) - H^{(p)}_{2\left[\frac{r+1}{2}\right]-1}. \tag{1.3}$$

For $p = 1$

$$\sum_{j=1}^{r} \frac{(-1)^j}{j} = H_{\left[\frac{r}{2}\right]} - H_r$$

where $[x]$ is the integer part of x. We also have the known results, for $0 < t \le 1$

$$\ln^2(1+t) = 2\sum_{n=1}^{\infty} \frac{(-t)^{n+1} H_n}{n+1}$$

and when $t = 1$

$$\ln^2 2 = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+1} = \zeta(2) - 2Li_2\left(\frac{1}{2}\right), \tag{1.4}$$

$$t\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-t)^{n+1}}{n},$$

hence

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{2^n}, \tag{1.5}$$

$$B(0) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n} = 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2). \tag{1.6}$$

Proof. The proof of (1.3) is given in the paper [15].

Firstly, (1.4) and (1.5) are standard known results. Next from the definition, for $p \in \mathbb{N}_0$

$$H_n^{(p+1)} = \frac{(-1)^p}{p!} \int\limits_0^1 \frac{\ln^p x}{1-x} (1 - x^n) \, dx \tag{1.7}$$

we can write

$$B(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(2)}}{n} = \int\limits_0^1 \frac{\log(x)}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(1 - x^{n-\frac{1}{2}}\right)}{n} dx =$$

$$= \int\limits_0^1 \frac{\log(x)}{1-x} \left(\frac{\ln(1+x)}{\sqrt{x}} - \ln 2 \right) dx =$$

$$= 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2),$$

where $G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approxeq 0.91596$ is Catalan's constant.

Lemma 1.1 is proved.

Lemma 1.2. *Let r be a positive integer, then we have the recurrence relation*

$$B(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(2)}}{n+r} = -B(r-1) - \frac{2\pi}{(2r-1)^2} + \frac{8G}{2r-1} - \frac{2\zeta(2)}{r} -$$

$$- \frac{4(-1)^r}{(2r-1)^2} \left(\ln 2 + H_{\left[\frac{r-1}{2}\right]} - H_{r-1} \right) \tag{1.8}$$

with solution

$$B(r) = (-1)^r B(0) - 2(-1)^r \zeta(2) \left(H_{\left[\frac{r}{2}\right]} - H_r \right) -$$

$$-(-1)^r \ln 2 \left(2\zeta(2) + H_{r-\frac{1}{2}}^{(2)} \right) +$$

$$+8(-1)^r G \sum_{j=1}^{r} \frac{(-1)^j}{2j-1} - 2(-1)^r \pi \sum_{j=1}^{r} \frac{(-1)^j}{(2j-1)^2} -$$

$$-4(-1)^r \sum_{j=1}^{r} \frac{\left(H_{\left[\frac{j-1}{2}\right]} - H_{j-1} \right)}{(2j-1)^2} \tag{1.9}$$

and $B(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(2)}}{n} = 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2).$

Proof. By a change of counter

$$B(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}^{(2)}}{n+r} = \sum_{n=2}^{\infty} \frac{(-1)^n H_{n-\frac{3}{2}}^{(2)}}{n+r-1} =$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{n+r-1} \left(H^{(2)}_{n-\frac{1}{2}} - \left(\frac{2}{2n-1} \right)^2 \right) =$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n+r-1} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{(2n-1)^2\,(n+r-1)} + \frac{H^{(2)}_{\frac{1}{2}}}{r} - \frac{4}{r} =$$

$$= -B\,(r-1) + \frac{1}{r}\left(H^{(2)}_{\frac{1}{2}} - 4 \right) + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)^2} \begin{pmatrix} \dfrac{2\,(2r-1)}{(2n-1)^2} \\ -\dfrac{2}{2n-1} + \dfrac{1}{n+r-1} \end{pmatrix} =$$

$$= -B(r-1) - \frac{2\zeta(2)}{r} - \frac{\pi}{2(2r-1)^2} + \frac{8G}{2r-1} +$$

$$+ 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)^2 (n+r-1)}.$$

From Lemma 1.1 and using the known results

$$B(r) = -B\,(r-1) - \frac{2\zeta(2)}{r} - \frac{\pi}{2\,(2r-1)^2} + \frac{8G}{2r-1} +$$

$$+ \frac{4(-1)^r}{(2r-1)^2} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n} \right)$$

which, for $r \geq 1$, results in the recurrence relation

$$B(r) + B\,(r-1) = -\frac{2\pi}{(2r-1)^2} + \frac{8G}{2r-1} - \frac{2\zeta(2)}{r} -$$

$$- \frac{4(-1)^r}{(2r-1)^2} \left(\ln 2 + H_{\left[\frac{r-1}{2}\right]} - H_{r-1} \right).$$

By the subsequent reduction of the $B(r),\ B(r-1),\ B(r-2), \ldots, B(1)$ terms in (1.8), we arrive at the identity (1.9).

