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SOME REMARKS ON SPECTRAL SYNTHESIS SETS

ДЕЯКІ ЗАУВАЖЕННЯ ЩОДО МНОЖИН СПЕКТРАЛЬНОГО СИНТЕЗУ

Relations between the difference spectra of unions and intersections are studied and their implications on some problems in spectral synthesis are observed.

Вивчаються взаємозв'язки між різницевими спектрами об'єднань та перетинів. Встановлено їхні наслідки для деяких проблем спектрального синтезу.

1. Introduction. The question whether the union of two spectral synthesis sets is again a spectral synthesis set and whether every spectral synthesis set is a Wiener – Ditkin set are outstanding open problems in commutative harmonic analysis. In connection with the study of such problems in spectral synthesis Stegeman [9] introduced the difference spectrum and later the idea was systematically exploited by him in [10]. Difference spectrum is a union of perfect sets and is closed if the group is metrizable. But, for non metrizable groups, Salinger and Stegeman [8] proved that the difference spectrum need not be closed. This article studies certain relations between the difference spectrum of the unions and intersections of closed sets, and the corresponding unions and intersections of the difference spectra. Their implications on some problems in spectral synthesis are also studied.

If G is the character group of the locally compact Abelian group, then $A(G)$, Fourier algebra, is the algebra of Fourier transforms of functions in the convolution algebra $L^1(\Gamma)$. If $f \in A(G)$, then $f = \hat{\phi}$, for some $\phi \in L^1(\Gamma)$. Define $\|f\|_{A(G)} = \|\phi\|_{L^1(\Gamma)}$, then $A(G)$ becomes a commutative regular semisimple Banach algebra isometrically isomorphic to $L^1(\Gamma)$, which is identified as a predual of a group von Neumann algebra [2]. For the notations we follow [4], [6] and [10].

Suppose $x \in G, f, g \in A(G)$ and I, J are ideals in $A(G)$. We use the notation $f =_x g$ if f and g agree in some neighborhood of $x, f \in_x I$ if $f =_x g$ for some $g \in I, I \subset_x J$ if $f \in_x J$ for every $f \in I$ and $I =_x J$ if $I \subset_x J$ and $J \subset_x I$. For a closed subset E of G, $I(E) = \{f \in A(G)\colon f = 0 \text{ on } E\}$, $j(E) = \{f \in A(G)\colon \operatorname{supp} f \cap E = \varnothing\}$ and $J(E) = \overline{j(E)}$. Then $J(E) \subset I \subset I(E)$ for every closed ideal I of $A(G)$ with zero set E. E is called a set of synthesis or spectral synthesis set if $I(E) = J(E)$ and E is said to be Wiener – Ditkin set (Calderon set or C-set) if $f \in \overline{fj(E)}$ for every $f \in I(E)$. Evidently, every Wiener – Ditkin set is a spectral set. We recall the definition of $\Delta(E)$, the difference spectrum of a closed set E : $\Delta(E) = \{x\colon I(E) \neq_x J(E)\}$ so that $\Delta(E)$ is a union of perfect subsets of $\partial(E)$, the boundary of E. E is a set of spectral synthesis if and only if $\Delta(E) = \varnothing$ [10]. Also the n-difference spectrum is $\Delta_n(E) = \bigcup_{f \in I(E)} \{x \in G\colon f^n \notin_x J(E)\}$ so that E is an n-weak synthesis set if and only if $\Delta_n(E) = \varnothing$ and $\Delta_m(E) \neq \varnothing$ for all $m < n$ [6]. The 1-difference spectrum is nothing but the difference spectrum $\Delta(E)$ and the 1-weak synthesis set is the synthesis set. We use the notation $B(x, r)$ for the closed ball with center x and radius $r \geq 0$.

2. Difference spectra of sets and its subsets. The entire set G itself is a synthesis set, so in general, it is not true that a subset of a spectral synthesis set is again a spectral synthesis set, unless G is discrete.

