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RELATIVE EXTENSIONS OF MODULES AND HOMOLOGY GROUPS*
ВІДНОСНІ РОЗШИРЕННЯ МОДУЛІВ ТА ГОМОЛОГІЧНИХ ГРУП

We introduce the concepts of relative (co)extensions of modules and explore the relationship between the relative (co)extensions of modules and relative (co)homology groups. Some applications are given.

Введено поняття відносних (спів)розширень модулів та вивчено взаємозв'язок між відносними (спів)розширеннями модулів та відносними (ко)гомологічними групами.

1. Introduction. In classical homological algebra, given right R-modules M, N and a left R-module L, the cohomology group $\mathrm{Ext}^n(M,N)$ is obtained by using a right injective resolution of N or a left projective resolution of M, and the homology group $\mathrm{Tor}_n(M,L)$ is obtained by using a left projective (flat) resolution of M or L. In relative homological algebra [5], if $\mathcal{G}$ is a preenveloping class of right R-modules, then we can get the relative cohomology group $\mathrm{Ext}^n_{\mathcal{G}}(M,N)$ computed by the right $\mathcal{G}$-resolution of N. Similarly, if $\mathcal{F}$ is a precovering class of right R-modules, then we can get the relative cohomology group ${}_{\mathcal{F}}\mathrm{Ext}^n(M,N)$ and the relative homology group ${}_{\mathcal{F}}\mathrm{Tor}_n(M,L)$ computed by the left $\mathcal{F}$-resolution of M.

The main goal of the present paper is to extend some important properties of classical (co)homology groups to relative (co)homological groups. We introduce the concepts of an $\mathcal{F}$-extension and a $\mathcal{G}$-coextension of modules, where $\mathcal{F}$ and $\mathcal{G}$ denote two classes of right R-modules. It is proven that the set of all equivalence classes of $\mathcal{F}$-extensions (resp. $\mathcal{G}$-coextensions) of A by C, denoted by ${}_{\mathcal{F}}E(C,A)$ (resp. $E_{\mathcal{G}}(C,A)$), is an Abelian group. Moreover, we prove that $\mathrm{Ext}^1_{\mathcal{G}}(C,A)\cong E_{\mathcal{G}}(C,A)$ if $\mathcal{G}$ is a monic preenveloping class and ${}_{\mathcal{F}}\mathrm{Ext}^1(C,A)\cong {}_{\mathcal{F}}E(C,A))$ if $\mathcal{F}$ is an epic precovering class. As applications, we obtain several properties of relative (co)homology groups. For example, if $\mathcal{F}$ is an epic precovering class of right R-modules, then we prove that: (1) there is a monomorphism ${}_{\mathcal{F}}\mathrm{Ext}^1(C,A)\to \mathrm{Ext}^1(C,A)$ for all right R-modules A and C; (2) there is an epimorphism $\mathrm{Tor}_1(A,B)\to {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)$ for any right R-module A and any left R-module B. In addition, we give a relative version of Wakamatsu's lemmas.

We next recall some notions and facts needed in the sequel.

Following [3], we say that a right R-module homomorphism $\phi\colon M\to G$ is a *$\mathcal{G}$-preenvelope* of M if $G\in\mathcal{G}$ and the Abelian group homomorphism $\phi^*\colon \mathrm{Hom}(G,G')\to\mathrm{Hom}(M,G')$ is surjective for each $G'\in\mathcal{G}$. A $\mathcal{G}$-preenvelope $\phi\colon M\to G$ is said to be a *$\mathcal{G}$-envelope* of M if every endomorphism $g\colon G\to G$ such that $g\phi=\phi$ is an isomorphism. Dually we have the definitions of an *$\mathcal{F}$-precover* and an *$\mathcal{F}$-cover*. $\mathcal{G}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

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We say that $\mathcal{G}$ is a (resp. *monic*) *preenveloping class* of right R-modules [8] if every right R-module has a (resp. monic) $\mathcal{G}$-preenvelope. Dually, $\mathcal{F}$ is called a (resp. *epic*) *precovering class* of right R-modules if every right R-module has an (resp. epic) $\mathcal{F}$-precover.

Let $\mathcal{G}$ be a preenveloping class. Then any right R-module N has a *right $\mathcal{G}$-resolution*, i.e., there is a cocomplex $0 \to N \to G^0 \to G^1 \to \ldots$ with each $G^i \in \mathcal{G}$ such that $\ldots \to \mathrm{Hom}(G^1, G) \to$ $\to \mathrm{Hom}(G^0, G) \to \mathrm{Hom}(N, G) \to 0$ is exact for any $G \in \mathcal{G}$. Let $\mathbf{G}^{\cdot}$ be the deleted cocomplex corresponding to a right $\mathcal{G}$-resolution of N, which is unique up to homotopy, then for a right R-module M, we obtain the nth cohomology group of the cocomplex $\mathrm{Hom}(M, \mathbf{G}^{\cdot})$, denoted by $\mathrm{Ext}^n_{\mathcal{G}}(M, N)$ (see [5], 8.2). Particularly, if $\mathcal{G}$ is the class of injective right R-modules, then $\mathrm{Ext}^n_{\mathcal{G}}(M, N)$ is just the classical cohomology group $\mathrm{Ext}^n(M, N)$.

Dually, let $\mathcal{F}$ be a precovering class, then any right R-module M has a *left $\mathcal{F}$-resolution*, i.e., there is a complex $\ldots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathcal{F}$ such that $\ldots \to \mathrm{Hom}(F, F_1) \to$ $\to \mathrm{Hom}(F, F_0) \to \mathrm{Hom}(F, M) \to 0$ is exact for any $F \in \mathcal{F}$. Let $\mathbf{F}_{\cdot}$ be the deleted complex corresponding to a left $\mathcal{F}$-resolution of M, which is unique up to homotopy. Then for a right R-module N, we obtain the nth cohomology group of the cocomplex $\mathrm{Hom}(\mathbf{F}_{\cdot}, N)$, denoted by ${}_{\mathcal{F}}\mathrm{Ext}^n(M, N)$ (see [5], 8.2). In addition, for a left R-module L, we get the nth homology group of the complex $\mathbf{F}_{\cdot} \otimes L$, denoted by ${}_{\mathcal{F}}\mathrm{Tor}_n(M, L)$. Particularly, if $\mathcal{F}$ is the class of projective right R-modules, then ${}_{\mathcal{F}}\mathrm{Ext}^n(M, N)$ is just the classical cohomology group $\mathrm{Ext}^n(M, N)$ and ${}_{\mathcal{F}}\mathrm{Tor}_n(M, L)$ is just the classical homology group $\mathrm{Tor}_n(M, L)$.

