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TOPOLOGICALLY MIXING MAPS AND THE PSEUDOARC*

ТОПОЛОГІЧНО ПЕРЕМІШУЮЧІ ВІДОБРАЖЕННЯ ТА ПСЕВДОДУГА

It is known that the pseudoarc can be constructed as the inverse limit of the copies of $[0,1]$ with one bonding map f which is topologically exact. On the other hand, the shift homeomorphism σ_f is topologically mixing in this case. Thus, it is natural to ask whether f can be only mixing or must be exact. It has been recently observed that, in the case of some hereditarily indecomposable continua (e.g., pseudocircles) the property of mixing of a bonding map implies its exactness. The main purpose of this article is to show that the indicated kind of forcing of recurrence is not the case for the bonding map defining the pseudoarc.

Відомо, що псевдодугу можна отримати як обернену границю копій відрізка $[0,1]$ з єдиним зв'язуючим відображенням f, що є топологічно точним. З іншого боку, в цьому випадку зсувний гомеоморфізм σ_f є топологічно перемішуючим. Таким чином, природно запитати чи може f бути тільки перемішуючим, чи воно обов'язково повинно бути точним. Нещодавно було встановлено, що для деяких спадково нерозкладних континуумів (наприклад, псевдокіл) точність зв'язуючого відображення є наслідком властивості перемішування. В даній статті показано, що розглянутий тип примусового повернення не реалізується для зв'язуючого відображення, що визначає псевдодугу.

1. Introduction. Inverse limits provide an excellent tool in theory of continua, allowing construction of continua of many different types by a careful selection of a sequence of bonding maps. In theory, it is enough to specify some sufficient conditions that a continuous map f should satisfy (e.g., some level of crookedness), and if succeeded, the space obtained as an inverse limit with f as the bonding map should have desired properties. In practice it is not that simple, since one property can interact with other, making the final construction hard or even impossible. Additionally, small modifications in the construction can have large impact on the resulting inverse limit. For example, in 1990s Tom Ingram asked a question if it is true that all the inverse limits induced by maps in the family of tent maps are nonhomeomorphic. This question, after around 20 years of research and numerous partial steps towards it, finally received complete affirmative answer in [1]. Note that while the graphs of all the maps in this family seem to be very similar, nevertheless the corresponding inverse limits are not homeomorphic.

Another question of this type was related with the pseudoarc, a classical example of hereditarily indecomposable continuum. The first construction leading to the pseudoarc was obtained by Knaster in [11], and many years later extended by Moise [13] who gave it the name pseudoarc (since the continuum he obtained was similar to the arc, in the sense that it was homeomorphic to each of its subcontinua). Finally, Bing proved [5] that observation of Moise was not accidental, since all pseudoarcs are homeomorphic. So from topological point of view the pseudoarc is unique. But it was not the end of the story. For a long time it was unclear, if pseudoarc can be obtained as an inverse limit of the unit interval with one bonding map. In [9], Henderson provided such example. While topologically significant, this example was quite trivial from dynamical point of view. Its nonwandering set was consisting of two fixed points (attractor-repeller pair). In fact, Henderson's

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approach is probably the only known technique of construction of a map on the unit interval which gives pseudoarc in its inverse limit and has zero topological entropy. It is also easy to check that most of topological properties of dynamics (e.g., value of topological entropy, transitivity, topological mixing etc.) are shared by bonding map and induced shift homeomorphism σ_f on the inverse limit.

There were made numerous attempts to make dynamics on the pseudoarc more rich. One of the most accessible techniques (and important form the point of view of present work) was developed by Minc and Transue in [12]. They provided a method of perturbation (in fact a sequence of perturbations), such that topologically exact map is transformed to a transitive map f such that inverse limit with f as bonding map gives the pseudoarc (it is worth noting, that a few years earlier a map on the pseudoarc with positive topological entropy was constructed by J. Kennedy in [10]). In fact, it can be proved that if f is a transitive map on a topological graph, and inverse limit of this graph with f as a bonding map is hereditarily indecomposable, then f must be mixing [15]. So in fact, all the maps constructed in [12] are topologically mixing. In [14] it was shown, that technique of Minc and Transue can be extended in such a way that map f is not only transitive but has shadowing property. The authors have also shown that if we start with properly chosen map f then it will lead to a topologically exact map. However, recently it was proved that on some topological graphs (e.g., circle), if f is transitive and its inverse limit is hereditarily indecomposable then map f is topologically exact. Then a natural question arises:

Can method of Minc and Transue lead to a topologically mixing but not exact map?

To convince the reader even more, that the answer to the above question is not that easy, let us mention another question related to Minc and Transue technique. In their paper [6], Block, Keesling and Uspenskij asked if there exists a map of the unit interval, which has a periodic point of odd period but does not have a point of period 3. This question remains unanswered so far, despite the fact that there are known numerous techniques of construction of maps of type 5, 7, etc. in the Sharkovsky's ordering. In fact there are many more open questions related to dynamical properties of maps admitted on pseudoarc. Of particular interest is the question on admissible values of topological entropy. It was proved by Mouron [7] that entropy of shift homeomorphism on pseudoarc (induced as inverse limit with one bonding map) is either zero or infinity. It is an open question if other values of entropy for homeomorphisms of pseudoarc can be attained. It is worth mentioning that entropy of transitive interval map is always positive, and entropy of shift homeomorphism on the inverse limit is the same as the entropy of bonding map [16], hence if a transitive interval map f gives pseudoarc in its inverse limit then by the above mentioned result of Mouron [7] implies that f has infinite entropy.