Lemma 1.2 is proved.

***Example* 1.1.**

$$B\,(5) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n+5} = -\frac{184939}{297675} - \frac{182738\pi}{99225} - \frac{47}{30}\zeta(2) - 7\zeta(3) +$$

$$+ \frac{469876 \ln 2}{99225} + 2\ln 2\zeta(2) + 2\pi G + \frac{2104G}{315}.$$

It is of some interest to note that $B(r)$ may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next lemma.

Lemma 1.3. *For $r \in \mathbb{N}$, we have the identity*

$$V(r) = \sum_{n=1}^{\infty} \frac{H^{(2)}_{2n-\frac{1}{2}}}{(2n+r)\,(2n+r-1)} =$$

$$= B(r) + \frac{1}{2r-1}\left(3\zeta(2) - 4G\right) + \frac{1}{(2r-1)^2}\left(\pi - 6\ln 2 - 2H_{\frac{r-1}{2}}\right).$$

For $r = 0$

$$V(0) = \sum_{n=1}^{\infty} \frac{H^{(2)}_{2n-\frac{1}{2}}}{2n(2n-1)} = B(0) + 4G - 3\zeta\left(2\right) + \pi - 6\ln 2 - 2H_{-\frac{1}{2}} =$$

$$= \int_0^1 \frac{\log x}{1-x}\left(\ln 2 - \sqrt{x}\tanh^{-1}\left(x\right) - \frac{\ln\left(1-x^2\right)}{2\sqrt{x}}\right)dx =$$

$$= 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2) + 4G - 3\zeta(2) + \pi - 2\ln 2.$$

Proof. By expansion

$$B(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n+r} = \sum_{n=1}^{\infty}\left(\frac{H^{(2)}_{2n-\frac{3}{2}}}{2n+r-1} - \frac{H^{(2)}_{2n-\frac{1}{2}}}{2n+r}\right) =$$

$$= \sum_{n=1}^{\infty}\left(\frac{H^{(2)}_{2n-\frac{1}{2}}}{\left(2n+r-1\right)\left(2n+r\right)} - \frac{4}{\left(4n-1\right)^2\left(2n+r-1\right)}\right),$$

and by rearrangement

$$\sum_{n=1}^{\infty} \frac{H^{(2)}_{2n-\frac{1}{2}}}{\left(2n+r-1\right)\left(2n+r\right)} = B(r) + \sum_{n=1}^{\infty} \frac{4}{\left(4n-1\right)^2\left(2n+r-1\right)} =$$

$$= B(r) + \frac{1}{2r-1}\left(3\zeta(2) - 4G\right) + \frac{1}{(2r-1)^2}\left(\pi - 6\ln 2 - 2H_{\frac{r-1}{2}}\right),$$

where $B(r)$ is given by (1.9).

From (1.1) we can write

$$\sum_{n=1}^{\infty} \frac{H^{(2)}_{2n-\frac{1}{2}}}{2n\left(2n-1\right)} = -\int_0^1 \frac{\log x}{1-x} \sum_{n=1}^{\infty} \frac{\left(1 - x^{2n-\frac{1}{2}}\right)}{2n(2n-1)}\,dx =$$

$$= \int_0^1 \frac{\log x}{1-x}\left(\sqrt{x}\tanh^{-1}\left(x\right) + \frac{\ln\left(1-x^2\right)}{2\sqrt{x}} - \ln 2\right)dx =$$

$$= 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2) + 4G - 3\zeta(2) + \pi - 2\ln 2 =$$

$$= B(0) + 4G - 3\zeta(2) + \pi - 6\ln 2 - 2H_{-\frac{1}{2}}.$$

Lemma 1.3 is proved.

***Remark* 1.1.** Since we have the relation

$$H^{(2)}_{n-\frac{1}{2}} = 4H^{(2)}_{2n} - H^{(2)}_{n} - 2\zeta(2)$$

it is possible to obtain identities for $p = 0, 1$ and $k \geq 1$

$$X_k\left(p\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{2n}}{n^p \binom{n+k}{k}}.$$

The next three theorems relate the main results of this investigation, namely the closed form and integral representation of (1.2).

