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Fengquan Li (School Math. Sci., Dalian Univ. Techology, China)

SOME REGULARITY OF ENTROPY SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH IRREGULAR DATA

РЕГУЛЯРНІСТЬ ЕНТРОПІЙНИХ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ З НЕРЕГУЛЯРНИМИ ДАНИМИ

We introduce new sets of functions which are different to the space introduced in [*Bénilan Ph., Boccardo L., Gallouët T., Gariepy R., Pierre M., Vazquez J. L.* An L^1-theory of existence and uniqueness of solutions of non-linear elliptic equations // Ann. Scuola norm. super. Pisa. – 1995. – **22**, № 2. – P. 241 – 273] and Rakotoson's T-set in [*Rakotoson J. M.* Generalized solutions in a new type of sets for problems with measures as data // Different. and Integr. Equat. – 1993. – **6**, № 1. – P. 27 – 36; T-sets and relaxed solutions for parabolic equations // J. Different. Equat. – 1994. – **111**, № 2. – P. 458 – 471]. In the new framework of sets, we give some summability results of entropy solutions for nonlinear parabolic equations.

Введено нові множини функцій, що відрізняються як від простору, що був введений у [*Bénilan Ph., Boccardo L., Gallouët T., Gariepy R., Pierre M., Vazquez J. L.* An L^1-theory of existence and uniqueness of solutions of non-linear elliptic equations // Ann. Scuola norm. super. Pisa. – 1995. – **22**, № 2. – P. 241 – 273], так і від T-множини Ракотосона, що була введена в [*Rakotoson J. M.* Generalized solutions in a new type of sets for problems with measures as data // Different. and Integr. Equat. – 1993. – **6**, № 1. – P. 27 – 36; T-sets and relaxed solutions for parabolic equations // J. Different. Equat. – 1994. – **111**, № 2. – P. 458 – 471]. В рамках нової колекції множин отримано нові результати про сумовність ентропійних розв'язків нелінійних параболічних рівнянь.

1. Introduction. In this paper we will consider the following problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,Du)) = f \quad && \text{in} \quad Q, \\ &u = 0 && \text{on} \quad \Sigma, \\ &u(x,0) = u_0 && \text{in} \quad \Omega, \end{aligned} \tag{P}$$

where Ω is a bounded open subset of R^N, $N \geq 2$, and $T > 0$, $Q = \Omega \times (0,T)$, Σ denotes the lateral surface of Q, $f \in \mathcal{M}_b(Q)$, $u_0 \in \mathcal{M}_b(\Omega)$. Here $\mathcal{M}_b(Q), \mathcal{M}_b(\Omega)$ denote the spaces of bounded Radon measures on Q and Ω respectively.

There are many works contributing to nonlinear elliptic equations and parabolic equations with measure data (see [1, 3 – 7, 9 – 21]).

If $f \in L^{p'}\big(0,T; W^{-1,p'}(\Omega)\big)$, $u_0 \in L^2(\Omega)$, the existence of solutions for this problem is a classical result due to Lions's methods in [8].

If $f \in \mathcal{M}_b(Q)$, $u_0 \in \mathcal{M}_b(\Omega)$ and $p > 2 - \dfrac{1}{N+1}$, Boccardo and Gallouët proved the existence of distributional solution to problem (P) in [6] (also see [5, 7, 10, 13 – 15]). If $1 < p \leq 2 - \dfrac{1}{N+1}$, one can't expect the solution to be in the classical Sobolev space. In order to deal with this case, Rakotoson has introduced T-sets and the notion of relaxed solutions in [12] (also see [11]). To study elliptic equations, $\mathcal{T}_0^{1,p}(\Omega)$ has been introduced in [3] too.

Dall' Aglio, Orsina have discussed the existence and regularity of solutions under three kinds of conditions about f and u_0 in a framework of Sobolev spaces as $p > 2 - \dfrac{1}{N+1}$ in [7] (also see

[6, 9]). Similar problem has been discussed in [5], but in which the summability of solutions with respect to space and time was considered separately. In [17] Segura de León and Toledo have given a precise summability result of entropy solutions to problem (P) and its gradient with respect to space and time in a framework of Lebesgue and Marcinkiewicz space under the assumptions of $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ or $f = 0$, $u_0 \in L^1(\Omega)$ (also see [16]).