Throughout this paper, R is an associative ring with identity and all modules are unitary. All classes of modules are closed under isomorphisms and direct summands. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+. The reader is referred to [5, 6, 8, 10, 12, 14] for unexplained concepts and notations.

2. Relative homology groups and relative extensions of modules. Let A and C be two right R-modules. Then an exact sequence $0 \to A \to B \to C \to 0$ is called an *extension* of A by C [10]. We first introduce the concepts of relative (co)extensions as follows.

Definition 2.1. *Given a class $\mathcal{F}$ of right R-modules, an exact sequence* $0 \to A \to B \to C \to 0$ *of right R-modules is said to be an* $\mathcal{F}$-extension *of A by C if*

$$0 \to \mathrm{Hom}(F, A) \to \mathrm{Hom}(F, B) \to \mathrm{Hom}(F, C) \to 0$$

is exact for any $F \in \mathcal{F}$.

Dually, given a class $\mathcal{G}$ of right R-modules, an exact sequence $0 \to A \to B \to C \to 0$ *of right R-modules is called a* $\mathcal{G}$-coextension *of A by C if* $0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0$ *is exact for any $G \in \mathcal{G}$.*

***Remark* 2.1.** (1) Let $\mathcal{F}$ (resp. $\mathcal{G}$) be the class of projective (resp. injective) right R-modules, then an $\mathcal{F}$-extension (resp. a $\mathcal{G}$-coextension) of A by C is just the usual extension of A by C.

(2) Let $\mathcal{F}$ (resp. $\mathcal{G}$) be the class of pure-projective (resp. pure-injective) right R-modules, then an $\mathcal{F}$-extension (resp. a $\mathcal{G}$-coextension) of A by C is just a pure exact sequence $0 \to A \to B \to C \to$ $\to 0$.

(3) Let $\mathcal{G}$ be the class of cotorsion right R-modules (A right R-module M is called *cotorsion* [4] if $\mathrm{Ext}^1(F, M) = 0$ for every flat right R-module F), then a $\mathcal{G}$-coextension of A by C is just an exact sequence $0 \to A \to B \to C \to 0$ with $A \to B$ a strongly pure monomorphism in the sense of [9].

Two $\mathcal{F}$-extensions ($\mathcal{G}$-coextensions) Δ and Δ' of A by C are called *equivalent* if there is $\sigma\colon B \to B'$ such that the following diagram is commutative:

$$\begin{tikzcd}
\Delta : 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dotted, "\sigma"'] & C \arrow[r] \arrow[d, equal] & 0 \\
\Delta' : 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 .
\end{tikzcd}$$

By the Five lemma, the middle homomorphism σ is an isomorphism. So the equivalence of $\mathcal{F}$-extensions ($\mathcal{G}$-coextensions) is a reflexive, symmetric and transitive relation. We write $_{\mathcal{F}}E(C, A)$ (resp. $E_{\mathcal{G}}(C, A)$) to be the set of all equivalence classes of $\mathcal{F}$-extensions (resp. $\mathcal{G}$-coextensions) of A by C.

If $\mathcal{F}$ (resp. $\mathcal{G}$) is the class of projective (resp. injective) right R-modules, it is well known that $_{\mathcal{F}}E(C, A)$ (resp. $E_{\mathcal{G}}(C, A)$) is an Abelian group using the so called Baer sum (see [10]). We can extend this result to a more general setting as follows.

Theorem 2.1. *The following are true for right R-modules A and C:*

(1) *$E_{\mathcal{G}}(C, A)$ is an Abelian group for any class $\mathcal{G}$ of right R-modules.*

(2) *$_{\mathcal{F}}E(C, A)$ is an Abelian group for any class $\mathcal{F}$ of right R-modules.*

Proof. (1) Let $\Delta_1 : 0 \to A \overset{i_1}{\to} B_1 \overset{\pi_1}{\to} C \to 0$ and $\Delta_2 : 0 \to A \overset{i_2}{\to} B_2 \overset{\pi_2}{\to} C \to 0$ be two $\mathcal{G}$-coextensions of A by C. Then we get the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, "i_2"] & B_1 \arrow[r, "\pi_1"] \arrow[d, "f"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_2 \arrow[r, "g"] \arrow[d, "\pi_2"] & H_{12} \arrow[r] \arrow[d] & C \arrow[r] & 0 , \\
 & C \arrow[r, equal] \arrow[d] & C \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $H_{12} = (B_1 \oplus B_2)/W, W = \{(i_1(a), -i_2(a)) : a \in A\}, f(b_1) = \overline{(b_1, 0)}$ for $b_1 \in B_1$, $g(b_2) = \overline{(0, b_2)}$ for $b_2 \in B_2$. Let $Q = \{(x, y) : \pi_1(x) = \pi_2(y), x \in B_1, y \in B_2\} \subseteq B_1 \oplus B_2$. Then $W \subseteq Q$. Put $Y_{12} = Q/W \subseteq H_{12}$. Then we get the sequence

$$\Psi_{12} : 0 \to A \overset{\lambda_{12}}{\to} Y_{12} \overset{\tau_{12}}{\to} C \to 0,$$

where $\lambda_{12}(a) = \overline{(i_1(a), 0)}$ for $a \in A$ and $\tau_{12}(\overline{(x, y)}) = \pi_1(x) = \pi_2(y)$ for $(x, y) \in Q$.

We first claim that Ψ_{12} is exact. In fact, it is clear that λ_{12} is monic, τ_{12} is epic and $\tau_{12}\lambda_{12} = 0$. If $\tau_{12}(\overline{(x, y)}) = 0$, then $x = i_1(a_1)$ and $y = i_2(a_2)$ for some $a_1, a_2 \in A$. Thus $\overline{(x, y)} = \overline{(i_1(a_1), i_2(a_2))} = \overline{(i_1(a_1 + a_2), 0)} = \lambda_{12}(a_1 + a_2)$. So Ψ_{12} is exact.