2. Closed form and integral identities. We now prove the following theorems.

Theorem 2.1. *Let k be real positive integer. Then from* (1.2) *with $p = 0$ we have*

$$W_k(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{\binom{n+k}{k}} = \sum_{r=1}^{k} (-1)^{1+r} r \binom{k}{r} B(r), \tag{2.1}$$

where $B(r)$ is given by (1.9)*.*

Proof. Consider the expansion

$$W_k(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} k! H^{(2)}_{n-\frac{1}{2}}}{(n+1)_k} =$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} k! H^{(2)}_{n-\frac{1}{2}} \sum_{r=1}^{k} \frac{M_r}{n+r},$$

where

$$M_r = \lim_{n \to -r} \left\{ \frac{n+r}{\prod_{r=1}^{k} n+r} \right\} = \frac{(-1)^{r+1} r}{k!} \binom{k}{r}.$$

Hence,

$$W_k(0) = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n+r} =$$

$$= \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} B(r).$$

Theorem 2.1 is proved.

The other case of $W_k(1)$ can be evaluated in a similar fashion. We list the results in the next corollary.

***Corollary* 2.1.** *Under the assumptions of Theorem* 2.1, *we have*

$$W_k(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{n \binom{n+k}{k}} = B(0) - \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} B(r) =$$

$$= 7\zeta(3) - 2\pi G - 2\ln 2\zeta(2) - \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} B(r). \tag{2.2}$$

Proof. The proof follows directly from Theorem 2.1 and using the same technique.

It is possible to represent the alternating harmonic number sums (2.1), (2.2) and (1.8) in terms of an integral, which is developed in the next theorem.

Theorem 2.2. *Let* k *be a positive integer. Then we have*

$$\frac{k}{1+k} \int_0^1 \frac{\sqrt{x}\ \log x}{(1-x)(1+x)} {}_2F_1 \left[\begin{matrix} 1,1 \\ 2+k \end{matrix} \middle| -x \right] dx = \tag{2.3}$$

$$= 2G - \zeta(2) - \frac{k\zeta(2)}{2(1+k)} {}_2F_1 \left[\begin{matrix} 1,1 \\ 2+k \end{matrix} \middle| -1 \right] - W_k(0), \tag{2.4}$$

where $W_k(0)$ *is given by* (2.1) *and* ${}_2F_1 \left[\begin{matrix} \cdot,\cdot \\ \cdot \end{matrix} \middle| z \right]$ *is the classical Gauss hypergeometric function.*

Proof. From (1.7)

$$H_n^{(2)} = -\int_0^1 \frac{\ln x}{1-x} (1-x^n)\, dx$$

we can therefore write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{\binom{n+k}{k}} = -\int_0^1 \frac{\log x}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(1 - x^{n-\frac{1}{2}}\right)}{\binom{n+k}{k}} dx =$$

$$= \sum_{r=1}^{k} (-1)^{1+r} r \binom{k}{r} B(r) = W_k(0),$$

hence,

$$\int_0^1 \frac{1}{1-x} \left(\begin{matrix} \dfrac{k}{2(1+k)} {}_2F_1 \left[\begin{matrix} 1,1 \\ 2+k \end{matrix} \middle| -1 \right] - \dfrac{(1-\sqrt{x})^2}{2(1+x)} \\ -\dfrac{k\sqrt{x}}{(1+k)(1+x)} {}_2F_1 \left[\begin{matrix} 1,1 \\ 2+k \end{matrix} \middle| -x \right] \end{matrix} \right) dx = \sum_{r=1}^{k} (-1)^{1+r} r \binom{k}{r} B(r).$$

Therefore, (2.3) and (2.4) follows.

Theorem 2.2 is proved.

A similar integral representation can be evaluated for $W_k(1)$ and $B(r)$, the results is recorded in the next theorem.

