UDC 517.5

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ON THE OPTIMAL RECOVERY OF INTEGRALS OF SET-VALUED FUNCTIONS

ПРО ОПТИМАЛЬНЕ ВІДНОВЛЕННЯ ІНТЕГРАЛІВ ВІД БАГАТОЗНАЧНИХ ФУНКЦІЙ

We consider the problem of optimization of the approximate integration of set-valued functions from the class defined by a given majorant of their moduli of continuity performed by using the values of these functions at n fixed or free points of their domain.

Розглядається задача оптимізації наближеного інтегрування на класах багатозначних функцій, що мають задану мажоранту модуля неперервності. При цьому використовуються значення функцій в n фіксованих або довільних точках області визначення.

1. Introduction. By $K(\mathbb{R}^m)$ we denote the space of nonempty compact subsets of $\mathbb{R}^m$. Let $K^c(\mathbb{R}^m)$ be the set of convex elements of $K(\mathbb{R}^m)$. We consider below set-valued functions with nonempty compact images, i.e., functions $f\colon [0,1] \to K(\mathbb{R}^m)$.

Considerations of integration of set-valued functions go back to Minkowski and currently there exist many different approaches to the definition of integrals of set-valued functions (see, e.g., [1 – 8]). Integrals of such functions were found to be very applicable in many mathematical fields, especially in mathematical economics, control theory, integral geometry, and statistics. One of the most useful is Aumann integral [1] because this integral has many good properties. At the same time it is proved in [2] that Riemann – Minkowski integral for any continuous and bounded set-valued function exists and coincides with Aumann integral.

Theory of numerical integration is important part of approximation theory and numerical analysis and works of many mathematicians were devoted to the problems of optimization of quadrature formulas in various settings for the classes of real-valued functions. For surveys of obtained results see, e.g., [9 – 12]. Estimates of deviation of Riemann sums and some other methods of approximate calculation of integrals from the corresponding integrals of set-valued functions were considered in the works [13 – 16]. Articles [17, 18] are devoted to the optimization of quadrature formulas on classes of monotone with respect to inclusion convex-valued functions.

The goal of this paper is to consider the problems of optimization of approximate calculation of Riemann – Minkowski integrals of set-valued functions from the class defined by given majorant of their moduli of continuity (not necessary convex-valued functions) using values of the functions at n fixed or n free points of their domain. Since Riemann – Minkowski integral is always a convex set, it is not natural to use direct analogs of usual quadrature formulas. Instead, we consider these problems from optimal recovery theory point of view.

Note that the theory of optimal recovery of functions, functionals, and operators has been incrementally developed since mid 1960. Statements of the problems and surveys of obtained results can be found in [19 – 27] and others.

Our paper is organized as follows. In Section 2 we present some necessary definitions and facts from set-valued analysis. Statements of problems of optimal recovery of weighted integrals of set-valued functions using exact values of the functions at n fixed or n free points of the domain are presented in Section 3. Solution of the problem of optimal recovery of weighted integrals of set-valued functions using exact values of the functions at n fixed points of the domain on the class of functions having given majorant of moduli of continuity is presented in Section 4. The problem of optimal recovery in the case when information is given at n free points of a domain is discussed in Section 5.

2. Preliminaries. In this section we present some definitions and facts from theory of set-valued functions.

As usual, a linear combination of sets $A, B \subset K(\mathbb{R}^m)$ is defined by

$$\lambda A + \mu B = \{\lambda a + \mu b \colon a \in A, b \in B\}, \quad \lambda, \mu \in \mathbb{R}.$$

Convex hull, denoted by $\mathrm{co}A$, of a set $A \subset K(\mathbb{R}^m)$ is the set of all elements of the form $\sum_{i=1}^{r} \lambda_i a_i$, where $r \geq 2$, $a_i \in A$, $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0$ for $i = 1, \ldots, r$, and $\sum_{i=1}^{r} \lambda_i = 1$. Convex hull has the following properties:

$$\mathrm{co}\,(\mu A) = \mu\, \mathrm{co}A \quad \forall \mu \in \mathbb{R},\ A \subset K(\mathbb{R}^m),$$

$$\mathrm{co}\,(A + B) = \mathrm{co}A + \mathrm{co}B \quad \forall A, B \subset K(\mathbb{R}^m).$$