Theorem 1. *If $E \subset \mathbb{R}, x \in E$ and $V = E \cap B(x,r)$, then $\Delta(V) \subset \Delta(E)$. Moreover if $r > 0$, then there are points $x_1, x_2, \ldots$ in E which satisfy $\Delta(E) \subset \cup\Delta(V_i)$, where $V_i = E \cap B(x_i, r)$.*

Proof. Let $y \in \Delta(V)$ then $y \in \partial E$. If $y \in \partial B(x,r)$, boundary of an interval, and since $\Delta(V)$ is a union of perfect sets there is a $y' \in \Delta(V)$ such that $|x - y'| < |x - y|$. Otherwise let $y' = y$. Choose a neighborhood U of y' such that $U \cap E \subset V$. Pick $k \in A(G)$ with $k = 1$ near y' and supp $k \subset U$. So there exists $f \in I(V)$ such that $f \notin_{y'} J(V)$. Then $fk \in I(E), f =_{y'} fk$ and since $J(V) \supset J(E)$, $y' \in \Delta(E)$. Since y is the limit of such points y' and $\Delta(E)$ is closed, the first part of the statement follows.

For the second part, let $x \in \Delta(E)$ and $V = E \cap B(x,r)$, choose $f \in I(E)$ and $f \notin_x J(E)$, and $r_1 = \dfrac{r}{2}$. Let U be the open ball with center x and radius r_1. Choose $k \in A(G)$ with $k = 1$ near x and supp $k \subset U$. Then $f =_x fk$ and $J(E) =_x J(V)$ hence $f \notin_x J(V)$. The same argument works not only for x but for all the points in the r_1 neighborhood of x and hence the proof.

The first part of the above theorem is not true for $\mathbb{R}^n$ if $n > 2$, for let $E = \{x \in \mathbb{R}^n \colon \|x\| \geq 1\} \cup \{0\}$. At the same time, the second part of the above theorem is true for $\mathbb{R}^n$, $n > 1$, as well (essentially the same proof works). So if $E \subset \mathbb{R}^n$ then $\Delta(E) \subset \bigcup_{x_i \in E} \Delta(V_i)$, where $V_i = E \cap B(x_i, r)$, which, in particular, implies the known result that if each point of a set has a relative synthesis neighborhood then the set itself is a synthesis set. For the n-difference spectrum, however, we have the following result: $\Delta_n(E) \subset \bigcup_{x_i \in E} \Delta_n(V_i)$, so that if each V_i is a weak synthesis set with characteristic less than or equal to n then E itself is a weak synthesis set with characteristic less than or equal to n.

Using the above theorem it is easy to construct examples of non synthesis sets in $\mathbb{R}^3$. Let G_1 and G_2 are locally compact Abelian groups and let $\alpha \colon G_1 \to G_2$ be injective, continuous homomorphism. Assume that $E_1 \subset G_1$ is compact and $E_2 = \alpha E_1$. Then it is proved in [1] that E_1 is a synthesis set if and only if E_2 is a synthesis set. This result shows that any sphere of positive radius is a non synthesis set in $\mathbb{R}^3$. Due to the symmetry of S^2, the unit sphere in $\mathbb{R}^3$, $\Delta(S^2) = S^2$. Now $S^2 \cap B(x,r), r > 0, x \in S^2$, is a non synthesis set in $\mathbb{R}^3$. So the union of pieces of different spheres is also a non-synthesis set in $\mathbb{R}^3$. Similarly many curves in $\mathbb{R}^2$ are examples of synthesis set in $\mathbb{R}^2$ since S^1 is a synthesis set.

For $E, F \subset G$ and $x \in G$, we recall that $E =_x F$ means $E \cap V = F \cap V$ for some neighborhood V of x. Now we use the following lemma to prove a theorem.

Lemma 1 [6]. *Let $E, F \subset G$ and $x \in G$. If $E =_x F$, then* (i) *$J(E) =_x J(F)$ and* (ii) $\Delta(E) =_x \Delta(F)$.