While maps of the unit interval which are topologically mixing but not exact are fairly special (e.g., they have infinitely many fixed points), it is not sufficient obstruction for inducing pseudoarc as an inverse limit. The main purpose of this article is to propose an update of Minc – Transue technique, so that a positive answer to the above question is obtained.

2. Preliminaries. In this article we consider continuous maps $f\colon [0,1] \to [0,1]$, where $[0,1]$ is endowed with the standard Euclidean metric (and this metric is induced on all subintervals). The space of all continuous $f\colon [a,b] \to [a,b]$, where $a < b$, is denoted by $C([a,b])$. We endow the space $C([a,b])$ with the standard supremum metric

$$\rho(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|.$$

Note that the space $C([a,b])$ is a complete metric space.

A map $f \in C([0,1])$ is:

topologically mixing if for all nonempty open sets U, V there is N such that $f^n(U) \cap V \neq \varnothing$ for every $n > N$,

topologically exact if for every nonempty open set U there is $n > 0$ such that $f^n(U) = [0,1]$.

The inverse limit of $f \in C([0,1])$ is the following subset of the Hilbert cube $\mathbb{I}$ (the Cartesian product of countably many copies of $[0,1]$ with metric induced by the Tikhonov product topology):

$$\mathbb{X}_f = \varprojlim(f, [0,1]) = \{(x_0, x_1, \ldots) \colon f(x_{n+1}) = x_n\} \subset \mathbb{I}.$$

In 1985 Barge and Martin (see [2]) have shown that many maps on the interval $[0,1]$ give rise to the inverse limit space $\mathbb{X}_f$ which is *indecomposable* (i.e., is not the union of two proper subcontinua) or at least which contains an indecomposable subcontinuum. In fact many of the inverse limits are *hereditarily indecomposable*, that is every subcontinuum of $\mathbb{X}_f$ is indecomposable. Any two nondegenerate hereditarily indecomposable continua are homeomorphic, so are in fact a topologically unique continuum known as the *pseudoarc* (see Knaster [11], Bing [3, 4] and Moise [13]). Actually, it is the motivation behind the name pseudoarc, since every nondegenerate subcontinuum of hereditarily indecomposable continuum C is itself hereditarily indecomposable, therefore homeomorphic with C by the above mentioned results. It is the situation similar to the interval $[0,1]$ where any nondegenerate subcontiunuum is a closed interval, in particular is homeomorphic to the original continuum $[0,1]$.

The main goal of this article is a construction of a continuous map $f\colon [0,1] \to [0,1]$ such that f is topologically mixing but not topologically exact and such that the inverse limit of copies $[0,1]$ with f as the bonding map is the pseudoarc.

Definition 2.1. *Let $f \in C([0,1])$, let $a, b \in [0,1]$ and let $\delta > 0$. We say that f is δ-crooked between a and b if, for every two points $c < d$ such that $f(c) = a$ and $f(d) = b$, there are points $c < c' < d' < d$ such that $|b - f(c')| < \delta$ and $|a - f(d')| < \delta$. We say that f is δ-crooked if it is δ-crooked between every pair of points.*

Assume that $f \in C([0,1])$ is such that $f(x) = a$ and $f(y) = b$. We say that f is *linear on* $[x, y]$ if $f(z) = a + (z - x)\dfrac{b-a}{y-x}$ and f is *piecewise linear* if there is a finite partition of $[0,1]$ such that f restricted to each element of the partition is a linear function. We say that $x \in [0,1]$ is a *critical point* of f if $f'(x)$ vanishes or is undefined.

Definition 2.2. *Fix any $0 \le c < d \le 1$. We say that a map $f \in C([c,d])$ is admissible on $[c,d]$ if it is piecewise linear, $|f'(t)| \ge 4$ for every noncritical $t \in (c,d)$, and for every $c \le a < b \le d$ there is positive integer m such that $f^m([a,b]) = [c,d]$. If $[c,d] = [0,1]$, then we simply say that f is admissible.*

Now, we recall some useful facts from [12], which provides a nice practical method for a construction of the pseudoarc:

Lemma 2.1 ([12], Proposition 2)**.** *Let $f, F \in C([0,1])$ be two maps such that $\rho(f, F) < \varepsilon$. If f is δ-crooked, then F is $(\delta + 2\varepsilon)$-crooked.*

Lemma 2.2 ([12], Proposition 3)**.** *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of δ-crooked continuous maps from $[0,1]$ into $[0,1]$. If the sequence converges uniformly, then the limit is also δ-crooked.*

Lemma 2.3 ([12], Proposition 4)**.** *Let $f \in C([0,1])$ be a map with the property that, for every $\delta > 0$ there is an integer $n > 0$ such that f^n is δ-crooked. Then the inverse limit of copies of $[0,1]$ with f as the bonding map is the pseudoarc.*

3. Mixing but not exact maps and pseudoarc. The following result is known and not hard to prove (see, e.g., [14], Theorem 10).