Up to now, I haven't found any regular results of solutions in the case of $1 < p \leq 2 - \dfrac{1}{N+1}$, $f \in L^1(0,T; L^1 \log L^1(\Omega))$. Since in the case of $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, one can prove the solution to problem (P) in a unified framework of Marcinkiewicz space $\mathcal{M}^q(Q)$ $\left(\text{here } q = \dfrac{N(p-1)+p}{N+1}\right)$ for all $p > 1$, and deduce that the solution belongs to the classical Sobolev space $L^q(0,T; W_0^{1,q}(\Omega))$, $q < \dfrac{N(p-1)+p}{N+1}$, in the case of $p > 2 - \dfrac{1}{N+1}$ (see [1, 17]). Can we prove the solution in a unified framework of Marcinkiewicz space in the case of $f \in L^1(0,T; L^1 \log L^1(\Omega))$ for all $p > 1$ as that of [1, 17]? The answer is negative because we have known in the case of $p > 2 - \dfrac{1}{N+1}$, $f \in L^1(0,T; L^1 \log L^1(\Omega))$, the solution to problem (P) belongs to the limit case $L^{\bar{q}}\big(0,T; W_0^{1,\bar{q}}(\Omega)\big)$, $\bar{q} = \dfrac{N(p-1)+p}{N+1}$, from [7] and [5], but we can't deduce that the limit case from the Marcinkiewicz space. Furthermore it's impossible to prove the solution in the framework of Rakotoson's T-set because it's difficult to obtain $\Phi(u) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big)$ (see [12]). Similar question also appears in the case of $f \in L^m(Q)$, $1 < m < \dfrac{(N+2)p}{(N+2)p-N}$.

To solve the above questions, here I will introduce a new type of functional sets which are different both from the ones introduced by Rakotoson's (T-set), $\mathcal{T}_0^{1,p}(Q)$ and the Marcinkiewicz space (see [4]). I will discuss the summability results of entropy solutions to problem (P) with respect to time and space in the framework of new functional sets. Similar case to elliptic equations has been discussed by the author in [20].

The author has also studied how the growth of $a(x,t,s,\xi)$ with respect to s affected the regularity of entropy solutions in [21].

The paper is organized as follows. In Section 2 we will give a new type of functional sets and specify the link with the classical Sobolev spaces. In Section 3 assumptions and statements of the main results will be given. In Section 4 we will complete the proof of the main results.

2. A new type of functional sets. In this paper we need to define four new types of functional sets.

For $k > 0$, we set $T_k(\sigma) = \max\{-k, \min\{k, \sigma\}\}$ $\forall \sigma \in R$.

For $1 < p < +\infty$, the definition of $\mathcal{T}_0^{1,p}(\Omega)$ can be found in [3]. $u \in L^p\big(0,T; \mathcal{T}_0^{1,p}(\Omega)\big)$ if and only if $T_k(u) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big)$ for every $k > 0$.

Let

$$\mathcal{T}_{0T}^{1,p}(Q) = \left\{ u \in L^p\big(0,T; \mathcal{T}_0^{1,p}(\Omega)\big) \;\middle|\; \sup_{k>0} \int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}} dxdt < +\infty \;\; \forall \delta > 0 \right\},$$

$$\mathcal{T}_{0T_1}^{1,p}(\Omega) = \left\{ u \in \mathcal{T}_{0T}^{1,p}(Q) \;\middle|\; \exists C > 0 \;\text{ and increasing function } \Theta \text{ such that} \right.$$

$$\left.\int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)}dxdt \le C[\Theta(\|T_k(u)\|_{L^\infty(0,T;L^1(\Omega))})+1] \ \ \forall k>0\right\}.$$

For any given $1 < m < +\infty$, let

$$\mathcal{T}^{1,p}_{0T_m}(Q) = \left\{ u \in \mathcal{T}^{1,p}_{0T}(Q) \ \middle| \ \exists C > 0 \ \text{ such that} \right.$$

$$\left.\int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda}dxdt \le C\left(\int\limits_Q (1+|T_k(u)|)^{\frac{(1-\lambda)m}{m-1}}dxdt\right)^{1-\frac{1}{m}} \quad \forall k>0 \ \text{ and } \ 0<\lambda<1\right\}.$$

For any given $0 < m < 1$, let

$$\mathcal{T}^{1,p}_{0T_m}(Q) = \left\{ u \in \mathcal{T}^{1,p}_{0T}(Q) \ \middle| \ \sup_{k>0}\int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^m}dxdt < +\infty \right\}.$$

***Remark* 2.1.** Here $\mathcal{T}^{1,p}_{0T}(Q)$ is different both from $\mathcal{T}^{1,p}_0(Q)$ introduced by Benilan, Boccardo, Gallouët, Gariepy, Pierre and Vazquez in [3] and Rakotoson's T-set in [12]. The above four types of functional sets are new. Similar functional sets are defined in [19].

The relations between the above four kind of functional spaces and the classical Sobolev spaces are stated as follows:

Proposition 2.1. *If* $p > 2 - \dfrac{1}{N+1}$, *then*

$$L^\infty(0,T;L^1(\Omega)) \cap \mathcal{T}^{1,p}_{0T}(Q) \subset L^r(0,T;W^{1,q}_0(\Omega)),$$

where r *and* q *satisfy the following inequalities*:

$$1 \le q < \min\left\{\frac{N(p-1)}{N-1}, p\right\}, \quad 1 \le r < p,$$

and

$$\frac{N(p-2)+p}{r} + \frac{N}{q} > N+1.$$

Furthermore, *as* $p > 2$, *we can take*

$$r = p \qquad \text{and} \qquad q < \frac{p}{2}.$$

Proof. Working as in the proof of Lemma 2.2 of [5] we can prove this propositionby by replacing u with $T_k(u)$ and taking the limit as k goes to $+\infty$.