Theorem 2. *For a closed subset E of $\mathbb{R}$ and $r > 0$, there is a collection of closed sets $V_i = E \cap B(x_i, r), i = 1, 2, 3, \ldots$, such that $\Delta(E) = \cup\Delta(V_i)$. Moreover for $i \neq j$, $\Delta(V_i) \cap \Delta(V_j)$ is either empty or a singleton set.*

Proof. Choose a closed interval F containing E. Let $x_1, x_2, \ldots$ be a sequence of real numbers in F such that for $i \neq j$, $B(x_i, r) \cap B(x_j, r)$ is either empty or a singleton set and $F = \cup B(x_i, r)$. Then $E = E \cap F = \cup(E \cap B(x_i, r)) = \cup V_i$. Let $\bigcup_{i,j,i\neq j} V_i \cap V_j = \{y_1, y_2, \ldots\}$. So $\Delta(E) = \Delta(\cup V_i) \subset \cup\Delta(V_i) \cup \{y_1, y_2, \ldots\}$. This follows from the fact that if $y \in \Delta(E)$ and $y \neq y_n, n = 1, 2, \ldots$, then $y \in V_k$ for some k. Now choose U, a neighborhood of y such that $U \cap V_j = \varnothing$ for $j \neq k$. Now $\cup V_i =_y V_k$ and $y \in \Delta(V_k)$ by Lemma 1.

Now if $y_n \in \Delta(\cup V_i)$, then $y_n \in \Delta(V_j)$ for some j so that $\Delta(\cup V_i) \subset \cup\Delta(V_i)$. To prove this, let $y_n \in \Delta(\cup V_i)$. Then y_n belongs to exactly one of the sets, say V_k or two such sets V_l and V_m. If $y_n \in V_k$ for a unique k, then as above $V_k =_{y_n} \cup V_i$. Otherwise if $y_n \in V_l \cap V_j$ and since $\Delta(\cup V_i)$ is a perfect set there is a sequence in $\Delta(\cup V_i)$ which converges to y_n. But such sequences are either in V_l or in V_m. As in the first case, y_n s are in $\Delta(V_l)$ or in $\Delta(V_m)$ so its limits also. Now the result follows from the above theorem.

Choose a non synthesis set E in $\mathbb{R}$, the existence of such sets follows from the Malliavin's theorem, so that $\Delta(E) \neq \varnothing$. For any $r > 0$, there are sets $V_1, V_2 \ldots$, as described above, which satisfy the relation $\Delta(E) \subset \cup\Delta(V_i)$. So for some i, $\Delta(V_i) \neq \varnothing$. Since translates of a non synthesis set are again non synthesis sets, any interval of positive length contains a non synthesis set. Due to the relation $\Delta(E_1 \times E_2) \supset E_1 \times \Delta(E_2) \cup \Delta(E_1) \times E_2$ [6], it is easy to see that if either E_1 or E_2 is a non synthesis set, then $E_1 \times E_2$ is a non synthesis set. So in $\mathbb{R}^n, n \geq 1$, any set of positive Lebesgue measure contains a non synthesis set.

For a closed set E of $\mathbb{R}$, let $V_1 = E \cap B(x, r_1)$ and $V_2 = E \cap B(x, r_2)$. Then $\Delta(V_1 \cap V_2) = \Delta(V_1) \cap$ $\cap\, \Delta(V_2)$. Now the natural question is whether $\Delta\left(\bigcap_{i=1}^{\infty} V_i\right) = \bigcap_{i=1}^{\infty} \Delta(V_i)$, where $V_1 \supset V_2 \supset \ldots$ and $V_i = E \cap B(x, r_i)$. In $\mathbb{R}^3$, the corresponding result is not true: for, let $E = \left\{x \in \mathbb{R}^3 : \|x\| \leq 2\right\} \setminus \left\{x \in \mathbb{R}^3 : \dfrac{1}{2} < \|x\| < 1\right\}$ and $V_n = E \cap B\left(x, 1 + \dfrac{1}{n}\right)$. Then $V_1 \supset V_2 \supset \ldots$ and $\Delta(\cap V_n) = S^2$, but $\cap\Delta(V_n) = \varnothing$. Even in $\mathbb{R}$ this result is not true. Consider a non synthesis set E in $\mathbb{R}$ so that $\varnothing \neq E' = \Delta(E)$, a union of perfect sets. Choose $x \in E'$ and for each positive integer, let $V_n = B\left(x, \dfrac{1}{n}\right) \cap E$. Each V_n is a non synthesis set so that $\Delta(V_n) \neq \varnothing$. As $n \to \infty$, $\Delta(\cap V_n) = \varnothing$ but $\cap\Delta(V_n) = \{x\}$.