If $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$, then $\|a\| := \sqrt{\sum_{j=1}^{m} {a_j}^2}$. For a point $a \in \mathbb{R}^m$, and a set $B \in K(\mathbb{R}^m)$, let $d(a, B) := \inf_{b \in B} \|a - b\|$ be the distance from the point a to the set B. For sets $A, B \in K(\mathbb{R}^m)$ let

$$d(A, B) := \sup_{a \in A} d(a, B)$$

be the distance from the set A to the set B. Hausdorff metric δ in the space $K(\mathbb{R}^m)$ is defined as follows. If $A, B \in K(\mathbb{R}^m)$, then

$$\delta(A, B) := \max\{d(A, B), d(B, A)\}.$$

Note that $K(\mathbb{R}^m)$ endowed with Hausdorff metric is a complete metric space.

Metric $\delta\,(A, B)$ has the following properties:

$$\delta\,(\lambda A, \lambda B) = \lambda \delta\,(A, B) \quad \forall \lambda > 0 \quad \forall A, B \in K(\mathbb{R}^m),$$

$$\delta\,(A + B, C + D) \leq \delta\,(A, C) + \delta\,(B, D) \quad \forall A,\ B,\ C,\ D \in K(\mathbb{R}^m),$$

$$\delta(\mathrm{co}\ A, \mathrm{co}\ B) \leq \delta(A, B) \quad \forall A, B \in K(\mathbb{R}^m).$$

One can find proofs of all properties presented above in [3] and [28].

The Aumann's integral of a globally bounded set-valued function $f \colon [0, 1] \to K(\mathbb{R}^m)$ is defined as the set of all integrals of integrable selections of f [1]:

$$I(f) = \int_0^1 f(x)dx := \left\{ \int_0^1 \phi(x)dx \ :\ \phi(x) \in f(x) \text{ a.e.}, \ \phi \text{ is integrable} \right\}.$$

The Riemann – Minkowski sum of f is defined in the following way. Let $P = \{x_0, x_1, \ldots, x_n\}$, $0 = x_0 < x_1 < \ldots < x_n = 1$, be some partition of the interval $[0, 1]$. We set $\Delta x_i = x_i - x_{i-1}$,

$\lambda(P) = \max\{|\Delta x_i| : i = 1, \ldots, n\}$, and $\xi = \{\xi_1, \ldots, \xi_n\}$, $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$. The Riemann–Minkowski sum of f relative to the pair (P, ξ) is defined as

$$\sigma(f; (P, \xi)) := \sum_{i=1}^{n} \Delta x_i \cdot f(\xi_i).$$

We define the standard base $\lambda(P) \to 0$ in the set of all pairs (P, ξ) as follows [29]:

$$\lambda(P) \to 0 := \{\mathcal{B}_\epsilon\}_{\epsilon > 0}, \ \ \mathcal{B}_\epsilon := \{(P, \xi) \ : \ \lambda(P) < \epsilon\}.$$

A function f is integrable in the Riemann–Minkowski sense if (see [2, 4]) there exists an element $I(f) \in K(\mathbb{R}^m)$ such that

$$\delta(\sigma(f; (P, \xi)), I(f)) \to 0 \ \text{ as } \ \lambda(P) \to 0.$$

It is proved in [2] that Riemann–Minkowski integral for any continuous and bounded set-valued function exists and coincides with Aumann integral.