We now prove that Ψ_{12} is a $\mathcal{G}$-coextension of A by C. In fact, let $G \in \mathcal{G}$ and $\alpha \in \mathrm{Hom}(A, G)$, then there exist $\beta_1 \in \mathrm{Hom}(B_1, G)$ and $\beta_2 \in \mathrm{Hom}(B_2, G)$ such that $\alpha = \beta_1 i_1$ and $\alpha = \beta_2 i_2$ by hypothesis. Thus by the property of a pushout, there exists $\xi \in \mathrm{Hom}(H_{12}, G)$ such that the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "i_2"'] & B_1 \arrow[d, "f"'] \arrow[ddr, "\beta_1"] & \\
B_2 \arrow[r, "g"] \arrow[drr, "\beta_2"'] & H_{12} \arrow[dr, dotted, "\xi"] & \\
 & & G\,.
\end{tikzcd}$$

Write $\epsilon\colon Y_{12} \to H_{12}$ to be the inclusion. Then

$$\alpha = \beta_1 i_1 = \xi f i_1 = (\xi\epsilon)\lambda_{12}.$$

So $\mathrm{Hom}(Y_{12}, G) \to \mathrm{Hom}(A, G)$ is epic, i.e., Ψ_{12} is a $\mathcal{G}$-coextension of A by C.

Define $[\Delta_1] + [\Delta_2] = [\Psi_{12}]$. It is obvious that $[\Delta_1] + [\Delta_2] = [\Delta_2] + [\Delta_1]$.

Let $\Delta_3\colon 0 \to A \overset{i_3}{\to} B_3 \overset{\pi_3}{\to} C \to 0$ be a $\mathcal{G}$-coextension of A by C. We next prove that $([\Delta_1] + [\Delta_2]) + [\Delta_3] = [\Delta_1] + ([\Delta_2] + [\Delta_3])$.

Let $([\Delta_1] + [\Delta_2]) + [\Delta_3] = [\Xi]$, where $\Xi\colon 0 \to A \overset{\omega}{\to} U/V \overset{\rho}{\to} C \to 0$ is a $\mathcal{G}$-coextension of A by C with $U = \{(\overline{(x,y)}, z)\colon \tau_{12}(\overline{(x,y)}) = \pi_3(z), \overline{(x,y)} \in Y_{12}, z \in B_3\} \subseteq Y_{12} \oplus B_3$, $V = \{(\lambda_{12}(a), -i_3(a))\colon a \in A\}$. Let $[\Delta_2] + [\Delta_3] = [\psi_{23}]$ and $[\Delta_1] + ([\Delta_2] + [\Delta_3]) = [\Lambda]$, where $\psi_{23}\colon 0 \to A \overset{\lambda_{23}}{\to} Y_{23} \overset{\tau_{23}}{\to} C \to 0$ is a $\mathcal{G}$-coextension of A by C and $\Lambda\colon 0 \to A \overset{\mu}{\to} M/N \overset{\nu}{\to} C \to 0$ is a $\mathcal{G}$-coextension of A by C with $M = \{(x, \overline{(y,z)})\colon \pi_1(x) = \tau_{23}(\overline{(y,z)}),\ x \in B_1,\ \overline{(y,z)} \in Y_{23}\} \subseteq B_1 \oplus Y_{23}$, $N = \{(i_1(a), -\lambda_{23}(a))\colon a \in A\}$.

Define $\sigma\colon U/V \to M/N$ by $\sigma(\overline{(\overline{(x,y)}, z)}) = \overline{(x, \overline{(y,z)})}$ for $\overline{(x,y)} \in Y_{12}, z \in B_3$. We claim that σ is well defined. In fact, if $\overline{(\overline{(x,y)}, z)} = \overline{0}$, then $(\overline{(x,y)}, z) = (\lambda_{12}(a), -i_3(a))$ for some $a \in A$. So $\overline{(x,y)} = \overline{(i_1(a), 0)}$, $z = -i_3(a)$. Thus $(x,y) - (i_1(a), 0) = (i_1(b), -i_2(b))$ for some $b \in A$. Hence $x = i_1(a+b)$, $y = -i_2(b)$. So $(x, \overline{(y,z)}) = (i_1(a+b), \overline{(-i_2(b), -i_3(a))}) = (i_1(a+b), \overline{(-i_2(a+b), 0)}) = (i_1(a+b), -\lambda_{23}(a+b)) \in N$. Thus $\overline{(x, \overline{(y,z)})} = \overline{0}$. Moreover, it is easy to verify that the following diagram is commutative:

$$\begin{tikzcd}
\Xi : 0 \arrow[r] & A \arrow[r, "\omega"] \arrow[d, equal] & U/V \arrow[r, "\rho"] \arrow[d, "\sigma"'] & C \arrow[r] \arrow[d, equal] & 0 \\
\Lambda : 0 \arrow[r] & A \arrow[r, "\mu"] & M/N \arrow[r, "\nu"] & C \arrow[r] & 0\,.
\end{tikzcd}$$

So $([\Delta_1] + [\Delta_2]) + [\Delta_3] = [\Delta_1] + ([\Delta_2] + [\Delta_3])$.

On the other hand, the split exact sequence $\mho\colon 0 \to A \overset{\iota}{\to} A \oplus C \overset{\kappa}{\to} C \to 0$ is clearly a $\mathcal{G}$-coextension of A by C. We claim that $[\Delta_1] + [\mho] = [\Delta_1]$. In fact, let $[\Delta_1] + [\mho] = [\psi_1]$, where $\psi_1\colon 0 \to A \overset{\lambda_1}{\to} Q_1/W_1 \overset{\tau_1}{\to} C \to 0$ is a $\mathcal{G}$-coextension of A by C with $W_1 = \{(i_1(a), -(a,0))\colon$

$a \in A\}, Q_1 = \{(x, (a, \pi_1(x))) \colon x \in B_1, a \in A\}$. Define $\sigma_1 \colon Q_1/W_1 \to B_1$ by $\sigma_1(\overline{(x, (a, \pi_1(x)))}) = x + i_1(a)$. It is easy to verify that σ_1 is well defined, $\sigma_1\lambda_1 = i_1$ and $\pi_1\sigma_1 = \tau_1$. Thus $[\mho]$ is the zero element in $E_{\mathcal{G}}(C, A)$.