***Corollary* 1.** *Let E be a closed subset of $\mathbb{R}$. Then $\Delta\left(\bigcap_{i=1}^{\infty} V_i\right) \subset \bigcap_{i=1}^{\infty} \Delta(V_i)$, where $V_i =$ $= E \cap B(x, r_i)$.*

Proof. $\Delta(\cap V_i) \subset \Delta(V_i)$ for each i since $\cap V_i \subset V_i$.

Let $E = E_{0,0} = [0, 1]$. Now see the following sequence of closed sets $\{V_i\}$ satisfying the condition $V_1 \supset V_2 \supset \ldots$. For any two disjoint closed balls $E_{(1,1)}, E_{(1,2)}$ in E, let $V_1 = E_{(1,1)} \cup E_{(1,2)}$. In general, for $n = 1, 2, 3, \ldots$, let $V_n = E_{(n,1)} \cup E_{(n,2)} \cup \ldots \cup E_{(n,2^n)}$, where $E_{(n,2i-1)}, E_{(n,2i)}$ are any two disjoint closed balls in $E_{(n-1,i)}$, $i = 1, 2, 3, \ldots, 2^{n-1}$. Clearly, $\Delta\left(\bigcap_{i=1}^{n} V_i\right) = \bigcap_{i=1}^{n} \Delta(V_i)$. Is it possible to conclude that $\Delta\left(\bigcap_{i=1}^{\infty} V_i\right) \subset \bigcap_{i=1}^{\infty} \Delta(V_i)$? If it is true, in particular, it shows that the Cantor set is a set of spectral synthesis, a result proved by Herz [3].

3. Difference spectra of unions and intersections. A closed subset E of G is said to be a Wiener–Ditkin set if whenever $f \in I(E), f \in \overline{fj(E)}$. For a subset E of G define $\Gamma(E) =$ $= \cup_{f \in I(E)} \{x \in G : f \notin_x \overline{fj(E)}\}$ so that, E is a Wiener–Ditkin set if and only if $\Gamma(E) = \varnothing$. Clearly $\Delta(E) \subset \Gamma(E)$, but the other inclusion, the C-set-S-set problem, is a major unsolved problem in commutative harmonic analysis. A weaker form of this, 'the weak C-set-S-set problem' is discussed recently in [7].

As we mentioned in the introduction, the question whether the union of two spectral synthesis sets is again a spectral synthesis set is a well known, difficult open problem. For any two closed sets E_1 and E_2, the relation $\Delta(E_1 \cup E_2) \subset \Delta(E_1) \cup \Delta(E_2)$ or a positive answer to the C-set-S-set problem gives an easy solution to the union problem. Our next aim is to find a relation between the difference spectrum of unions and unions of difference spectra. For closed subsets E_1 and E_2 of G, we introduce two sets $E_{1,2}$ and $E_{2,1}$ as follows. Let $E_{1,2} = \{x \in \partial E_1 \cap \partial E_2 \cap \partial(E_1 \cup E_2) :$

x is a limit point of $E_1 \setminus E_2\}$ and $E_{2,1} = \{x \in \partial E_1 \cap \partial E_2 \cap \partial(E_1 \cup E_2) \colon x$ is a limit point of $E_2 \setminus E_1\}$.