Proposition 2.2. *If* $p > 2 - \dfrac{1}{N+1}$, *then*

$$L^\infty(0,T;L^1(\Omega)) \cap \mathcal{T}^{1,p}_{0T_1}(Q) \subset L^r(0,T;W^{1,q}_0(\Omega)),$$

where r *and* q *satisfy*

$$1 \le q \le \frac{N(p-1)}{N-1}, \quad \textit{if} \quad p < N,$$

$$1 \le q < p, \qquad \textit{if} \quad p \ge N,$$

$$1 \le r < p,$$

and

$$\frac{N(p-2)+p}{r} + \frac{N}{q} = N+1.$$

Furthermore, *as* $p > 2$, *we can take*

$$r = p \qquad \textit{and} \qquad q = \frac{p}{2}.$$

Proof. For any given $k > 0$ and $u \in L^\infty(0,T;L^1(\Omega)) \cap \mathcal{T}^{1,p}_{0T_1}(Q)$, Hölder's inequality implies that

$$\int\limits_\Omega |DT_k(u)|^q dx \le$$

$$\le \left(\int\limits_\Omega \frac{|DT_k(u)|^p}{(1+|T_k(u)|)} dx \right)^{\frac{q}{p}} \left(\int\limits_\Omega (1+|T_k(u)|)^{\frac{q}{p-q}} dx \right)^{1-\frac{q}{p}}.$$

For any $1 \le r < p$, using Hölder's inequality again, we get

$$\int\limits_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt \le$$

$$\le \left(\int\limits_0^T \int\limits_\Omega \frac{|DT_k(u)|^p}{(1+|T_k(u)|)} dxdt \right)^{\frac{r}{p}} \left(\int\limits_0^T \left(\int\limits_\Omega (1+|T_k(u)|)^{\frac{q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \le$$

$$\le C \left[\Theta\left(\|T_k(u)\|_{L^\infty(0,T;L^1(\Omega))} \right) + 1 \right]^{\frac{r}{p}} \left(\int\limits_0^T \left(\int\limits_\Omega (1+|T_k(u)|)^{\frac{q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \le$$

$$\le C_1 \left[1 + \left(\int\limits_0^T \|T_k(u)\|^{\frac{r}{p-r}}_{L^{\frac{q}{p-q}}(\Omega)} dt \right)^{1-\frac{r}{p}} \right],$$

where C_1 denotes the positive constant independent of u and k. From then on, C_i denote analogous constants, which can vary from line to line. Applying Gagliardo–Nirenberg embedding inequality (also see [2]), we have

$$\|T_k(u(t))\|_{L^{\frac{q}{p-q}}(\Omega)} \leq C_2 \|DT_k(u(t))\|^{\theta}_{L^q(\Omega)} \|T_k(u(t))\|^{1-\theta}_{L^1(\Omega)} \leq$$

$$\leq C_3 \|DT_k(u(t))\|^{\theta}_{L^q(\Omega)}, \tag{2.1}$$

where θ satisfies

$$\frac{p-q}{q} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{1}. \tag{2.2}$$

Thus, from (2.1) and (2.2) it follows that

$$\int_0^T \|DT_k(u)\|^{\frac{r}{\theta}}_{L^{\frac{q}{p-q}}(\Omega)} dt \leq C_4 \int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt.$$

Let

$$\frac{r}{\theta} = \frac{r}{p-r}, \tag{2.3}$$

then

$$\int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt \leq C_5 \left[1 + \left(\int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt\right)^{1-\frac{r}{p}}\right] \leq$$

$$\leq C_6 + \varepsilon \int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt. \tag{2.4}$$

Taking $\varepsilon = \dfrac{1}{2}$ in (2.4), then

$$\int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt \leq C_7. \tag{2.5}$$

Taking $k \to \infty$ in (2.5) and using Fatou's lemma, it is easy to see that $u \in L^r(0,T; W_0^{1,q}(\Omega))$. Combining (2.2) with (2.3), we can deduce that

$$1 \leq q \leq \frac{N(p-1)}{N-1}, \quad \text{if} \quad p < N,$$

$$1 \leq q < p, \qquad \text{if} \quad p \geq N,$$

$$1 \leq r < p,$$

and

$$\frac{N(p-2)+p}{r}+\frac{N}{q}=N+1.$$

By checking the above estimate, it is possible to choose $r=p$, $q=\dfrac{p}{2}$ in the case of $p\geq 2$.

Proposition 2.2 is proved.