Finally consider the exact sequence $\Delta_1' \colon 0 \to A \xrightarrow{-i_1} B_1 \xrightarrow{\pi_1} C \to 0$, which is obviously a $\mathcal{G}$-coextension of A by C. We claim that $[\Delta_1] + [\Delta_1'] = [\mho]$. In fact, let $[\Delta_1] + [\Delta_1'] = [\psi']$, where $\psi' \colon 0 \to A \xrightarrow{\lambda'} Q'/W' \xrightarrow{\tau'} C \to 0$ is a $\mathcal{G}$-coextension of A by C with $W' = \{(i_1(a), i_1(a)) \colon a \in A\}, Q' = \{(x, y) \colon \pi_1(x) = \pi_1(y), x, y \in B_1\} = \{(y + i_1(a), y) \colon a \in A, y \in B_1\}$. Define $\sigma' \colon Q'/W' \to A \oplus C$ by $\sigma'(\overline{(y + i_1(a), y)}) = (a, \pi_1(y))$. It is easy to verify that σ' is well defined, $\sigma'\lambda' = \iota$ and $\kappa\sigma' = \tau'$. So Δ_1' is the negative element of Δ_1 in $E_{\mathcal{G}}(C, A)$.

It follows that $E_{\mathcal{G}}(C, A)$ is an Abelian group.

(2) can be proved dually.

Theorem 2.1 is proved.

It is well known that, in standard homological algebra, the cohomological group $\mathrm{Ext}^1(C, A)$ is isomorphic to the group of all equivalence classes of extensions of A by C. This result can be generalized as follows.

Theorem 2.2. *The following are true*:

(1) *If $\mathcal{G}$ is a monic preenveloping class of right R-modules, then there is an Abelian group isomorphism $\mathrm{Ext}^1_{\mathcal{G}}(C, A) \cong E_{\mathcal{G}}(C, A)$ for all right R-modules A and C.*

(2) *If $\mathcal{F}$ is an epic precovering class of right R-modules, then there is an Abelian group isomorphism ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) \cong {}_{\mathcal{F}}E(C, A)$ for all right R-modules A and C.*

Proof. (1) Let $0 \to A \xrightarrow{d^0} G^0 \xrightarrow{d^1} G^1 \xrightarrow{d^2} G_2 \to \ldots$ be a right $\mathcal{G}$-resolution of A. Then we get the cocomplex

$$0 \to \mathrm{Hom}(C, G^0) \xrightarrow{d^1_*} \mathrm{Hom}(C, G^1) \xrightarrow{d^2_*} \mathrm{Hom}(C, G^2) \to \ldots .$$

So $\mathrm{Ext}^1_{\mathcal{G}}(C, A) = \ker(d^2_*)/\mathrm{im}(d^1_*)$.

Let $\Gamma \colon 0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a $\mathcal{G}$-coextension of A by C, then there exist $\varepsilon_0 \colon B \to G^0$ and $\varepsilon_1 \colon C \to G^1$ such that the following diagram with exact rows is commutative:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "\pi"] \arrow[d, dotted, "\varepsilon_0"'] & C \arrow[r] \arrow[d, dotted, "\varepsilon_1"'] & 0 \\
0 \arrow[r] & A \arrow[r, "d^0"] & G^0 \arrow[r, "d^1"] & G^1 \arrow[r] & G^2.
\end{tikzcd}$$

Note that $d^2\varepsilon_1\pi = d^2d^1\varepsilon_0 = 0$. Thus $d^2\varepsilon_1 = 0$, and so $\varepsilon_1 \in \ker(d^2_*)$.

Define $\Theta \colon E_{\mathcal{G}}(C, A) \to \mathrm{Ext}^1_{\mathcal{G}}(C, A)$ by $\Theta([\Gamma]) = \overline{\varepsilon_1}$. We claim that Θ is well defined. In fact, if there exist $\varepsilon_0' \colon B \to G^0$ and $\varepsilon_1' \colon C \to G^1$ such that the above diagram also commutes, then $(\varepsilon_0' - \varepsilon_0)i = 0$, so there exists $\chi \colon C \to G^0$ such that $\varepsilon_0' - \varepsilon_0 = \chi\pi$. Thus $(\varepsilon_1' - \varepsilon_1)\pi = d^1(\varepsilon_0' - \varepsilon_0) = d^1\chi\pi$. So $\varepsilon_1' - \varepsilon_1 = d^1\chi \in \mathrm{im}(d^1_*)$. Hence $\overline{\varepsilon_1} = \overline{\varepsilon_1'}$.

We now prove that Θ is a group homomorphism.

Let $\Upsilon \colon 0 \to A \xrightarrow{\iota} H \xrightarrow{\rho} C \to 0$ be a $\mathcal{G}$-coextension of A by C. Then there is the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, equal] & H \arrow[r, "\rho"] \arrow[d, dashed, "\gamma_0"'] & C \arrow[r] \arrow[d, dashed, "\gamma_1"'] & 0 \\
0 \arrow[r] & A \arrow[r, "d^0"] & G^0 \arrow[r, "d^1"] & G^1 \arrow[r] & G^2\,.
\end{tikzcd}$$

Let $[\Gamma]+[\Upsilon]=[\Psi]$, where $\Psi\colon 0\to A\xrightarrow{\lambda} Q/W\xrightarrow{\tau} C\to 0$ is a $\mathcal{G}$-coextension of A by C with $Q=\{(x,y)\colon \pi(x)=\rho(y),\ x\in B,\ y\in H\}$ and $W=\{(i(a),-\iota(a))\colon a\in A\}$ by Theorem 2.1.

Define $\eta\colon Q/W\to G^0$ by $\eta(\overline{(x,y)})=\varepsilon_0(x)+\gamma_0(y)$ for $(x,y)\in Q$. Then η is well defined and $\eta\lambda(a)=\eta(\overline{(i(a),0)})=\varepsilon_0 i(a)=d^0(a)$, $(\varepsilon_1+\gamma_1)\tau(\overline{(x,y)})=(\varepsilon_1+\gamma_1)(\pi(x))=d^1\varepsilon_0(x)+d^1\gamma_0(y)=$ $=d^1\eta(\overline{(x,y)})$. So we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\lambda"] \arrow[d, equal] & Q/W \arrow[r, "\tau"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "\varepsilon_1+\gamma_1"] & 0 \\
0 \arrow[r] & A \arrow[r, "d^0"] & G^0 \arrow[r, "d^1"] & G^1 \arrow[r] & G^2\,.
\end{tikzcd}$$

Thus $\Theta([\Gamma]+[\Upsilon])=\Theta([\Gamma])+\Theta([\Upsilon])$. We next prove that Θ is a group isomorphism.