By $\mathcal{RM}([0,1], K(\mathbb{R}^m))$ we denote the set of functions which are integrable in the Riemann–Minkowski sense. Note that bounded and continuous functions $f \colon [0,1] \to K(\mathbb{R}^m)$ belong to $\mathcal{RM}([0,1], K(\mathbb{R}^m))$, and the product $P \cdot f$ of a continuous real-valued function P and a function $f \in \mathcal{RM}([0,1], K(\mathbb{R}^m))$ belongs to $\mathcal{RM}([0,1], K(\mathbb{R}^m))$. Below we denote by $\int_0^1 f(x)dx$ the Riemann–Minkowski integral for functions $f \in \mathcal{RM}([0,1], K(\mathbb{R}^m))$.

Riemann–Minkowski integral has the following properties (see [3, 4, 28])

$$\int_0^1 f(x)dx \in K^c(\mathbb{R}^m) \quad \forall f \in \mathcal{RM}([0,1], K(\mathbb{R}^m)),$$

$$\int_0^1 \operatorname{co}(f(x))dx = \int_0^1 f(x)dx \quad \forall f \in \mathcal{RM}([0,1], K(\mathbb{R}^m)),$$

$$\int_0^1 \lambda f(x)dx = \lambda \int_0^1 f(x)dx \quad \forall f \in \mathcal{RM}([0,1], K(\mathbb{R}^m)) \quad \forall \lambda \in \mathbb{R},$$

$$\int_0^1 (f(x) + g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx \quad \forall f, g \in \mathcal{RM}([0,1], K(\mathbb{R}^m)),$$

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx = \int_0^1 f(x)\, dx \quad \forall f \in \mathcal{RM}([0,1], K(\mathbb{R}^m)),$$

$$\delta\left(\int_0^1 f(x)dx, \int_0^1 g(x)dx\right) \le \int_0^1 \delta(f(x), g(x))dx \quad \forall f, g \in \mathcal{RM}([0,1], K(\mathbb{R}^m)).$$

3. Setting of the problems. Let $\mathcal{M}$ be some class of Riemann–Minkowski integrable functions $f\colon [0,1] \to K(\mathbb{R}^m)$, i.e., $\mathcal{M} \subset \mathcal{RM}([0,1], K(\mathbb{R}^m))$. Let continuous, nonnegative almost everywhere function $P\colon [0,1] \to \mathbb{R}$ be given. Let also a set of points $\overline{x} = \{x_1, \ldots, x_n\}$, $0 \le x_1 < x_2 < \ldots$ $\ldots < x_n \le 1$, be given. We consider a problem of optimal recovery of the integral

$$\int_0^1 P(x)f(x)dx$$

on the class $\mathcal{M}$, using information $f(x_1), \ldots, f(x_n)$.

Arbitrary convex-valued mapping

$$\Phi\colon \underbrace{K(\mathbb{R}^m) \times \ldots \times K(\mathbb{R}^m)}_{n \text{ times}} \to K^c(\mathbb{R}^m)$$

is called a method of recovery of this integral.

The problem of finding the optimal method of recovery is formulated in the following way. Let

$$R(\mathcal{M}, \overline{x}, \Phi) \;:=\; \sup_{f \in \mathcal{M}} \delta\left(\int_0^1 P(x)f(x)dx, \Phi(f(x_1), \ldots, f(x_n))\right).$$

This value is called the error of a method Φ on the class $\mathcal{M}$. Let also

$$R(\mathcal{M}, \overline{x}) \;:=\; \inf_{\Phi} R(\mathcal{M}, \overline{x}, \Phi). \tag{1}$$

The mapping $\overline{\Phi}$, that realizes $\inf\limits_{\Phi}$ on the right-hand side of (1), is called optimal for the class $\mathcal{M}$ for a fixed set of knots $\overline{x}$.

Problem 1. *Find the value*

$$R(\mathcal{M}, \overline{x}) \;=\; \inf_{\Phi} R(\mathcal{M}, \overline{x}, \Phi),$$

and optimal method $\overline{\Phi}$.