Proposition 2.3. *If* $1<m<\dfrac{(N+2)p}{(N+2)p-N}$, $p>2-\dfrac{1}{N+1}$, *then*

$$L^{\infty}(0,T;L^{\frac{[(2-m)-(1-m)p]N}{N+p-pm}}(\Omega))\cap\mathcal{T}^{1,p}_{0T_m}(Q)\subset L^r(0,T;W^{1,q}_0(\Omega)),$$

for every pair (r,q) *satisfying*

$$1\leq q\leq N\frac{Nm(p-1)-p(p-2)(m-1)}{2p(m-1)+N(N-m)},\quad \textit{if}\quad p<N,$$

$$1\leq q<p,\qquad \textit{if}\quad p\geq N,$$

$$1\leq r\leq p,$$

and

$$\frac{N(p-2)+mp}{r}+\frac{N[1+(p-1)(m-1)]}{q}=N+2-m.$$

Proof. For any given $u\in L^{\infty}\Big(0,T;L^{\frac{[(2-m)-(1-m)p]N}{N+p-pm}}(\Omega)\Big)\cap\mathcal{T}^{1,p}_{0T_m}(Q)$, $0<\lambda<1$, $1\leq q<p$, by Hölder's inequality we get

$$\int\limits_{\Omega}|DT_k(u)|^q dx\leq$$

$$\leq\left(\int\limits_{\Omega}\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{\lambda}}dx\right)^{\frac{q}{p}}\left(\int\limits_{\Omega}(1+|T_k(u)|)^{\frac{\lambda q}{p-q}}dx\right)^{1-\frac{q}{p}}. \tag{2.6}$$

For any $1\leq r<p$, Hölder's inequality and (2.6) yield

$$\int\limits_0^T\|DT_k(u)\|^r_{L^q(\Omega)}dt\leq$$

$$\leq\left(\int\limits_0^T\int\limits_{\Omega}\frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{\lambda}}dxdt\right)^{\frac{r}{p}}\left(\int\limits_0^T\left(\int\limits_{\Omega}(1+|T_k(u)|)^{\frac{\lambda q}{p-q}}dx\right)^{\frac{(p-q)r}{(p-r)q}}dt\right)^{1-\frac{r}{p}}\leq$$

$$\leq C_8\left(\int\limits_{Q}(1+|T_k(u)|)^{(1-\lambda)m'}\right)^{\frac{r}{pm'}}\left(\int\limits_0^T\left(\int\limits_{\Omega}(1+|T_k(u)|)^{\frac{\lambda q}{p-q}}dx\right)^{\frac{(p-q)r}{(p-r)q}}dt\right)^{1-\frac{r}{p}}\leq$$

$$\leq C_9 \left[1 + \left(\int\limits_Q |T_k(u)|^{(1-\lambda)m'} dxdt \right)^{\frac{r}{pm'}} \left(\int\limits_0^T \left(\int\limits_\Omega |T_k(u)|^{\frac{\lambda q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \right]. \tag{2.7}$$

Taking

$$\lambda = \frac{Nm - p(m-1)(N+2)}{N + p - pm} \tag{2.8}$$

and working as in the proof of Lemma 2.4 in [5] we can prove

$$\int\limits_Q |T_k(u)|^{(1-\lambda)m'} dxdt \leq C_{10}. \tag{2.9}$$

Using Gagliardo – Nirenberg embedding inequality, then

$$\|T_k(u(t))\|_{L^{\frac{\lambda q}{p-q}}(\Omega)} \leq C_{11} \|DT_k(u(t))\|^{\theta}_{L^q(\Omega)} \|T_k(u(t))\|^{1-\theta}_{L^{2-\lambda}(\Omega)} \leq$$

$$\leq C_{12} \|DT_k(u(t))\|^{\theta}_{L^q(\Omega)}, \tag{2.10}$$

where $0 \leq \theta \leq 1$ satisfies

$$\frac{p-q}{\lambda q} = \theta \left(\frac{1}{q} - \frac{1}{N} \right) + \frac{1-\theta}{2-\lambda}. \tag{2.11}$$

Applying (2.9) and (2.10) to (2.7), we get

$$\int\limits_0^T \|DT_k(u(t))\|^r_{L^q(\Omega)} dt \leq C_{13} + C_{14} \left(\int\limits_0^T \|DT_k(u(t))\|^{\frac{\theta \lambda r}{p-r}}_{L^q(\Omega)} dt \right)^{1-\frac{r}{p}}.$$

Choosing

$$\frac{\theta \lambda r}{p-r} = r, \tag{2.12}$$

then we have

$$\int\limits_0^T \|DT_k(u(t))\|^r_{L^q(\Omega)} dt \leq C_{15}. \tag{2.13}$$

Taking $k \to \infty$ in (2.13) and using Fatou's lemma, we obtain $u \in L^r(0,T; W_0^{1,q}(\Omega))$. From (2.8), (2.11) and (2.12) it follows that

$$\frac{N(p-2) + mp}{r} + \frac{N[1 + (p-1)(m-1)]}{q} = N + 2 - m. \tag{2.14}$$

By (2.8), (2.12), (2.14) and $0 \leq \theta \leq 1$, we can deduce that

$$q \leq N \frac{Nm(p-1) - p(p-2)(m-1)}{2p(m-1) + N(N-m)}.$$

This inequality and $q < p$ yield

$$1 \leq q \leq N \frac{Nm(p-1) - p(p-2)(m-1)}{2p(m-1) + N(N-m)}, \quad \text{if} \quad p < N,$$

$$1 \leq q < p, \qquad \text{if} \quad p \geq N.$$

Moreover, by checking the above proof, (2.14) still hold for the case of $r = p$.