Write $\mu\colon \mathrm{im}(d^1)\to G^1$ to be the inclusion. Then there exists $\nu\colon G^0\to \mathrm{im}(d^1)$ such that $\mu\nu=d^1$.

Let $\beta\in\ker(d_*^2)$. Then $d^2\beta=0$. So $\mathrm{im}(\beta)\subseteq\ker(d^2)=\mathrm{im}(d^1)$. Thus there exists $\widehat{\beta}\colon C\to\mathrm{im}(d^1)$ such that $\beta=\mu\widehat{\beta}$. We obtain the following pullback diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & D \arrow[r, "g"] \arrow[d, "\omega"'] & C \arrow[r] \arrow[d, "\widehat{\beta}"'] & 0 \\
0 \arrow[r] & A \arrow[r, "d^0"] & G^0 \arrow[r, "\nu"] & \mathrm{im}(d^1) \arrow[r] & 0\,.
\end{tikzcd}$$

For any $M\in\mathcal{G}$ and any homomorphism $h\colon A\to M$, there is $j\colon G^0\to M$ such that $h=jd^0$. So $(jw)f=jd^0=h$. Thus the sequence $\mathrm{Hom}(D,M)\to\mathrm{Hom}(A,M)\to 0$ is exact. Hence the exact sequence $\Delta\colon 0\to A\xrightarrow{f} D\xrightarrow{g} C\to 0$ is a $\mathcal{G}$-coextension of A by C. Since $\beta=\mu\widehat{\beta}$, we have $\Theta([\Delta])=\overline{\beta}$. So Θ is an epimorphism.

On the other hand, let $\Theta([\Gamma])=\overline{\varepsilon_1}=\overline{0}$, then $\varepsilon_1=d^1\kappa$ for some $\kappa\in\mathrm{Hom}(C,G^0)$. Since $d^2\varepsilon_1=d^2d^1\kappa=0$, $\mathrm{im}(\varepsilon_1)\subseteq\ker(d^2)=\mathrm{im}(d^1)$. Thus there exists $\widehat{\varepsilon_1}\colon C\to\mathrm{im}(d^1)$ such that $\varepsilon_1=\mu\widehat{\varepsilon_1}$. So $\mu\nu\kappa=d^1\kappa=\varepsilon_1=\mu\widehat{\varepsilon_1}$. Thus $\nu\kappa=\widehat{\varepsilon_1}$.

Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "\pi"] \arrow[d, "\varepsilon_0"'] & C \arrow[r] \arrow[d, "\widehat{\varepsilon_1}"'] \arrow[dl, "\kappa"'] & 0 \\
0 \arrow[r] & A \arrow[r, "d^0"] & G^0 \arrow[r, "\nu"] & \mathrm{im}(d^1) \arrow[r] & 0.
\end{tikzcd}$$

Then there exists $\delta\colon B\to A$ such that $\delta i=1$ by [6, p. 44] (Lemma 8.4). Therefore Γ is a split exact sequence, and so $[\Gamma]=0$. Hence Θ is a monomorphism.

(2) can be proved dually.

Theorem 2.2 is proved.

As an immediate consequence of Theorem 2.2, we have the following corollary.

***Corollary* 2.1.** *The following are true*:

(1) *If $\mathcal{G}$ is a monic preenveloping class of right R-modules, then there is a monomorphism* $\mathrm{Ext}^1_{\mathcal{G}}(C, A) \to \mathrm{Ext}^1(C, A)$ *for all right R-modules A and C.*

(2) *If $\mathcal{F}$ is an epic precovering class of right R-modules, then there is a monomorphism* ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) \to \mathrm{Ext}^1(C, A)$ *for all right R-modules A and C.*

Obviously, a preenveloping class $\mathcal{G}$ of right R-modules is monic if and only if $\mathcal{G}$ contains all injective right R-modules and a precovering class $\mathcal{F}$ of right R-modules is epic if and only if $\mathcal{F}$ contains all projective right R-modules. Furthermore, we have the following result.

***Corollary* 2.2.** *The following are true*:

(1) *Let $\mathcal{G}$ be a monic preenveloping class of right R-modules, then* $\mathrm{Ext}^1_{\mathcal{G}}(C, A) \cong \mathrm{Ext}^1(C, A)$ *for all right R-modules A and C if and only if $\mathcal{G}$ is the class of injective right R-modules.*

(2) *Let $\mathcal{F}$ be an epic precovering class of right R-modules, then* ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) \cong \mathrm{Ext}^1(C, A)$ *for all right R-modules A and C if and only if $\mathcal{F}$ is the class of projective right R-modules.*

Proof. (1) $\Rightarrow$. For any $M \in \mathcal{G}$, there is an exact sequence $0 \to M \to E \to C \to 0$ with E injective. By Theorem 2.2(1), the exact sequence is a $\mathcal{G}$-coextension of M by C, and so is split. Thus M is injective.

$\Leftarrow$ is trivial.

(2) can be proved dually.

Corollary 2.2 is proved.

Now we characterize when $\mathrm{Ext}^n_{\mathcal{G}}(-,-)$ and ${}_{\mathcal{F}}\mathrm{Ext}^n(-,-)$ $(n = 1, 2)$ vanish.

Proposition 2.1. *The following are true*:

(1) *Let $\mathcal{G}$ be a monic preenveloping class of right R-modules, then any $\mathcal{G}$-coextension of A by C* $0 \to A \to B \to C \to 0$ *is split if and only if* $\mathrm{Ext}^1_{\mathcal{G}}(C, A) = 0$.

(2) *Let $\mathcal{F}$ be an epic precovering class of right R-modules, then any $\mathcal{F}$-extension of A by C* $0 \to A \to B \to C \to 0$ *is split if and only if* ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) = 0$.