Lemma 3.1. *Let $f \in C([0,1])$ be a topologically exact map such that $f(0) = 0$ and $f(1) = 1$. For every $\varepsilon > 0$ there exists an admissible map $F \in C([0,1])$ such that $F(0) = 0$, $F(1) = 1$ and $\rho(f, F) < \varepsilon$.*

Lemma 3.2. *Let $f \in C([0,1])$ be a topologically exact map such that $f(0) = 0$ and $f(1) = 1$ and let $\varepsilon > 0$. There exists $\tau \in (0, \varepsilon)$ and $g\colon [0,1] \to [0,1]$ such that $g|_{[\tau,1-\tau]}$ is admissible, $g(t) = t$ for every $t \in [0, \tau] \cup [1-\tau, 1]$ and $\rho(f, g) < \varepsilon$.*

Proof. By Lemma 3.1 there exists an admissible map $F \in C([0,1])$ with $F(0) = 0$, $F(1) = 1$ and $\rho(F, f) < \varepsilon/2$. Let $m > 0$ be an integer such that $|F'(t)| < m$ for every noncritical $t \in [0,1]$. Denote $\tau = \varepsilon/20m$. We define $g\colon [0,1] \to [0,1]$ by putting:

$$g(x) = \begin{cases} (1-2\tau)F\left(\dfrac{x-\tau}{1-2\tau}\right) + \tau, & \text{if } x \in [\tau, 1-\tau], \\ x, & \text{otherwise.} \end{cases}$$

Note that $g(\tau) = (1-2\tau)F(0) + \tau = \tau$ and $g(1-\tau) = (1-2\tau)F(1) + \tau = 1-\tau$ so g is continuous. Easy calculation yields that $g'(t) = F'\left(\dfrac{t-\tau}{1-2\tau}\right)$ for every noncritical $t \in [\tau, 1-\tau]$. It is also easy to note that $\hat{g} = g|_{[\tau,1-\tau]}$ is topologically exact. Namely $\hat{g} = \pi \circ F \circ \pi^{-1}$, where $\pi\colon [0,1] \to [\tau, 1-\tau]$ is the linear homeomorphism $\pi(x) = (1-2\tau)x + \tau$, and hence for any open set $U \subset [\tau, 1-\tau]$ and any $s \geq 0$ we have $\hat{g}^s(U) = \pi(F^s(\pi^{-1}(U)))$. In particular, if $s > 0$ is sufficiently large then $\hat{g}^s(U) = \pi([0,1]) = [\tau, 1-\tau]$, so indeed $\hat{g}$ is topologically exact.

Finally for every $x \in [\tau, 1-\tau]$, by mean value theorem we have (with some point $c_x \in [0,1]$)

$$|F(x) - g(x)| \leq \left|(1-2\tau)F\left(\frac{x-\tau}{1-2\tau}\right) + \tau - F(x)\right| \leq$$

$$\leq \left|(1-2\tau)F\left(\frac{x-\tau}{1-2\tau}\right) - (1-2\tau)F(x)\right| + |(1-2\tau)F(x) - F(x)| + \tau \leq$$

$$\leq (1-2\tau)|F'(c_x)|\left|\frac{x-\tau}{1-2\tau} - x\right| + 2\tau|F(x)| + \tau \leq$$

$$\leq (1-2\tau)\frac{m\tau}{1-2\tau} + 2\tau + \tau \leq$$

$$\leq \tau(m+3) \leq 4m\tau \leq \frac{\varepsilon}{5}.$$

If $x \in [0, \tau]$ then we see that (with some $c_x \in [0,1]$)

$$|F(x) - g(x)| = |F(x) - x| \leq |F(x) - F(0)| - |F(0) - x| \leq$$

$$\leq |F'(c_x)| \cdot \tau + \tau \leq \tau(m+1) \leq \frac{\varepsilon}{5}.$$

Similar calculations yield that $|F(x) - g(x)| \leq \dfrac{\varepsilon}{5}$ for $x \in [1-\tau, 1]$. Combining together the above three cases we see that $\rho(F, g) \leq \dfrac{\varepsilon}{5}$ and hence $\rho(f, g) < \varepsilon$.

Lemma 3.2 is proved.

Lemma 3.3. *Let $\gamma > 0$ and let $0 \le a < b \le 1$. There is a piecewise linear map $h \in C([a,b])$ such that h is admissible on $[a,b]$, $h(a) = a$, $h(b) = b$ and $\sup_{t\in[a,b]} |h(t) - t| < \gamma$.*

Proof. Decrease γ if necessary, so that $\gamma < 1$. Let $q = \left\lfloor \dfrac{1}{3\gamma} \right\rfloor + 1$, where $\lfloor t \rfloor$ is the integer part of t. Let $\mu = \dfrac{b-a}{q}$ and $\xi = \dfrac{\mu}{5}$ for each $i = 0, 1, \ldots, q-1$, let $u_i = a + i\mu$. Let h be a piecewise linear map on $[a,b]$ which is defined over each interval $[u_i, u_{i+1}]$ as the connect-the-dots map by points (u_i, u_i), $(u_i + \xi, u_{i+1})$, $(u_i + 2\xi, \max\{a, u_i - \xi\})$, $(u_{i+1} - 2\xi, \min\{b, u_{i+1} + \xi\})$, $(u_{i+1} - \xi, u_i)$, (u_{i+1}, u_{i+1}).