Proposition 2.4. (i) *If* $1 < \gamma \leq \frac{N}{N-1}$, $1 + \frac{(2-\gamma)(N-1)}{N} < p$, *then*

$$L^\infty(0,T;L^\gamma(\Omega)) \cap \mathcal{T}^{1,p}_{0T_{2-\gamma}}(Q) \subset L^r(0,T;W^{1,q}_0(\Omega)), \tag{2.15}$$

for every pair (r,q) *satisfies*

$$1 \leq q \leq \frac{N(p-2+\gamma)}{N-2+\gamma}, \quad \textit{if} \quad p < N,$$

$$1 \leq q < p, \qquad \textit{if} \quad p \geq N,$$

$$1 \leq r < p,$$

and

$$\frac{N\gamma}{q} + \frac{(N+\gamma)p - 2N}{r} = N + \gamma. \tag{2.16}$$

(ii) *If* $\frac{N}{N-1} < \gamma < 2$, $1 + \frac{(2-\gamma)(N-1)}{N} < p < \frac{2N}{N+1}$, *then* (2.15) *holds and* r *and* q *satisfy*

$$\frac{N(p-2+\gamma)}{N-2+\gamma} < q \leq \frac{p\gamma}{2},$$

$$1 \leq r < p,$$

and (2.16). *Furthermore, as* $p \geq \frac{2}{\gamma}$, *we can take*

$$r = p \qquad \textit{and} \qquad q = \frac{p\gamma}{2}.$$

Proof. For any given $u \in L^\infty(0,T;L^\gamma(\Omega)) \cap \mathcal{T}^{1,p}_{0T_{2-\gamma}}(Q)$, $k > 0, 1 \leq q < p$, by Hölder's inequality we get

$$\int\limits_\Omega |DT_k(u)|^q dx \leq$$

$$\leq \left(\int\limits_\Omega \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{2-\gamma}} dx \right)^{\frac{q}{p}} \left(\int\limits_\Omega (1+|T_k(u)|)^{\frac{(2-\gamma)q}{p-q}} dx \right)^{1-\frac{q}{p}}. \tag{2.17}$$

For $1 \leq r < p$, Hölder's inequality and inequality (2.17) imply that

$$\int_0^T \|DT_k(u)\|^r_{L^q(\Omega)} dt \leq$$

$$\leq \left(\int_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{2-\gamma}} dxdt \right)^{\frac{r}{p}} \left(\int_0^T \left(\int_\Omega (1+|T_k(u)|)^{\frac{(2-\gamma)q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \leq$$

$$\leq C_{16} \left(\int_0^T \left(\int_\Omega (1+|T_k(u)|)^{\frac{(2-\gamma)q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \leq$$

$$\leq C_{17} \left[1 + \left(\int_0^T \left(\int_\Omega |T_k(u)|^{\frac{(2-\gamma)q}{p-q}} dx \right)^{\frac{(p-q)r}{(p-r)q}} dt \right)^{1-\frac{r}{p}} \right]. \tag{2.18}$$

Applying Gagliardo – Nirenberg embedding inequality to $T_k(u(t))$, we have

$$\|T_k(u(t))\|_{L^{\frac{(2-\gamma)q}{p-q}}(\Omega)} \leq C_{18} \|DT_k(u(t))\|^\theta_{L^q(\Omega)} \|T_k(u(t))\|^{1-\theta}_{L^\gamma(\Omega)} \leq$$

$$\leq C_{19} \|DT_k(u(t))\|^\theta_{L^q(\Omega)}, \tag{2.19}$$

where $0 \leq \theta \leq 1$ and satisfies

$$\frac{p-q}{(2-\gamma)q} = \theta \left(\frac{1}{q} - \frac{1}{N} \right) + \frac{1-\theta}{\gamma}. \tag{2.20}$$

(2.18), (2.19) yield

$$\int_0^T \|DT_k(u(t))\|^r_{L^q(\Omega)} dt \leq C_{20} + C_{21} \left(\int_0^T \|DT_k(u(t))\|^{\frac{\theta(2-\gamma)r}{p-r}}_{L^q(\Omega)} dt \right)^{1-\frac{r}{p}}.$$

Let

$$\frac{\theta(2-\gamma)r}{p-r} = r. \tag{2.21}$$

Thus we obtain

$$\int_0^T \|DT_k(u(t))\|^r_{L^q(\Omega)} dt \leq C_{22}. \tag{2.22}$$

Taking $k \to \infty$ in (2.22) and by Fatou's lemma, it is easy to see that $u \in L^r(0,T; W^{1,q}_0(\Omega))$. From (2.20) and (2.21) it follows that

$$\frac{(N+\gamma)p - 2N}{r} + \frac{N\gamma}{q} = N + \gamma. \tag{2.23}$$

By (2.20), (2.21), (2.23) and $0 \leq \theta \leq 1$, we can deduce that the conditions of q and r in (i) and (ii).