Proof. (1) $\Leftarrow$. Since $\mathcal{G}$ is a monic preenveloping class of right R-modules, we have $\mathrm{Ext}^0_{\mathcal{G}}(C, -) \cong$ $\cong \mathrm{Hom}(C, -)$ (see [5, p. 170]) and $\mathcal{G}$ is closed under finite direct sums by [1] (Lemma 1). Thus by [5] (Theorem 8.2.5(1)), the $\mathcal{G}$-coextension of A by C $0 \to A \to B \to C \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \to \mathrm{Ext}^1_{\mathcal{G}}(C, A) = 0.$$

So $0 \to A \to B \to C \to 0$ is split.

$\Rightarrow$. Since any $\mathcal{G}$-coextension of A by C $0 \to A \to B \to C \to 0$ is equivalent to the exact sequence $0 \to A \to A \oplus C \to C \to 0$, $E_{\mathcal{G}}(C, A) = 0$. So $\mathrm{Ext}^1_{\mathcal{G}}(C, A) \cong E_{\mathcal{G}}(C, A) = 0$ by Theorem 2.2(1).

(2) $\Leftarrow$. Since $\mathcal{F}$ is an epic precovering class of right R-modules, we have ${}_{\mathcal{F}}\mathrm{Ext}^0(-, A) \cong$ $\cong \mathrm{Hom}(-, A)$ (see [5, p. 170]) and $\mathcal{F}$ is closed under direct sums by [7] (Proposition 1). So by [5] (Theorem 8.2.3(2)), the $\mathcal{F}$-extension of A by C $0 \to A \to B \to C \to 0$ induces the exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(A, A) \to {}_{\mathcal{F}}\mathrm{Ext}^1(C, A) = 0.$$

Thus $0 \to A \to B \to C \to 0$ is split.

$\Rightarrow$. Since any $\mathcal{F}$-extension of A by C $0 \to A \to B \to C \to 0$ is equivalent to the exact sequence $0 \to A \to A \oplus C \to C \to 0$, we have ${}_{\mathcal{F}}E(C, A) = 0$. Thus ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) \cong {}_{\mathcal{F}}E(C, A) = 0$ by Theorem 2.2(2).

Proposition 2.1 is proved.

***Corollary* 2.3.** *The following are true*:

(1) *Let $\mathcal{G}$ be a monic preenveloping class of right R-modules, then a right R-module A belongs to $\mathcal{G}$ if and only if* $\mathrm{Ext}^1_{\mathcal{G}}(C, A) = 0$ *for any right R-module C.*

(2) *Let $\mathcal{F}$ be an epic precovering class of right R-modules, then a right R-module C belongs to $\mathcal{F}$ if and only if* ${}_{\mathcal{F}}\mathrm{Ext}^1(C, A) = 0$ *for any right R-module A.*

Proof. It is easy by Proposition 2.1.

Proposition 2.2. *The following are true*:

(1) *Let $\mathcal{G}$ be a monic preenveloping class of right R-modules, then* $\mathrm{Ext}^2_{\mathcal{G}}(N, M) = 0$ *for all right R-modules M and N if and only if $C \in \mathcal{G}$ for any $\mathcal{G}$-coextension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{G}$.*

(2) *Let $\mathcal{F}$ be an epic precovering class of right R-modules, then* ${}_{\mathcal{F}}\mathrm{Ext}^2(N, M) = 0$ *for all right R-modules M and N if and only if $A \in \mathcal{F}$ for any $\mathcal{F}$-extension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{F}$.*

Proof. (1) $\Rightarrow$. By [5] (Theorem 8.2.5(1)), for any right R-module N, any $\mathcal{G}$-coextension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{G}$ induces the exact sequence

$$0 = \mathrm{Ext}^1_{\mathcal{G}}(N, B) \to \mathrm{Ext}^1_{\mathcal{G}}(N, C) \to \mathrm{Ext}^2_{\mathcal{G}}(N, A) = 0.$$

So $\mathrm{Ext}^1_{\mathcal{G}}(N, C) = 0$. Thus $C \in \mathcal{G}$ by Corollary 2.3(1).

$\Leftarrow$. For any right R-module M, by hypothesis, there exists a $\mathcal{G}$-coextension $0 \to M \to B \to$ $\to C \to 0$ with $B \in \mathcal{G}$. So $C \in \mathcal{G}$. Thus by [5] (Theorem 8.2.5(1)), for any right R-module N, we get the induced exact sequence

$$0 = \mathrm{Ext}^1_{\mathcal{G}}(N, C) \to \mathrm{Ext}^2_{\mathcal{G}}(N, M) \to \mathrm{Ext}^2_{\mathcal{G}}(N, B) = 0.$$

So $\mathrm{Ext}^2_{\mathcal{G}}(N, M) = 0$.

(2) can be proved dually.

Proposition 2.2 is proved.

The following result may be viewed as a relative version of Wakamatsu's lemmas.

Theorem 2.3. *The following are true*:

(1) *Suppose that $\mathcal{G}$ is a monic preenveloping class of right R-modules and $\mathcal{C}$ is a class of right R-modules closed under $\mathcal{G}$-coextensions. If $\alpha\colon N \to M$ is a $\mathcal{C}$-envelope of N, then* $\mathrm{Ext}^1_{\mathcal{G}}(\mathrm{coker}(\alpha), C) =$ $= 0$ *for any $C \in \mathcal{C}$.*

(2) *Suppose that $\mathcal{F}$ is an epic precovering class of right R-modules and $\mathcal{D}$ is a class of right R-modules closed under $\mathcal{F}$-extensions. If $\alpha\colon N \to M$ is a $\mathcal{D}$-cover of M, then* ${}_{\mathcal{F}}\mathrm{Ext}^1(D, \ker(\alpha)) = 0$ *for any $D \in \mathcal{D}$.*

Proof. (1) By Proposition 2.1(1), it is enough to show that any $\mathcal{G}$-coextension $0 \to C \to B \overset{\rho}{\to}$ $\overset{\rho}{\to} \mathrm{coker}(\alpha) \to 0$ with $C \in \mathcal{C}$ is split.

Let $\lambda\colon \mathrm{im}(\alpha) \to M$ be the inclusion and $\pi\colon M \to \mathrm{coker}(\alpha)$ the canonical map. Then there exists $\gamma\colon N \to \mathrm{im}(\alpha)$ such that $\lambda\gamma = \alpha$.

Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{im}(\alpha) \arrow[r, equal] \arrow[d, "i"] & \mathrm{im}(\alpha) \arrow[d, "\lambda"] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "\theta"] & M \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & C \arrow[r] & B \arrow[r, "\rho"] \arrow[d] & \text{coker}(\alpha) \arrow[r] \arrow[d] & 0\,. \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $0 \to C \to B \xrightarrow{\rho} \text{coker}(\alpha) \to 0$ is a $\mathcal{G}$-coextension, it is easy to see that $0 \to C \to X \to M \to 0$ is also a $\mathcal{G}$-coextension. Thus $X \in \mathcal{C}$ since $\mathcal{C}$ is closed under $\mathcal{G}$-coextensions. Because $\alpha\colon N \to M$ is a $\mathcal{C}$-envelope, there exists $g\colon M \to X$ such that $i\gamma = g\alpha$. Thus $\alpha = \lambda\gamma = \beta i\gamma = \beta g\alpha$. Hence βg is an isomorphism.

Define $\varphi\colon \text{coker}(\alpha) \to B$ by $\varphi(\overline{x}) = \theta g(\beta g)^{-1}(x)$ for $x \in M$. Since $\theta g(\beta g)^{-1}\alpha = \theta g\alpha = = \theta i\gamma = 0$, φ is well defined. Note that

$$\rho\varphi(\overline{x}) = \rho\theta g(\beta g)^{-1}(x) = \pi\beta g(\beta g)^{-1}(x) = \pi(x) = \overline{x}$$

for $x \in M$. Thus $\rho\varphi = 1$. Hence $0 \to C \to B \xrightarrow{\rho} \text{coker}(\alpha) \to 0$ is split, and so $\text{Ext}^1_{\mathcal{G}}(\text{coker}(\alpha), C) = 0$.

(2) can be proved dually.

Theorem 2.3 is proved.

***Remark* 2.2.** (1) Let $\mathcal{F}$ (resp. $\mathcal{G}$) in Theorem 2.3 be the class of projective (resp. injective) right R-modules, then Theorem 2.3 is just the usual Wakamatsu's lemmas (see [5], Corollary 7.2.3 and Proposition 7.2.4 or [14], Section 2.1).

(2) Following [13], an exact sequence $0 \to A \to B \to C \to 0$ of left R-modules is called *RD-exact* if the sequence $\text{Hom}(R/Ra, B) \to \text{Hom}(R/Ra, C) \to 0$ is exact for every $a \in R$. A left R-module G is called *RD-injective* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to 0$ is exact. According to [2], a right R-module F is called *RD-flat* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is exact.

Let $\mathcal{F}$ be the class of pure-projective right R-modules. It is well known that $\mathcal{F}$ is an epic precovering class of right R-modules (see [5], Example 8.3.2). Let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence of right R-modules with X and Z RD-flat, then we get the split exact sequence $0 \to Z^+ \to Y^+ \to X^+ \to 0$. Since X^+ and Z^+ are RD-injective by [2] (Proposition 1.1), we have Y^+ is RD-injective. Hence Y is RD-flat. So the class of RD-flat right R-modules is closed under $\mathcal{F}$-extensions by Remark 2.1(2). Note that any right R-module M has an RD-flat cover $\alpha\colon N \to M$ by [11] (Theorem 2.6(2)). Thus ${}_{\mathcal{F}}\text{Ext}^1(D, \ker(\alpha)) = 0$ for any RD-flat right R-module D by Theorem 2.3(2).

We next give some isomorphism formulas about relative (co)homological groups.

Theorem 2.4. *The following are true*:

(1) *Let $\mathcal{G}$ be a preenveloping class of right R-modules*, *A_S a projective right S-module*, *${}_SB_R$ an (S, R)-bimodule, C_R a right R-module and $n \geq 0$. Then*

$$\mathrm{Ext}^n_{\mathcal{G}}(A \otimes_S B, C) \cong \mathrm{Hom}_S(A, \mathrm{Ext}^n_{\mathcal{G}}(B, C)).$$

(2) *Let $\mathcal{F}$ be a precovering class of right R-modules*, *A_R a right R-module*, *${}_RB_S$ an (R, S)-bimodule*, *E_S an injective right S-module and $n \geq 0$. Then*

$${}_{\mathcal{F}}\mathrm{Ext}^n(A, \mathrm{Hom}_S(B, E)) \cong \mathrm{Hom}_S({}_{\mathcal{F}}\mathrm{Tor}_n(A, B), E).$$

Proof. (1) Let $\mathbf{G}^{\cdot} : 0 \to G^0 \to G^1 \to \ldots$ be a deleted right $\mathcal{G}$-resolution of C. Then we obtain the cocomplex $\mathrm{Hom}_R(A \otimes_S B, \mathbf{G}^{\cdot})$:

$$0 \to \mathrm{Hom}_R(A \otimes_S B, G^0) \to \mathrm{Hom}_R(A \otimes_S B, G^1) \to \ldots,$$

which is isomorphic to the cocomplex $\mathrm{Hom}_S(A, \mathrm{Hom}_R(B, \mathbf{G}^{\cdot}))$:

$$0 \to \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, G^0)) \to \mathrm{Hom}_S(A, \mathrm{Hom}_R(B, G^1)) \to \ldots.$$

Note that $\mathrm{Hom}_S(A, -)$ is an exact functor. So by [12, p. 170] (Exercise 6.4), we have $\mathrm{Ext}^n_{\mathcal{G}}(A \otimes$ $\otimes\, {}_SB, C) = H^n(\mathrm{Hom}_R(A \otimes_S B, \mathbf{G}^{\cdot})) \cong H^n(\mathrm{Hom}_S(A, \mathrm{Hom}_R(B, \mathbf{G}^{\cdot}))) \cong \mathrm{Hom}_S(A, H^n(\mathrm{Hom}_R(B,$ $\mathbf{G}^{\cdot}))) = \mathrm{Hom}_S(A, \mathrm{Ext}^n_{\mathcal{G}}(B, C))$.