Let now

$$R_n(\mathcal{M}) := \inf_{\#(\overline{x})=n} R(\mathcal{M}, \overline{x}), \tag{2}$$

where $\#(\overline{x})$ is the number of elements in the set $\overline{x}$.

This value is called the optimal error of recovery using n knots on the class $\mathcal{M}$, and the set $\overline{x}^*$ that realizes $\inf\limits_{\#(\overline{x})=n}$ on the right-hand part of (2) is called an optimal set of knots.

Problem 2. *Find the value* $R_n(\mathcal{M})$, *optimal set of knots* $\overline{x}^*$, *and optimal on the class* $\mathcal{M}$ *method* $\overline{\Phi}$ *that uses values* $f(x_1^*), \ldots, f(x_n^*)$.

We solve Problems 1 and 2 for the following classes of set-valued functions. Given modulus of continuity $\omega(t)$, we denote by $H^\omega([0,1], K(\mathbb{R}^m))$ the class of functions $f\colon [0,1] \to K(\mathbb{R}^m)$ such that,

$$\delta(f(x'), f(x'')) \le \omega(|x' - x''|) \quad \forall\, x', x'' \in [0,1].$$

In Section 6 we consider the problems of optimal recovery of integrals on the class $H^\omega([0,1], K(\mathbb{R}^m))$ using information with error.

4. Solution of the Problem 1 for $\mathcal{M} = H^\omega([0,1], K(\mathbb{R}^m))$. Let the set of knots $\overline{x} = \{x_1, \ldots, x_n\}$ be given. We define

$$\Pi_i(\overline{x}) = \{x \in [0,1] \colon \min_{j=1,\ldots,n} |x - x_j| = |x - x_i|\},$$

$$c_i^* = c_i^*(P, \overline{x}) = \int\limits_{\Pi_i(\overline{x})} P(x)dx, \quad i = 1, \ldots, n.$$

In particular if $P(x) \equiv 1$, then

$$c_1^* = c_1^*(\overline{x}) = \frac{x_1 + x_2}{2},$$

$$c_i^* = c_i^*(\overline{x}) = \frac{x_{i+1} - x_{i-1}}{2}, \text{ if } 1 < i < n,$$

$$c_n^* = c_n^*(\overline{x}) = 1 - \frac{x_{n-1} + x_n}{2}.$$

In addition, we define $f_{\omega,\overline{x}}(x) := \omega\left(\min_{i=1,\ldots,n} |x - x_i|\right)$.

Theorem 1. *Let a modulus of continuity $\omega(t)$ and a set of points $\overline{x} = \{x_1, \ldots, x_n\}$ be given. The optimal method of the recovery of integral $\int_0^1 P(x)f(x)dx$ on the class $H^\omega([0,1], K(\mathbb{R}^m))$, using information $f(x_1), \ldots, f(x_n)$, is*

$$\Phi^*(f(x_1), \ldots, f(x_n)) = \mathrm{co}\left(\sum_{k=1}^n c_k^*(P, \overline{x}) f(x_k)\right),$$

and optimal error of recovery is

$$R(H^\omega([0,1], K(\mathbb{R}^m)), \overline{x}) = \int\limits_0^1 P(x) f_{\omega,\overline{x}}(x)dx.$$

Proof. Using properties of Riemann–Minkowski integral, Hausdorff metric, and convex hull presented in Section 2, we have for any $f \in H^\omega([0,1], K(\mathbb{R}^m))$

$$\delta\left(\int\limits_0^1 P(x)f(x)dx, \mathrm{co}\left(\sum_{i=1}^n c_i^*(P; \overline{x}) f(x_i)\right)\right) =$$

$$= \delta\left(\sum_{i=1}^n \int\limits_{\Pi_i(\overline{x})} P(x)\mathrm{co}f(x)dx, \sum_{i=1}^n \int\limits_{\Pi_i(\overline{x})} P(x)dx \cdot \mathrm{co}f(x_i)\right) \le$$