3. Assumptions and statements of the main results. Let $a\colon Q \times R \times R^N \to R^N$ be a Carathéodory function satisfying for almost every $(x,t) \in Q$ and every $(s,\xi) \in R^{N+1}$, $\xi \in R^N$, $\xi' \in R^N$, $\xi \neq \xi'$,

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \tag{3.1}$$

$$|a(x,t,s,\xi)| \leq \beta(a_0(x,t) + |s|^{p-1} + |\xi|^{p-1}), \tag{3.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi')][\xi - \xi'] > 0, \tag{3.3}$$

where α, β are two positive constants, a_0 is a nonnegative function belonging to $L^{p'}(Q)$, $p' = \dfrac{p}{p-1}$.

Definition 3.1. *A measurable function* $u \in L^\infty(0,T;L^1(\Omega))$ *will be called an entropy solution to problem* (P) *if* $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, $S_k(u(\cdot,t)) \in L^1(\Omega)$ $\forall k > 0$ $\forall t \in [0,T]$, *and* u *satisfies*

$$\int\limits_\Omega S_k(u(t) - \phi(t))dx + \int\limits_0^t \langle \phi_\tau, T_k(u-\phi)\rangle\, d\tau +$$

$$+ \int\limits_Q a(x,\tau,u,Du)DT_k(u-\phi)dxd\tau \leq$$

$$\leq \int\limits_\Omega S_k(u_0 - \phi(0))dx + \int\limits_Q fT_k(u-\phi)dxd\tau$$

$\forall k > 0$ $\forall \phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ *such that* $\phi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$.

We denote by $L^1 \log L^1(\Omega)$ Orlicz space on Ω. Now, we state the main results of this paper.

Theorem 3.1. *Let* u *be an entropy solution to problem* (P) *and* $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$. *Then under the hypothesis* (3.1), $u \in \mathcal{T}_{0T}^{1,p}(Q)$.

Theorem 3.2. *Let* u *be an entropy solution to problem* (P) *and* $f \in L^1(0,T;L^1\log L^1(\Omega))$, $u_0 \in L^1 \log L^1(\Omega)$. *Then under the hypothesis* (3.1), $u \in \mathcal{T}_{0T_1}^{1,p}(Q)$.

Theorem 3.3. *Let* u *be an entropy solution to problem* (P) *and* $f \in L^m(Q)$, $1 < m < \dfrac{(N+2)p}{(N+2)p-N}$, $u_0 = 0$. *Then under the hypothesis* (3.1), $u \in \mathcal{T}_{0T_m}^{1,p}(Q)$.

Theorem 3.4. *Let* u *be an entropy solution to problem* (P) *and* $f = 0, u_0 \in L^\gamma(\Omega), 1 < \gamma < 2$. *Then under the hypothesis* (3.1), $u \in \mathcal{T}_{0T_{2-\gamma}}^{1,p}(Q)$.

Remark **3.1.** In Theorems 3.1 – 3.4, we obtain the precise summability results of entropy solutions to problem (P) by using four types of functional spaces.

Remark **3.2.** If $f \in L^m(Q)$, $u_0 = 0$, the existence and regular results of solutions to problem (P) were obtained in Theorem 1.9 and Remark 2.5 of [5] as $p \geq 2$. However, here Theorem 3.3 and Proposition 2.3 get rid of the condition of $p \geq 2$ and give a precise summability result of entropy solutions to problem (P).

Remark **3.3.** Theorem 3.4 and Proposition 2.4 improve those results obtained in [9, 16] and Theorem 5.5 (1) in [17] (in the case of bounded domain).

4. Proofs of the main results. In order to prove the main results of this paper, we need the following lemmas.

Lemma 4.1. *If* $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ *and suppose that* u *is an entropy solution to problem* (P), *then*

$$\int\limits_{\{h \le |u| < h+k\}} |Du|^p dxdt \le k \left(\int\limits_{\{|u| \ge h\}} |f| dxdt + \int\limits_{\{|u_0| \ge h\}} |u_0| dx \right) \quad \forall k, h > 0. \tag{4.1}$$

Proof. The proof can be seen in [9].

For convenience, let

$$A_m = \big\{ (x,t) \in Q \colon m \le |u(x,t)| < m+1 \big\}$$

and

$$A_{0m} = \big\{ x \in \Omega \colon m \le |u(x)| < m+1 \big\}.$$

For any given $k > 0$, let $K = [k]$ denote the maximal integer not beyond k.