Theorem 2.3. *Let the conditions of Theorem 2.2 hold. Then we have*

$$\frac{1}{1+k}\int_0^1 \frac{\sqrt{x}\log x}{1-x}\,{}_2F_1\left[\begin{matrix}1,1\\2+k\end{matrix}\,\middle|\,-x\right]dx =$$

$$= W_k(1) - \frac{\zeta(2)}{1+k}\,{}_2F_1\left[\begin{matrix}1,1\\2+k\end{matrix}\,\middle|\,-1\right]. \tag{2.5}$$

Also for $B(r)$

$$\int_0^1 \frac{\sqrt{x}\log x}{1-x}\Phi(-x,1,1+r)\,dx = B(r) - \frac{\zeta(2)}{2}\left(H_{\frac{r}{2}} - H_{\frac{r-1}{2}}\right), \tag{2.6}$$

where $\Phi(-x,1,1+r)$ *is the Lerch transcendent.*

Proof. The proof follows the same pattern as that employed in Theorem 2.2.

***Example* 2.1.** From (2.6), for $r=2$,

$$\int_0^1 \frac{\sqrt{x}\ln x}{1-x}\Phi\left(-x,1,3\right)dx = B(2) - \frac{\zeta(2)}{2}\left(2\ln 2 - 1\right),$$

which reduces to

$$\lim_{\epsilon\longrightarrow 0}\int_\epsilon^1 \frac{\ln x}{x^{\frac{5}{2}}\left(1-x\right)}\left(\ln\left(1+x\right) - x\left(1-\frac{x}{2}\right)\right)dx =$$

$$= 7\zeta(3) - 2\left(\pi + \frac{8}{3}\right)G + 3\left(\frac{1}{2} - \ln 2\right)\zeta(2) - \frac{40}{9}\ln 2 + \frac{16}{9}\pi + \frac{4}{9}.$$

From (2.5), for $k=2$,

$$\frac{1}{3}\int_0^1 \frac{\sqrt{x}\log x}{1-x}\,{}_2F_1\left[\begin{matrix}1,1\\4\end{matrix}\,\middle|\,-x\right]dx = W_2(1) - \frac{\zeta(2)}{2}\,{}_2F_1\left[\begin{matrix}1,1\\4\end{matrix}\,\middle|\,-1\right],$$

which reduces to

$$\lim_{\epsilon\longrightarrow 0}\int_\epsilon^1 \frac{\ln x}{x^{\frac{5}{2}}\left(1-x\right)}\left(\left(1+x\right)^2\ln\left(1+x\right) - x\right)dx =$$

$$= 28\zeta(3) + 3\left(1 - 4\ln 2\right)\zeta(2) - 8\left(\pi + \frac{8}{3}\right)G - \frac{112}{9}\ln 2 + \frac{52}{9}\pi + \frac{4}{9}.$$

From (2.4), for $k = 2$,

$$\frac{2}{3}\int_0^1 \frac{\sqrt{x}\log x}{1-x\,(1+x)}\,{}_2F_1\left[\begin{matrix}1,1\\4\end{matrix}\middle| -x\right]dx =$$

$$= 2G - \zeta(2) - \frac{\zeta(2)}{3}\,{}_2F_1\left[\begin{matrix}1,1\\4\end{matrix}\middle| -1\right] - W_2(0)$$

which reduces to

$$\lim_{\epsilon\longrightarrow 0}\int_\epsilon^1 \frac{\ln x}{x^{\frac{5}{2}}\,(1-x)}\left((1+x)\ln(1+x) - \frac{x}{1+x}\right)dx =$$

$$= 14\zeta(3) + 3\left(\frac{1}{2} - 2\ln 2\right)\zeta(2) - \left(\frac{46}{3} + 4\pi\right)G - \frac{76}{9}\ln 2 + \frac{34}{9}\pi + \frac{4}{9}.$$

The Wolfram online integrator yields no solution to these integrals.

Remark **2.1.** It appears that, for $r \in \mathbb{N}_0,\ p \geq 3$

$$Y(p,r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{n-\frac{1}{2}}}{(n+r)^p}$$

may not have a closed form solution, in terms of some common special functions. Remarkably, however, the sum of two consecutive terms of $Y(p,r)$ does have a closed form solution, this result is pursued in the next lemma.