Observe that by the construction $|h'(t)| \ge 4$ for every noncritical t and

$$\sup_{t\in[0,1]} |h(t) - t| \le \max_{0\le i<q} |u_{i+1} + \xi - (u_i - \xi)| \le 3\mu < \gamma.$$

Fix a nondegenerate closed interval $J \subset [a,b]$. We claim that there is an integer $m > 0$ and $0 \le i < q$ such that $[u_i, u_{i+1}] \subset h^m(J)$. Observe that if J contains two critical points, then there is i such that two consecutive points in the sequence $u_i,\ u_i + \xi,\ u_i + 2\xi,\ u_{i+1} - 2\xi,\ u_{i+1} - \xi,\ u_{i+1}$ belong to J. But then $[u_i, u_{i+1}] \subset h(J)$. So the remaining possibility is that J contains at most one critical point. But then there is an interval $[p,q] \subset J$ such that h is linear on $[p,q]$ and $q - p \ge \operatorname{diam}(J)/2$. Then

$$\operatorname{diam}(h(J)) \le 4|q-p| \le 2\operatorname{diam}(J)$$

and so for some $k > 0$ the interval $h^k(J)$ must contain two critical points, in particular $[u_i, u_{i+1}] \subset h^{k+1}(J)$ for some i. The proof of the claim is finished.

Finally, observe that if $0 \le i < j \le q$ are such that $[u_i, u_j] \subset J$ then by the definition of h we obtain that $[u_{i'}, u_{j'}] \subset h^2(J)$ where $i' = \max\{0, i-1\}$ and $j' = \min\{1, j+1\}$.

Combining all the previous observations we see that for every non-degenerate closed interval $J \subset [0,1]$ there are integers $s, l \ge 0$ and $0 \le i < q$ such that $[u_i, u_{i+1}] \subset h^s(J)$ and $h^l([u_i, u_{i+1}]) = [0,1]$. In particular, if we put $m = s + l$ then $h^m(J) = [0,1]$ which shows that h is admissible.

Lemma 3.3 is proved.

To prove Lemma 3.5, let us first present the following fact which is [12] (Lemma on p. 1167):

Lemma 3.4. *Let $f \in C([0,1])$ be an admissible map. Let η and δ be two positive numbers. Then there is an admissible map $F \in C([0,1])$, and there is a positive integer n such that F^n is δ-crooked and $|F(t) - f(t)| < \eta$ for every $t \in [0,1]$. Moreover, if $0 \le a < b \le 1$ and $b - a \ge \eta$, then $f([a,b]) \subset F([a,b])$ and $F^n([a,b]) = [0,1]$.*

Lemma 3.5. *Let $0 < \theta < \dfrac{1}{2}$ and let η and δ be two positive numbers. Let $f \in C([\theta, 1-\theta])$ be an admissible map on $[\theta, 1-\theta]$ with the property that, for every $\varepsilon > 0$, there is an integer $n > 0$ such that $f^n([a,b]) = [\theta, 1-\theta]$ provided that $b - a \ge \varepsilon$ and $\theta \le a < b \le 1-\theta$.*

Then there is an admissible map $F\colon [\theta, 1-\theta] \to [\theta, 1-\theta]$ and an integer $m > 0$ such that F^m is δ-crooked and $|F(t) - f(t)| < \eta$ for every $t \in [\theta, 1-\theta]$.

Moreover, if $\theta \le a < b \le 1-\theta$ and $b - a \ge \eta$, then $f([a,b]) \subset F([a,b])$ and $F^m([a,b]) = [\theta, 1-\theta]$.

Proof. Let us define a function $f^* = \phi^* \circ f \circ \phi$ from $[0,1]$ into $[0,1]$, where ϕ and ϕ^* are two linear homeomorphisms defined by: $\phi\colon [0,1] \to [\theta, 1-\theta]$, $\phi(t) = (1-2\theta)t + \theta$ and $\phi^*\colon [\theta, 1-\theta] \to [0,1]$, $\phi^*(t) = \phi^{-1}(t) = \dfrac{1}{1-2\theta}t - \dfrac{\theta}{1-2\theta}$. Then f^* is a continuous and piecewise linear function and additionally, for each noncritical t we have

$$\left|(f^*)'(t)\right| = \left|\phi^*\left(f\left(\phi(t)\right)\right)\right| \cdot \left|f'\left(\phi(t)\right)\right| \cdot \left|\phi'(t)\right| \geq \frac{1}{1-2\theta} \cdot 4 \cdot (1-2\theta) = 4.$$

In order to prove that f^* is an admissible we need to show that for every $0 \leq a < b \leq 1$ there is a positive integer m such that $(f^*)^m([a,b]) = [0,1]$. Let us notice that

$$(f^*)^m = \phi^* \circ f^m \circ \phi$$

and from the assumption that f is an admissible map on $[\theta, 1-\theta]$ we get that f^* is an admissible map. Take $\delta^* = \delta/(1-2\theta)$ and $\eta^* = \eta/(1-2\theta)$. Let $F^*\colon [0,1] \to [0,1]$ be a map provided for f^* by application of Lemma 3.4, that is $(F^*)^n$ is δ^*-crooked for some integer $n > 0$, $|F^*(t) - f^*(t)| < \eta^*$ for every $t \in [0,1]$ and if $0 \leq a < b \leq 1$ and $b - a \geq \eta$, then $f^*([a,b]) \subset F^*([a,b])$ and $(F^*)^n([a,b]) = [0,1]$.