$$\le \sum_{i=1}^n \delta\left(\int\limits_{\Pi_i(\overline{x})} P(x)\mathrm{co}f(x)dx, \int\limits_{\Pi_i(\overline{x})} P(x)\mathrm{co}f(x_i)dx\right) \le$$

$$\leq \sum_{i=1}^{n} \int_{\Pi_i(\overline{x})} P(x)\delta(\mathrm{co}f(x), \mathrm{co}f(x_i))dx \leq \sum_{i=1}^{n} \int_{\Pi_i(\overline{x})} P(x)\delta(f(x), f(x_i))dx \leq$$

$$\leq \sum_{i=1}^{n} \int_{\Pi_i(\overline{x})} P(x)\omega(|x - x_i|)dx = \int_0^1 P(x)f_{\omega,\overline{x}}(x)dx.$$

Consequently,

$$R(H^\omega([0,1], K(\mathbb{R}^m)), \overline{x}) \leq R(H^\omega([0,1], K(\mathbb{R}^m)), \overline{x}, \Phi^*) \leq \int_0^1 P(x)f_{\omega,\overline{x}}(x)dx. \qquad (3)$$

We obtained the estimate from above for the value $R(H^\omega([0,1], K(\mathbb{R}^m)), \overline{x})$. Next we obtain the estimate from below.

We choose an arbitrary $a \in \mathbb{R}^m$ such that $\delta(\{a\}, \{\theta\}) = \|a\| = 1$, where $\theta = (0, \ldots, 0) \in \mathbb{R}^m$, and define $f_{\omega,\overline{x},a} \colon [0,1] \to K(\mathbb{R}^m)$ with the help of the equality

$$f_{\omega,\overline{x},a}(x) := f_{\omega,\overline{x}}(x) \cdot \{a\}.$$

Note that $f_{\omega,\overline{x},a}(x) \in H^\omega([0,1], K(\mathbb{R}^m))$, $f_{\omega,\overline{x},a}(x_k) = \{\theta\}$, $k = 1, \ldots, n$, and

$$\int_0^1 P(x)f_{\omega,\overline{x},a}(x)dx = \int_0^1 P(x)f_{\omega,\overline{x}}(x)dx \cdot \{a\}.$$

For an arbitrary method of recovery Φ, we get

$$R(H^\omega([0,1], K(\mathbb{R}^m)), \overline{x}, \Phi) =$$

$$= \sup_{f \in H^\omega([0,1], K(\mathbb{R}^m))} \left(\delta \int_0^1 P(x)f(x)dx, \Phi\left(f(x_1), \ldots, f(x_n)\right) \right) \geq$$

$$\geq \max\left\{ \delta\left(\int_0^1 P(x)f_{\omega,\overline{x},a}(x)dx, \Phi\left(\{\theta\}, \ldots, \{\theta\}\right) \right), \right.$$

$$\left. \delta\left(-\int_0^1 P(x)f_{\omega,\overline{x},a}(x)dx, \Phi\left(\{\theta\}, \ldots, \{\theta\}\right) \right) \right\} =$$

$$= \max\left\{ \delta\left(\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx \cdot \{a\}, \Phi\left(\{\theta\}, \ldots, \{\theta\}\right) \right), \right.$$

$$\left. \delta\left(-\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx \cdot \{a\}, \Phi\left(\{\theta\}, \ldots, \{\theta\}\right) \right) \right\} \geq$$

$$\geq \frac{1}{2}\left\{\delta\left(\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\cdot\{a\},\Phi\left(\{\theta\},\ldots,\{\theta\}\right)\right)+\right.$$

$$\left.+\delta\left(-\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\cdot\{a\},\Phi\left(\{\theta\},\ldots,\{\theta\}\right)\right)\right\}\geq$$

$$\geq \frac{1}{2}\delta\left(\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\cdot\{a\},-\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\cdot\{a\}\right)=$$

$$=\frac{1}{2}\delta\left(2\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\cdot\{a\},\{\theta\}\right)=\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx\ \cdot\delta\left(\{a\},\{\theta\}\right)=$$

$$=\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx.$$

Therefore, for an arbitrary method Φ

$$R(H^\omega([0,1],K(\mathbb{R}^m)),\overline{x},\Phi)\geq\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx. \tag{4}$$

Comparing relations (3) and (4), we obtain the statement of Theorem 1.