(2) Let $\mathbf{F}_{.} : \ldots \to F_1 \to F_0 \to 0$ be a deleted left $\mathcal{F}$-resolution of A. Then we obtain the cocomplex $\mathrm{Hom}_S(\mathbf{F}_{.} \otimes_R B, E)$:

$$0 \to \mathrm{Hom}_S(F_0 \otimes_R B, E) \to \mathrm{Hom}_S(F_1 \otimes_R B, E) \to \ldots,$$

which is isomorphic to the cocomplex $\mathrm{Hom}_R(\mathbf{F}_{.}, \mathrm{Hom}_S(B, E))$:

$$0 \to \mathrm{Hom}_R(F_0, \mathrm{Hom}_S(B, E)) \to \mathrm{Hom}_R(F_1, \mathrm{Hom}_S(B, E)) \to \ldots.$$

Note that $\mathrm{Hom}_S(-, E)$ is an exact functor. So by [12, p. 170] (Exercise 6.4), we have $\mathrm{Hom}_S({}_{\mathcal{F}}\mathrm{Tor}_n(A,$ $B), E) = \mathrm{Hom}_S(H_n(\mathbf{F}_{.} \otimes_R B), E) \cong H^n(\mathrm{Hom}_S(\mathbf{F}_{.} \otimes_R B), E) \cong H^n(\mathrm{Hom}_R(\mathbf{F}_{.}, \mathrm{Hom}_S(B, E))) =$ $= {}_{\mathcal{F}}\mathrm{Ext}^n(A, \mathrm{Hom}_S(B, E))$.

Theorem 2.4 is proved.

***Corollary* 2.4.** *Let $\mathcal{F}$ be a precovering class of right R-modules, A_R a right R-module, ${}_RB$ a left R-module and $n \geq 0$. Then ${}_{\mathcal{F}}\mathrm{Ext}^n(A, B^+) \cong {}_{\mathcal{F}}\mathrm{Tor}_n(A, B)^+$.*

Proof. Let $S = \mathbb{Z}$ and $E = \mathbb{Q}/\mathbb{Z}$ in Theorem 2.4(2). Then we get the isomorphism ${}_{\mathcal{F}}\mathrm{Ext}^n(A, B^+) \cong {}_{\mathcal{F}}\mathrm{Tor}_n(A, B)^+$.

Finally we discuss the relationship between $\mathrm{Tor}_n(A, B)$ and ${}_{\mathcal{F}}\mathrm{Tor}_n(A, B)$.

Suppose that $\mathcal{F}$ is an epic precovering class of right R-modules. Let

$$\ldots \to F_2 \to F_1 \to F_0 \to A \to 0$$

be a left $\mathcal{F}$-resolution of a right R-module A and let

$$\ldots \to P_2 \to P_1 \to P_0 \to A \to 0$$

be a left projective resolution of A. Then there exist $f_i : P_i \to F_i$ such that the following diagram is commutative:

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r] \arrow[d, dotted, "f_2"'] & P_1 \arrow[r] \arrow[d, dotted, "f_1"'] & P_0 \arrow[r] \arrow[d, dotted, "f_0"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\ldots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Applying $-\otimes B$ to the above diagram, we have the following commutative diagram of complexes:

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \otimes B \arrow[r] \arrow[d, "f_2\otimes 1"'] & P_1 \otimes B \arrow[r] \arrow[d, "f_1\otimes 1"'] & P_0 \otimes B \arrow[r] \arrow[d, "f_0\otimes 1"'] & 0 \\
\ldots \arrow[r] & F_2 \otimes B \arrow[r] & F_1 \otimes B \arrow[r] & F_0 \otimes B \arrow[r] & 0.
\end{tikzcd}$$

Then it is easy to check that there exist group homomorphisms:

$$\eta_n : \mathrm{Tor}_n(A,B) \to {}_{\mathcal{F}}\mathrm{Tor}_n(A,B), \quad n \geq 0.$$

Theorem 2.5. *If $\mathcal{F}$ is an epic precovering class of right R-modules, then* $\eta_1 : \mathrm{Tor}_1(A,B) \to$ $\to {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)$ *is an epimorphism for any right R-module A and any left R-module B.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
{}_{\mathcal{F}}\mathrm{Ext}^1(A,B^+) \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \mathrm{Ext}^1(A,B^+) \arrow[d, "\beta"'] \\
{}_{\mathcal{F}}\mathrm{Tor}_1(A,B)^+ \arrow[r, "\eta_1^+"] & \mathrm{Tor}_1(A,B)^+.
\end{tikzcd}$$

Note that α and β are isomorphisms by Corollary 2.4 and γ is a monomorphism by Corollary 2.1(2). So $\eta_1^+ : {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)^+ \to \mathrm{Tor}_1(A,B)^+$ is a monomorphism. Thus $\eta_1 : \mathrm{Tor}_1(A,B) \to {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)$ is an epimorphism.

Theorem 2.5 is proved.

***Remark* 2.3.** Let $\mathcal{F}$ be an epic precovering class of right R-modules. Although $\eta_1 : \mathrm{Tor}_1(A,B) \to$ $\to {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)$ is an epimorphism by Theorem 2.5, this is not an isomorphism in general. For example, if $\mathcal{F}$ is the class of pure-projective $\mathbb{Z}$-modules, then ${}_{\mathcal{F}}\mathrm{Tor}_1(\mathbb{Z}_2,\mathbb{Z}_2) = 0$, but $\mathrm{Tor}_1(\mathbb{Z}_2,\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Note that $\mathrm{Tor}_0(A,B) \cong {}_{\mathcal{F}}\mathrm{Tor}_0(A,B) \cong A \otimes B$ for any right R-module A and any left R-module B. It is natural to ask when $\mathrm{Tor}_n(A,B) \to {}_{\mathcal{F}}\mathrm{Tor}_n(A,B)$ is an isomorphism. We give the following answer which is easy to verify.

Proposition 2.3. *Let $\mathcal{F}$ be an epic precovering class of right R-modules. Then the following are equivalent*:

(1) $\mathrm{Tor}_n(A,B) \cong {}_{\mathcal{F}}\mathrm{Tor}_n(A,B)$ *for any right R-module A, left R-module B and $n \geq 1$.*

(2) $\mathrm{Tor}_1(A,B) \cong {}_{\mathcal{F}}\mathrm{Tor}_1(A,B)$ *for any right R-module A and left R-module B.*

(3) *Every $M \in \mathcal{F}$ is flat.*

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