Lemma 2.1. *For* $r \in \mathbb{N},\ p \in \mathbb{N}$

$$Y(p,r) + Y(p,r+1) = \sum_{n=1}^{\infty} (-1)^{n+1}\, H^{(2)}_{n-\frac{1}{2}}\left(\frac{1}{(n+r)^p} + \frac{1}{(n+r+1)^p}\right) =$$

$$= 4\left(\frac{2}{2r+1}\right)^p G - \frac{p\pi}{2}\left(\frac{2}{2r+1}\right)^{p+1} - \frac{2\zeta(2)}{(r+1)^p} +$$

$$+ (-1)^r p\left(\frac{2}{2r+1}\right)^{p+1}\left(\ln 2 + H_{[\frac{r}{2}]} - H_r\right) +$$

$$+(-1)^r \sum_{j=2}^{p} (p+1-j)\left(\frac{2}{2r+1}\right)^{p+2-j}\left(1 - 2^{1-j}\right)\zeta(j) +$$

$$+ (-1)^r \sum_{j=2}^{p} (p+1-j)\left(\frac{2}{2r+1}\right)^{p+2-j}\left(\frac{1}{2^j}\left(H^{(j)}_{[\frac{r}{2}]} + H^{(j)}_{[\frac{r-1}{2}]}\right) - H^{(j)}_{2[\frac{r+1}{2}]-1}\right), \tag{2.7}$$

and for $r = 0$

$$Y(p,0) + Y(p,1) = \sum_{n=1}^{\infty} (-1)^{n+1}\, H^{(2)}_{n-\frac{1}{2}}\left(\frac{1}{n^p} + \frac{1}{(n+1)^p}\right) =$$

$$= 2^{p+2}G - 2^{p}p\pi + 2^{p+1}p\ln 2 + \left(2^{p-1}(p-1) - 2\right)\zeta(2) +$$
$$+ \sum_{j=3}^{p}(p+1-j)2^{-j}\left(1-2^{1-j}\right)\zeta(j).$$

Proof. Consider

$$Y(p,r) + Y(p,r+1) = \sum_{n=1}^{\infty}(-1)^{n+1}H_{n-\frac{1}{2}}^{(2)}\left(\frac{1}{(n+r)^p} + \frac{1}{(n+r+1)^p}\right)$$

and by a change of counter in the second sum we can write

$$Y(p,r) + Y(p,r+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{1}{2}}^{(2)}}{(n+r)^p} - \frac{2\zeta(2)}{(r+1)^p} -$$
$$-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+r)^p}\left(H_{n-\frac{1}{2}}^{(2)} - \frac{4}{(2n-1)^2}\right) =$$
$$= \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{(2n-1)^2(n+r)^p} - \frac{2\zeta(2)}{(r+1)^p} =$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\left(\begin{array}{c}\left(\frac{2}{2r+1}\right)^p\frac{4}{(2n-1)^2} - \left(\frac{2}{2r+1}\right)^{p+1}\frac{2p}{2n-1} + \\ + \left(\frac{2}{2r+1}\right)^{p+1}\frac{p}{n+r} + \sum_{j=2}^{p}\left(\frac{2}{2r+1}\right)^{p+2-j}\frac{p+1-j}{(n+r)^j}\end{array}\right) - \frac{2\zeta(2)}{(r+1)^p} =$$
$$= 4\left(\frac{2}{2r+1}\right)^p G - \frac{p\pi}{2}\left(\frac{2}{2r+1}\right)^{p+1} - \frac{2\zeta(2)}{(r+1)^p} +$$
$$+ (-1)^r p\left(\frac{2}{2r+1}\right)^{p+1}\left(\ln 2 + H_{\left[\frac{r}{2}\right]} - H_r\right) +$$
$$+ (-1)^r\sum_{j=2}^{p}(p+1-j)\left(\frac{2}{2r+1}\right)^{p+2-j}\left(1-2^{1-j}\right)\zeta(j) +$$
$$+(-1)^r\sum_{j=2}^{p}(p+1-j)\left(\frac{2}{2r+1}\right)^{p+2-j}\left(\frac{1}{2^j}\left(H_{\left[\frac{r}{2}\right]}^{(j)} + H_{\left[\frac{r-1}{2}\right]}^{(j)}\right) - H_{2\left[\frac{r+1}{2}\right]-1}^{(j)}\right),$$

and a rearrangement leads to (2.7). The case of $r = 0$ follows.

Lemma 2.1 is proved.

Example 2.2.

$$Y(5,2) + Y(5,3) = \frac{128}{3125}G - \frac{32}{3125}\pi + \frac{64}{3125}\ln 2 - \frac{9579}{25000} +$$
$$+ \frac{9302}{759375}\zeta(2) + \frac{36}{625}\zeta(3) + \frac{14}{125}\zeta(4) + \frac{3}{20}\zeta(5),$$

$$Y(5,0)+Y(5,1)=128G-160\pi+320\ln 2+$$
$$+\,62\zeta(2)+36\zeta(3)+14\zeta(4)+\frac{15}{4}\zeta(5).$$

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