Put $F = \phi \circ F^* \circ \phi^*$. Note that if $|a-b| < \eta$ then $|\phi^*(a) - \phi^*(b)| < \dfrac{\eta}{1-2\theta} \leq \eta^*$. Similarly, if $|a-b| < \eta^*$ then $|\phi(a) - \phi(b)| < (1-2\theta)\eta^* \leq \eta$. Therefore

$$\sup_{t\in[\theta,1-\theta]} |F(t) - f(t)| \leq \sup_{s\in[0,1]} |\phi(F^*(s)) - \phi(f^*(s))| \leq \eta.$$

Similar calculations show that all the other required conditions on F are satisfied.

Lemma 3.5 is proved.

Lemma 3.6. *Let* $0 < \theta < \dfrac{1}{2}$, *let* $\delta > 0$ *and let* $f \in C([\theta, 1-\theta])$ *satisfy* $f(\theta) = \theta$, $f(1-\theta) = 1-\theta$. *Let* $F \in C([0,1])$ *be a continuous map defined by*

$$F(t) = \begin{cases} f(t) & \text{for } t \in [\theta, 1-\theta], \\ t & \text{for } t \notin [\theta, 1-\theta]. \end{cases}$$

If there is an integer $k > 0$ *such that* f^k *is* δ*-crooked on* $[\theta, 1-\theta]$ *then* F^k *is* $(\theta+\delta)$*-crooked on* $[0,1]$.

Proof. Fix any nondegenerate interval $J = [a,b] \subset [0,1]$. We need to find points $a < c < d < b$ such that $|F^k(a) - F^k(d)| < \theta + \delta$ and $|F^k(b) - F^k(c)| < \theta + \delta$. If $J \subset [\theta, 1-\theta]$ then there is nothing to prove, since f^k is δ-crooked. Similarly, if $J \subset [0,\theta]$ or $J \subset [1-\theta, 1]$ we are done by taking any $a < c < d < b$ since then $|F^k(a) - F^k(d)| = |a-d| \leq \theta$ and $|F^k(b) - F^k(c)| = |b-c| \leq \theta$. In the remaining case, let $p = \max\{\theta, a\}$ and $q = \min\{p, 1-\theta\}$. Then $[p,q] \subset [\theta, 1-\theta]$ is a nondegenerate interval, and so there are $p < c < d < q$ such that $|f^k(p) - f^k(d)| = |F^k(p) - F^k(d)| < \delta$ and $|f^k(q) - f^k(c)| = |F^k(q) - F^k(c)| < \delta$. Finally, note that if $a < p$ then $|F^k(a) - F^k(p)| = |a-p| < \theta$ and so

$$|F^k(a) - F^k(d)| \leq |F^k(a) - F^k(p)| + |F^k(p) - F^k(d)| \leq \theta + \delta$$

and similarly $|F^k(c) - F^k(b)| \leq \theta + \delta$.

Lemma 3.6 is proved.

Definition 3.1. *A piecewise linear map* $f\colon [0,1] \to [0,1]$ *is Markov if there is a sequence* $0 = a_0 < a_1 < \ldots < a_n = 1$ *such that* f *is linear on each interval* $[a_i, a_{i+1}]$ *and* $f([a_i, a_{i+1}]) = [a_k, a_m]$ *for some* $k < m$.

The main application of Markov maps is that it is relatively easy to verify if such a map is transitive or topologically exact. It will be a useful tool in many constructions presented later.

Lemma 3.7. *For every admissible map $f \in C([a,b])$ and every $\varepsilon > 0$ there exists an admissible Markov map $F \in C([a,b])$ such that $\sup_{t\in[a,b]} |f(t) - F(t)| < \varepsilon$ and $F(a) \le f(a)$ and $F(b) \ge f(b)$.*

Proof. Let $\varepsilon > 0$ and $a = u_0 < u_1 < \ldots < u_n = b$ be a partition of $[a,b]$ into n intervals of equal length, that is $|u_{i+1} - u_i| = |u_{j+1} - u_j|$ for any $0 \le i < j < n$. Assume additionally that n is large enough, so that the following condition is satisfied:

$$\max_{i=0,1,\ldots,n-1} \{|u_i - u_{i+1}|, \operatorname{diam} f([u_i, u_{i+1}])\} < \frac{\varepsilon}{6}.$$