Lemma 4.2. *Let* u *be an entropy solution to problem* (P), *then for any fixed* $0 < \tau < 1$ *and large enough positive integer* $l > 1$, *we have*

$$\sum_{m=1}^{l} \int\limits_{\{|u| \ge m-1\}} |f| m^{-\tau} dxdt \le \frac{1}{1-\tau} \int\limits_{Q} |f| (1 + |T_l(u)|)^{1-\tau} dxdt, \tag{4.2}$$

$$\sum_{m=1}^{l} \int\limits_{\{|u_0| \ge m-1\}} |u_0| m^{-\tau} dx \le \frac{1}{1-\tau} \int\limits_{\Omega} |u_0| (1 + |T_l(u_0)|)^{1-\tau} dx \tag{4.3}$$

and

$$\sum_{m=1}^{l} m^{-1} \left(\int\limits_{\{|u| \ge m-1\}} |f| dxdt + \int\limits_{\{|u_0| \ \ge m-1\}} |u_0| dx \right) \le$$

$$\le \int\limits_{Q} |f| (1 + \ln(1 + |T_l(u)|)) dxdt + \int\limits_{\Omega} |u_0| (1 + \ln(1 + |T_l(u_0)|)) dx. \tag{4.4}$$

Proof. From the formula of Abel's summation it follows that

$$\sum_{m=1}^{l} \int\limits_{\{|u| \ge m-1\}} |f| m^{-\tau} dxdt = \sum_{m=1}^{l} \sum_{h=m-1}^{\infty} \int\limits_{A_h} |f| m^{-\tau} dxdt =$$

$$= \int\limits_{\{|u| \ge l-1\}} |f| \sum_{m=1}^{l} m^{-\tau} dxdt + \sum_{m=1}^{l-1} \int\limits_{A_{m-1}} |f| \sum_{h=1}^{m} h^{-\tau} dxdt \le$$

$$\leq \frac{1}{1-\tau} \int\limits_{\{|u| \geq l-1\}} |f| l^{1-\tau} dxdt + \frac{1}{1-\tau} \sum_{m=1}^{l-1} \int\limits_{A_{m-1}} |f| m^{1-\tau} dxdt \leq$$

$$\leq \frac{1}{1-\tau} \int\limits_{\{|u| \geq l-1\}} |f|(1+|T_l(u)|)^{1-\tau} dxdt +$$

$$+ \frac{1}{1-\tau} \sum_{m=1}^{l-1} \int\limits_{A_{m-1}} |f|(|1+T_m(u)|)^{1-\tau} dxdt =$$

$$= \frac{1}{1-\tau} \int\limits_{\{|u| \geq l-1\}} |f|(1+|T_l(u)|)^{1-\tau} dxdt +$$

$$+ \frac{1}{1-\tau} \sum_{m=1}^{l-1} \int\limits_{A_{m-1}} |f|(|1+T_l(u)|)^{1-\tau} dxdt =$$

$$= \frac{1}{1-\tau} \int\limits_{Q} |f|(1+|T_l(u)|)^{1-\tau} dxdt.$$

Working as in the proof of (4.2), we can prove (4.3) and (4.4).

***Proof of Theorem* 3.1.** Let u be an entropy solution to problem (P), then $T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega))$ $\forall k > 0$, thus $u \in L^p\big(0,T; \mathcal{T}_0^{1,p}(\Omega)\big)$.

By (4.1) (here $h = m, k = 1$), we get

$$\int\limits_{Q} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}} dxdt = \sum_{m=0}^{K} \int\limits_{A_m} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^{1+\delta}} dxdt \leq$$

$$\leq \sum_{m=0}^{K} \frac{1}{(1+m)^{1+\delta}} \int\limits_{A_m} |Du|^p dxdt \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{1}{(1+m)^{1+\delta}} (\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) \leq C(\delta),$$

where $C(\delta)$ is a positive constant independent of k. Thus it easy to see that $u \in \mathcal{T}_{0T}^{1,p}(Q)$.

***Proof of Theorem* 3.2.** If $f \in L^1(0,T; L^1 \log L^1(\Omega))$, $u_0 \in L^1 \log L^1(\Omega)$, let u be an entropy solution to problem (P). By virtue of $L^1(0,T; L^1 \log L^1(\Omega)) \subset L^1(Q)$, $L^1 \log L^1(\Omega) \subset L^1(\Omega)$, we can deduce $u \in \mathcal{T}_{0T}^{1,p}(Q)$ from Theorem 3.1.

For any given $k > 0$, if $0 < k \leq 1$, it's obvious. If $k > 1$, by (4.1) (here $h = m$, $k = 1$) and (4.4) (here $l = K$), we get

$$\int\limits_{Q} \frac{|DT_k(u)|^p}{1+|T_k(u)|} dxdt = \sum_{m=0}^{K} \int\limits_{A_m} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)} dxdt \leq$$

$$\leq \sum_{m=0}^{K} \frac{1}{1+m} \int\limits_{A_m} |Du|^p dxdt \leq$$

$$\leq \sum_{m=0}^{K} \frac{1}{1+m} \left(\int\limits_{\{|u|\geq m\}} |f| dxdt + \int\limits_{\{|u_0|\geq m\}} |u_0| dx \right) =$$

$$= \sum_{m=1}^{K} \frac{1}{m} \left(\int\limits_{\{|u|\geq m-1\}} |f| dxdt + \int\limits_{\{|u_0|\geq m-1\}} |u_0| dx \right) +$$