Remark. Theorem 1 generalizes results of N. P. Korneichuk [30] and G. K. Lebed' [31] for real-valued functions. Multivariate analogs of their results were obtained by V. F. Babenko (see [32–34]).

5. Optimal recovery of integrals using n free knots. It follows from Korneichuk and Lebed' results, that Problem 2 will be solved for the class $H^\omega([0,1],K(\mathbb{R}^m))$ if we find the set of knots $\overline{x}^0$ that realizes

$$\inf_{\overline{x}}\int_0^1 P(x)f_{\omega,\overline{x}}(x)dx.$$

Then the optimal method is

$$\Phi^*(f(x_1^0),\ldots,f(x_n^0))=\mathrm{co}\left(\sum_{k=1}^n c_k^*(P,\overline{x}^0)f(x_k^0)\right).$$

In addition,

$$R_n(H^\omega([0,1],K(\mathbb{R}^m)))=\int_0^1 P(x)f_{\omega,\overline{x}^0}(x)dx.$$

Comparing this fact and Korneichuk's result from [30], we obtain that in the case $P(x)\equiv 1$ the following theorem holds.

Theorem 2. *Let a modulus of continuity $\omega(t)$ and a number $n \in \mathbb{N}$ be given. Then*

$$R_n(H^\omega([0,1], K(\mathbb{R}^m))) = 2n \int\limits_0^{\frac{1}{2n}} \omega(t)dt,$$

optimal set of knots is $\overline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) := \left(\frac{1}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}\right)$, *and the method*

$$\Phi(f(x_1^*), \ldots, f(x_n^*)) = \mathrm{co}\left(\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right)\right)$$

is optimal on the class $H^\omega([0,1], K(\mathbb{R}^m))$ among all methods of recovery of the integral $\int_0^1 f(x)dx$ *that use the information of the form* $f(x_1), f(x_2), \ldots, f(x_n)$.

In the case when $P(x)$ is not identically equal to 1 we can not obtain the explicit expressions for the optimal knots and the explicit value for $R_n(H^\omega([0,1], K(\mathbb{R}^m)))$. However, using the results from [35] we can obtain the exact asymptotics for this value (under some additional assumptions) when n tends to ∞.

Let $c > 0$ be given. Let $\Omega(x) := \int_0^x \omega(t/2)dt$, $\gamma_c(x) := \Omega^{-1}(c\Omega(x))$, and

$$B(P, \omega) := \lim_{n\to\infty} \sum_{k=1}^{n} \Omega^{-1}\left(P\left(\frac{2k-1}{2n}\right)\Omega\left(\frac{1}{n}\right)\right).$$

Theorem 3. *Let a modulus of continuity $\omega(x)$ be such that for all $c > 0$ the function $\gamma_c(x)/x$ is monotone in the right neighborhood of zero. Let also weight function P be continuous and positive almost everywhere on $[0,1]$. Then*

$$\limsup_{n\to\infty} \frac{R_n\left(H^\omega([0,1], K(\mathbb{R}^m))\right)}{n\Omega(B/n)} = 1.$$

***Corollary* 1.** *Let $P(x)$ be the same as in the previous theorem. Let also $\omega(x) = x^\alpha$, $\alpha \in (0,1]$. Then*

$$R_n\left(H^\omega([0,1], K(\mathbb{R}^m))\right) = \frac{(2n)^{-\alpha}}{\alpha+1}\left(\int\limits_0^1 P(x)^{\frac{1}{1+\alpha}}dx\right)^{\alpha+1} + o\left(\frac{1}{n^\alpha}\right), \quad n\to\infty.$$

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Received 29.05.14