We are going to define a piecewise linear map $F\colon [a,b] \to [a,b]$. First, we define F at the points of the partition by the formula

$$F(u_i) = \max\{u_j : u_j \le \min f([u_i, u_{i+1}])\} \qquad \text{for} \quad i < n,$$

$$F(u_n) = \min\{u_j : u_j \ge \max f([u_{n-1}, u_n])\}.$$

Now, we are going to define F on each of the intervals $[u_i, u_{i+1}]$ for $i = 0, 1, \ldots, n-1$. Fix $0 \le i < n$ and denote $s = s(i) = \max\{j\colon u_j \le \min f([u_i, u_{i+1}])\}$ and $t = t(i) = \min\{j\colon u_j \ge \ge \max f([u_i, u_{i+1}])\}$. Note f is not constant on any interval, hence we have $s < t$ and

$$f([u_i, u_{i+1}]) \subset [u_s, u_t].$$

Denote $\alpha_i = F(u_{i+1}) - u_s$. We are going to define F on $[u_i, u_{i+1}]$ depending on the value of α_i. If $\alpha_i \le 0$ then we divide interval $[u_i, u_{i+1}]$ into four intervals of equal length, that is we introduce 3 middle points $p_j = u_i + j \cdot (u_{i+1} - u_i)/4$, for $j = 1, 2, 3$. Next, we define F as the connect-the-dots map by points $(u_i, F(u_i))$, (p_1, u_t), (p_2, u_s), (p_3, u_t), $(u_{i+1}, F(u_{i+1}))$. Note that each interval in the partition $u_i < p_1 < p_2 < p_3 < u_{i+1}$ has length $|u_{i+1} - u_i|/4$ and $|u_t - F(u_{i+1})| \ge |u_t - u_s|$, so F has slope 4 in each of the intervals of the partition.

In the second case $\alpha_i > 0$, we divide interval $[u_i, u_{i+1}]$ into 5 pieces of equal length by introducing additional partition points $q_j = u_i + j \cdot (u_{i+1} - u_i)/5$, for $j = 1, 2, 3, 4$. Then we define function F as the connect-the-dots map given by points $(u_i, F(u_i))$, (q_1, u_t), (q_2, u_s), (q_3, u_t), (q_4, u_s), $(u_{i+1}, F(u_{i+1}))$. Note that $|F(u_{i+1}) - u_s| \ge |u_{i+1} - u_i|$ so again F has slope at least 4 on each interval of linearity in $[u_i, u_{i+1}]$.

Therefore the function F defined above is a piecewise linear continuous map on $[a,b]$. Additionally, for every $0 \le i < n$ there are $k \le s(i) < t(i) \le m$ such that $F([u_i, u_{i+1}]) = [u_k, u_m]$, and $|F'(t)| \ge 4$ for every noncritical $t \in (a,b)$. To prove that F is admissible, it remains to show that F is topologically exact.

Denote $J(i) = \{m\colon [u_m, u_{m+1}] \subset F([u_i, u_{i+1}])\}$ and observe that by the definition of F we have

$$f([u_i, u_{i+1}]) \subset [u_s, u_t] \subset F([u_i, u_{i+1}]) = \bigcup_{j\in J(i)} [u_j, u_{j+1}].$$

Then, using mathematical induction, we obtain that

$$f^{k+1}([u_i, u_{i+1}]) \subset f^k(F([u_i, u_{i+1}])) \subset f^k\left(\bigcup_{j\in J(i)} [u_j, u_{j+1}]\right) =$$

$$= \bigcup_{j\in J(i)} f^k([u_j,u_{j+1}]) \subset \bigcup_{j\in J(i)} F^k([u_j,u_{j+1}]) = F^k\left(\bigcup_{j\in J(i)}[u_j,u_{j+1}]\right) =$$

$$= F^{k+1}([u_i,u_{i+1}]).$$

In particular, since f is an exact map, for every $0 \le i < n$ there is $k > 0$ such that $F^k([u_i,u_{i+1}]) = [a,b]$.

Now let J be an open interval. Since $F'(t) \ge 4$ for every noncritical value $t \in (a,b)$ there is $j > 0$ such that $F^j(J)$ contains three consecutive critical points. But then, by the definition of F there is $0 \le i < n$ such that $[u_i,u_{i+1}] \subset F^{j+1}(J)$. Indeed, F is an exact map.

Observe that for every $0 \le i < n$ we have

$$\sup_{t\in[u_i,u_{i+1}]} |F(t)-f(t)| \le \operatorname{diam} F([u_i,u_{i+1}]).$$

Denote $r(i) = i+2$ if $i < n-1$ and put $r(i) = n$ otherwise. Then by the definition of F we obtain that

$$\operatorname{diam} F([u_i,u_{i+1}]) \le 2\cdot\frac{\varepsilon}{6} + \operatorname{diam} f([u_i,u_{r(i)}]) \le \frac{\varepsilon}{3} + 2\cdot\frac{\varepsilon}{6} < \varepsilon.$$

This yields that $\sup_{t\in[a,b]}|f(t)-F(t)| < \varepsilon$ and so the proof is finished.

Lemma 3.7 is proved.