$$+ \frac{1}{K+1} \left(\int\limits_{\{|u|\geq K\}} |f| dxdt + \int\limits_{\{|u_0|\geq K\}} |u_0| dx \right) \leq$$

$$\leq 2 \int\limits_{Q} |f|(1+\ln(1+|T_K(u)|))dxdt +$$

$$+2 \int\limits_{\Omega} |u_0|(1+\ln(1+|T_K(u_0)|))dx \leq$$

$$\leq 2 \int\limits_{Q} |f|(1+\ln(1+|T_k(u)|))dxdt +$$

$$+2 \int\limits_{\Omega} |u_0|(1+\ln(1+|T_k(u_0)|))dx \leq$$

$$\leq 2 \int\limits_{Q} |f|(1+\ln(1+|T_k(u)|))dxdt +$$

$$+2 \int\limits_{\Omega} |u_0|(1+\ln(1+|u_0|))dx \leq$$

$$\leq C\big[\Theta(\|T_k(u)\|_{L^\infty(0,T;L^1(\Omega))}) + 1\big], \tag{4.5}$$

where C is a positive constant independent of k. The final inequality in (4.5) is due to $f \in \in L^1(0,T;L^1\log L^1(\Omega))$, $u_0 \in L^1\log L^1(\Omega)$. The details can be seen in [7]. Thus we get $u \in \mathcal{T}^{1,p}_{0T_1}(Q)$.

Proof of Theorem **3.3.** In the following, we only need to prove that for any given $k > 0$, $0 < \lambda < 1$, there exists a positive constant C independent of k such that

$$\int\limits_{Q} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda} dx \leq C \left(\int\limits_{Q} (1+|T_k(u)|)^{\frac{(1-\lambda)m}{m-1}} dxdt \right)^{1-\frac{1}{m}}.$$

In fact, by (4.1) (here $h = n$, $k = 1$, $u_0 = 0$), (4.2) (here $l = K$, $\tau = \lambda$) and Hölder's inequality, we obtain

$$\int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda} dxdt = \sum_{n=0}^{K} \int\limits_{A_n} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda} dxdt \le$$

$$\le \sum_{n=0}^{K} \frac{1}{(n+1)^\lambda} \int\limits_{A_n} |Du|^p dxdt \le$$

$$\le \sum_{n=0}^{K} \frac{1}{(n+1)^\lambda} \int\limits_{\{|u|\ge n\}} |f| dxdt =$$

$$= \sum_{n=1}^{K} \frac{1}{n^\lambda} \int\limits_{\{|u|\ge n-1\}} |f| dxdt + \frac{1}{(K+1)^\lambda} \int\limits_{\{|u|\ge K\}} |f| dxdt \le$$

$$\le \frac{2}{1-\lambda} \int\limits_Q |f|(1+|T_K(u)|)^{1-\lambda} dxdt \le$$

$$\le \frac{2}{1-\lambda} \int\limits_Q |f|(1+|T_k(u)|)^{1-\lambda} dxdt \le$$

$$\le \frac{2}{1-\lambda} \|f\|_{L^m(Q)} \left(\int\limits_Q (1+|T_k(u)|)^{\frac{(1-\lambda)m}{m-1}} dxdt \right)^{1-\frac{1}{m}}.$$

Proof of Theorem **3.4.** For any given $k > 0$, taking $\lambda = 2-\gamma$, (4.1) (here $h = m$, $k = 1$, $f = 0$), (4.3) (here $l = K$, $\tau = \lambda$) and Hölder's inequality imply that

$$\int\limits_Q \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda} dxdt = \sum_{m=0}^{K} \int\limits_{A_m} \frac{|DT_k(u)|^p}{(1+|T_k(u)|)^\lambda} dxdt \le$$

$$\le \sum_{m=0}^{K} \frac{1}{(1+m)^\lambda} \int\limits_{A_m} |Du|^p dxdt \le$$

$$\le \sum_{m=0}^{K} \frac{1}{(1+m)^\lambda} \int\limits_{\{|u_0|\ge m\}} |u_0| dx =$$

$$= \sum_{m=1}^{K} \frac{1}{m^\lambda} \int\limits_{\{|u_0|\ge m-1\}} |u_0| dx + \frac{1}{(K+1)^\lambda} \int\limits_{\{|u_0|\ge K\}} |u_0| dx \le$$

$$\leq \frac{2}{1-\lambda}\int\limits_{\Omega}|u_0|(1+|T_K(u_0)|)^{1-\lambda}dx \leq \frac{2}{1-\lambda}\int\limits_{\Omega}|u_0|(1+|T_k(u_0)|)^{1-\lambda}dx \leq$$

$$\leq \frac{2}{1-\lambda}\int\limits_{\Omega}|u_0|(1+|u_0|)^{1-\lambda}dx \leq \frac{2}{\gamma-1}\left(\int\limits_{\Omega}|u_0|dx+\int\limits_{\Omega}|u_0|^{\gamma}dxdt\right).$$

Hence $u \in \mathcal{T}^{1,p}_{0T_{2-\gamma}}(Q)$.

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