Lemma 2. *Let E_1 and E_2 be closed sets in G. Then $\Delta(E_1 \cup E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup \Gamma(E_{1,2} \cap \cap E_{2,1})$.*

Proof. Assume that x does not belong to the right-hand side of the desired inclusion and $x \in \partial(E_1 \cup E_2)$. Then $I(E_1 \cup E_2) j(E_{1,2} \cap E_{2,1}) \subset_x J(E_1 \cup E_2)$. To prove this, if $x \in (E_{1,2} \cap E_{2,1})$ then $j(E_{1,2} \cap E_{2,1}) \in_x j(E_1 \cup E_2)$ and if $x \in (E_{1,2} \cap E_{2,1})^c \cap E_1, I(E_1 \cup E_2) \subset I(E_1) =_x =_x J(E_1) =_x J(E_1 \cup E_2)$. Similar proofs hold for $x \in (E_{1,2} \cap E_{2,1})^c \cap E_2$. Now let $f \in I(E_1 \cup E_2)$. Then $f \in_x \overline{fj(E_{1,2} \cap E_{2,1})} \subset_x \overline{I(E_1 \cup E_2) J(E_{1,2} \cap E_{2,1})} \subset_x J(E_1 \cup E_2)$. So $x \notin \Delta(E_1 \cup E_2)$ and the result follows.

Theorem 3. *If E_1, E_2 are synthesis sets and if $E_{1,2} \cap E_{2,1}$ is a Wiener–Ditkin set, then $E_1 \cup E_2$ is a synthesis set.*

Proof. Since $\Delta(E_1) = \Delta(E_2) = \Gamma(E_{1,2} \cap E_{2,1}) = \varnothing$, the result follows.

In the above relation one can replace $E_{1,2} \cap E_{2,1}$ with any set F that lies in between $E_{1,2} \cap E_{2,1}$ and $E_1 \cup E_2$. So if E_1 and E_2 are synthesis sets and F is a Wiener–Ditkin set, then $E_1 \cup E_2$ is a synthesis set. Whether $\Gamma(E_{1,2} \cap E_{2,1})$ can be replaced with $\Delta(E_{1,2} \cap E_{2,1})$ in the above lemma is another interesting problem (see also [10]).

For results about intersections, we define two sets as follows. For two closed sets E_1, E_2 in G, let $E'_{1,2} = \{x \in \partial E_1 \cap \partial E_2 \colon x$ is a limit point of $E_1 \setminus E_2\}$ and $E'_{2,1} = \{x \in \partial E_1 \cap \partial E_2 \colon$ x is a limit point of $E_2 \setminus E_1\}$.

Lemma 3. *If E_1, E_2 are closed sets in G, then $\Delta(E_1 \cap E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup \Gamma(E'_{1,2} \cap E'_{2,1})$.*

Proof. Assume that $x \notin \Delta(E_1) \cup \Delta(E_2) \cup \Gamma(E'_{1,2} \cap E'_{2,1})$ and $x \in \partial(E_1 \cap E_2) \subset \partial(E_1) \cap \partial(E_2)$. Then $I(E_1 \cap E_2) j(E'_{1,2} \cap E'_{2,1}) \subset_x J(E_1 \cap E_2)$. To prove this let $x \in E'_{1,2} \cap E'_{2,1}$. Then $j(E'_{1,2} \cap \cap E'_{2,1}) \subset_x j(E_1 \cap E_2)$. If $x \in (E'_{1,2} \cap E'_{2,1})^c \cap E_1$, then $I(E_1 \cap E_2) =_x I(E_1) =_x J(E_1) \subset_x \subset_x J(E_1 \cap E_2)$. If $f \in I(E_1 \cap E_2) \subset I(E'_{1,2} \cap E'_{2,1})$, then $f \in_x \overline{fj(E'_{1,2} \cap E'_{2,1})} \in_x J(E_1 \cap E_2)$. Consequently $x \notin \Delta(E_1 \cap E_2)$ and the theorem follows.

Now we have the following result for the intersection of two synthesis sets, which is an improvement of a result in [4] and the proof follows immediately from Lemma 3.