Lemma 3.8. *Let $\varepsilon > 0$. Let $f \in C([a,b])$ and $g \in C([b,c])$, be two admissible maps on $[a,b]$ and $[b,c]$, respectively, such that $f(b) = g(b) = b$. Then there is an admissible map $F \in C([a,c])$ such that for every $t \in [a,b]$ we have $|F(t)-f(t)| < \varepsilon$ and for every $t \in [b,c]$ we have $|F(t)-g(t)| < \varepsilon$.*

Proof. Note that if we use Lemma 3.7 to perturb f, g to Markov maps, then the condition $f(b) = g(b)$ remains satisfied. Hence, by Lemma 3.7, without loss of generality we may assume that both maps f,g are Markov. Let $\{l_i\}_{i=1}^t$ be an increasing sequence giving Markov partition for functions f and g, that is f is Markov with respect to partition $a = l_0 < l_1 < \ldots < l_s = b$ and g is Markov with respect to partition $b = l_s < l_{s+1} < \ldots < l_t = c$. Since periodic points are dense in $[a,b]$ and in $[c,d]$, including periodic orbits in the partition if necessary, we may assume that $|l_i - l_{i+1}| < \varepsilon/3$, $\operatorname{diam} f([l_i,l_{i+1}]) < \varepsilon/3$ for $0 \le i < s$ and $\operatorname{diam} g([l_i,l_{i+1}])) < \varepsilon/3$ for $s \le i < t$.

Fix points $p < q$ in $[l_{s-1},b]$ dividing it into three intervals of equal length (i.e., $p = l_{s-1} + (b-l_{s-1})/3$, $q = l_{s-1} + 2(b-l_{s-1})/3$) and let $p' < q'$ divide $[b,l_{s+1}]$ into three intervals of equal length. We define $F(t) = f(t)$ for every $t \in [a,l_{s-1}]$ and $F(t) = g(t)$ for every $t \in [l_{s+1},c]$. We also put $F(p) = F(p') = g(l_{s+1}) > l_{s+1}$ and $F(q) = F(q') = f(l_{s-1}) > l_{s-1}$ (these inequalities are obtained by the fact that both f,g have slope 4 on intervals of linearity). Finally, we put $F(b) = b$ and define F as the connect-the-dots map, so it becomes piecewise linear also on the interval $[l_{s-1},l_{s+1}]$. Note that $F'(t) \ge 4$ in every noncritical point and F is Markov with respect to partition $\{l_i\}_{i=0}^t \cup \{p,q,p',q'\}$. Note that $F([l_{s-1},p]) \cap F([p,q]) \cap F([q,b]) \supset f([l_{s-1},b])$ and $F([b,p']) \cap F([p',q']) \cap F([q',l_{s+1}]) \supset g([b,l_{s+1}])$.

Also, since $F'(t) \ge 4$, for every nondegenerate interval J there is an integer $k > 0$ such that $F^k(J)$ contains three consecutive points of the partition and hence there is j such that $[l_j,l_{j+1}] \subset F^{k+1}(J)$ for some $0 \le j < t$. Additionally, since for every $0 \le i < s$ we get $f([l_i,l_{i+1}]) \subset F([l_i,l_{i+1}])$, and for $s \le i < t$ we have $g([l_i,l_{i+1}]) \subset F([l_i,l_{i+1}])$ then there exists $m > 0$ such that $F^m([l_i,l_{i+1}]) \supset [a,b]$ for every $0 \le i < s$ and $F^m([l_i,l_{i+1}]) \supset [b,c]$ for every $s \le i < t$. But $[l_{s-1},l_{s+1}] \subset F([a,b]) \cap F([b,c])$. By the above observations we obtain that there is $0 \le j < t$ such that

$$F^{k+2m+2}(J) \supset F^{2m+1}([l_j, l_{j+1}]) \supset F^m([l_{s-1}, l_{s+1}]) \supset$$

$$\supset F^m([l_{s-1}, b]) \cup F^m([b, l_{s+1}]) \supset [a, b] \cup [b, c] = [a, c].$$

Indeed F is topologically exact.

Lemma 3.8 is proved.

Theorem 3.1. *For every $\varepsilon > 0$ and every topologically exact map $g \in C([0,1])$ such that $g(0) = 0$ and $g(1) = 1$ there exists a mixing but not topologically exact map $f \in C([0,1])$ such that $\rho(f, g) < \varepsilon$ and the inverse limit of copies of $[0,1]$ with f as the bonding map is the pseudoarc.*

Proof. Fix $\varepsilon > 0$. First, use Lemma 3.2 to obtain $0 < \tau < \varepsilon/16$ and a map $F \in C([0,1])$ such that $\rho(F, g) < \varepsilon/2$, F restricted to $[\tau, 1-\tau]$ is admissible and it is an identity mapping outside this interval.