Theorem 4. *If E_1, E_2 are synthesis sets and if $E'_{1,2} \cap E'_{2,1}$ is a Wiener–Ditkin set, then $E_1 \cap E_2$ is a synthesis set.*

Consider $\mathbb{R}^3$ and let $E_1 = B((0,0,0),2)$ and $E_2 = B((3,0,0),2)$. Then $E_{1,2} \cap E_{2,1}$ is a circle which contains perfect sets. Similar conclusion holds for $n > 3$ also. In [11], Varopoulose proved that every compact Abelian group G contains two sets of spectral synthesis whose intersection is not of synthesis. Let E_1 and E_2 be non synthesis sets in a compact group G such that $E_1 \cap E_2$ is not of synthesis. So, $\mathbb{R} \times E_1$ and $\mathbb{R} \times E_2$ are synthesis sets and $\mathbb{R} \times E_1 \cap \mathbb{R} \times E_2 = \mathbb{R} \times (E_1 \cap E_2)$ is not a synthesis set. Now by the Structure theorem, the compact Abelian group in Varopoulose result can be replaced by any non discrete locally compact Abelian group (see [5]). Recall that any set which does not contain a perfect set is a Wiener–Ditkin set. So there are synthesis sets E_1 and E_2 in $\mathbb{R}$ such that $E'_{1,2} \cap E'_{2,1}$ contains perfect sets. But in $\mathbb{R}$, can the set $E_{1,2} \cap E_{2,1}$, a still smaller set, ever contain a perfect set? If the answer is no, it shows that the union of two synthesis sets is again a synthesis set in $\mathbb{R}$. In $\mathbb{R}^2$ also the details of such sets are not known.

The idea of weak Wiener–Ditkin set (weak C-set) has been introduced and studied in [7]. A closed subset E of G is called an n-weak Wiener–Ditkin set if $f^n \in \overline{fj(E)}$ for all $f \in I(E)$. Thus E is an n-weak Wiener–Ditkin set if and only if $\Gamma_n(E) = \varnothing$ and $\Gamma_m(E) \neq \varnothing$ for all $m < n$, where $\Gamma_n(E) = \bigcup_{f \in I(E)} \{x \colon f^n \notin_x \overline{fj(E)}\}$. Note that the 1-weak Wiener–Ditkin set is nothing but the

Wiener – Ditkin set and $\Gamma_1(E) = \Gamma(E)$. Now for the n-difference spectrum we have the following result.

Theorem 5. *Let E_1 and E_2 be closed sets in G. Then* $\Delta_{2n-1}(E_1 \cup E_2) \subset \Delta_n(E_1) \cup \Delta_n(E_2) \cup \cup \Gamma_n(E_1 \cap E_2)$.

Proof. Assume that x does not belong to the right-hand side of the inclusion and $x \in \partial(E_1 \cup E_2)$. Then $\overline{I^n(E_1 \cup E_2) j(E_1 \cap E_2)} \subset_x J(E_1 \cup E_2)$ as in Lemma 2. Let $f \in I(E_1 \cup E_2)$, then $f^n \in_x \in_x \overline{fj(E_1 \cap E_2)}$. So $f^{2n-1} \in \overline{f^n j(E_1 \cap E_2)} \subset_x \overline{I^n(E_1 \cup E_2) j(E_1 \cap E_2)} \subset_x J(E_1 \cup E_2)$. So x does not belong to the left side of the inclusion and the result follows.

Similar arguments give the following improvement of 3.3 in [6].

Theorem 6. *Let E_1, E_2 be closed subsets of G. Then*

i) $\Delta_{2n-1}(E_1 \cap E_2) \subset \Delta_n(E_1) \cup \Delta_n(E_2) \cup \Gamma_n(E'_{1,2} \cap E'_{2,1})$,

ii) $\Delta_{2n-1}(E_1 \cup E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup \Gamma_n(E_{1,2} \cap E_{2,1})$,

iii) $\Delta_{2n-1}(E_1) \cup \Delta_{2n-1}(E_2) \subset \Delta_n(E_1 \cup E_2) \cup \Gamma_n(E_{1,2} \cap E_{2,1})$.

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