We claim that there is a sequence of maps $f_1, f_2, \ldots \in C([0,1])$, an increasing sequence of positive integers $n(1) < n(2) < \ldots$, and three sequences of positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \delta_1, \delta_2, \ldots, \tau_1, \tau_2, \ldots$ such that $\rho(f_1, F) < \varepsilon/4$, $\tau_1 = \tau$, $\delta_1 = \dfrac{\tau_1}{4}$ and additionally the following conditions are satisfied:

(i) $|f_{i+1}(t) - f_i(t)| < \varepsilon_i < 2^{-i}$ for all $i = 1, 2, \ldots$ and for each $t \in [0,1]$,

(ii) f_i restricted to $[\tau_i, 1-\tau_i]$ is an admissible mapping, $f_i([\tau_i, 1-\tau_i]) = [\tau_i, 1-\tau_i]$ and $f_i(t) = t$ for each $t \in [0, \tau_i] \cup [1-\tau_i, 1]$ and for $i = 1, 2, \ldots$,

(iii) for every $i = 1, 2, \ldots$ and every $1 \le k \le i$ the map $f_i^{n(k)}$ is $(2^{-k} - 2^{-k-i})$-crooked,

(iv) for every $i = 1, 2, \ldots$ and every $1 \le k \le i$, if $b - a \ge 2^{-k}$, then $f_i^{n(k)}([a, b]) \supset [\tau_k + \xi, 1 - \xi - \tau_k]$, where $\xi = \sum_{j=k}^{i} \delta_j$,

(v) $\delta_{i+1} < \dfrac{\delta_i}{2}$, $\varepsilon_{i+1} < \dfrac{\varepsilon_i}{2}$, $\tau_{i+1} < \dfrac{\tau_i}{2}$, $\varepsilon_i < \dfrac{\tau_i}{4}$.

We prove the above claim by the induction on i. For $i = 1$ apply Lemma 3.5 with $\theta = \tau_1$, $\eta = \varepsilon_1 = \min\{\tau_1/8, 2^{-2}\}$, $\delta = 2^{-1} - 2^{-1-1} = 1/4$ and map $F|_{[\tau_1, 1-\tau_1]}$ obtaining an admissible map $f_1 \colon [\tau_1, 1-\tau_1] \to [\tau_1, 1-\tau_1]$ and an integer $n(1) > 0$ such that (3) and (3) are satisfied. Next we extend f_1 onto whole $[0,1]$ putting $f_1(t) = t$ for all $t \in [0, \tau_1] \cup [1-\tau_1, 1]$, hence by our construction also (3) holds. This completes the first step of induction.

Next assume that the claim holds for all $i = 1, \ldots, s$ for some $s \ge 1$. We will prove that it also holds for $i = s+1$. Put $\tau_{s+1} = \tau_s/4$ and fix any $\delta_{s+1} < \min\left\{\delta_s/2, 2^{-2s-4}\right\}$. By uniform continuity of f_s we can find $\varepsilon_{s+1} < \min\left\{2^{-s-1}, \varepsilon_s/2, \tau_{s+1}/4\right\}$ such that if $\rho(h, f_s) < \varepsilon_{s+1}$ then $\rho(h^j, f_s^j) < \delta_{s+1}$ for $j = 0, 1, \ldots, n(s)$. In particular, for any $1 \le k \le s$ we have $(2^{-k} - 2^{-k-s} + 2\delta_{s+1}) < (2^{-k} - 2^{-k-s-1})$. Hence, by Lemma 2.1, for every $1 \le k \le s$ the map $h^{n(k)}$ is $(2^{-k} - 2^{-k-s-1})$-crooked. Similarly $h^{n(k)}([a, b]) \supset [\tau_k + \xi, 1 - \xi - \tau_k]$ where $\xi = \sum_{j=k}^{s+1} \delta_j$.

Note that f_s is an admissible map on $[\tau_s, 1-\tau_s]$, endpoints of this interval are fixed points of f_s and by Lemma 3.3 there are also admissible maps on $[\tau_{s+1}, \tau_s]$ and $[1-\tau_s, 1-\tau_{s+1}]$ which are arbitrarily small perturbations of identity on these intervals (and endpoints of these intervals are fixed points for them). Therefore, we can apply twice Lemma 3.8 obtaining an admissible map $G \colon [\tau_{s+1}, 1-\tau_{s+1}]$ such that $\sup_{t \in [\tau_{s+1}, 1-\tau_{s+1}]} |G(t) - f_s(t)| < \varepsilon_{s+1}/2$. We can extend G to the whole $[0,1]$ putting $G(t) = t$ for all $t \in [0, \tau_{s+1}] \cup [1-\tau_{s+1}, 1]$ and then $G \in C([0,1])$ and additionally $\rho(G, f_s) < \varepsilon_{s+1}/2$. Next we apply Lemma 3.5 with $\theta = \tau_{s+1}$, $\eta = \min\{\varepsilon_{s+1}/2, 2^{-s-2}\}$, $\delta = 2^{-s-1} - 2^{-2s-2}$ and map $G|_{[\tau_{s+1}, 1-\tau_{s+1}]}$ obtaining an admissible map $f_{s+1} \colon [\tau_{s+1}, 1-\tau_{s+1}] \to [\tau_{s+1}, 1-\tau_{s+1}]$ and an integer $n(s+1) > 0$ such that (3) and (3) are satisfied with $k = s+1$.

As before, we extend f_{s+1} to $[0,1]$ putting identity on the points outside $[\tau_{s+1}, 1-\tau_{s+1}]$. Since $\rho(f_{s+1}, f_s) < \varepsilon_{s+1}$, conditions (3) and (3) are satisfied for every $1 \leq k \leq s+1$ and also (3) holds. This completes the induction.

Note that $\sum_{j=1}^{\infty} \varepsilon_j \leq \tau/2$, that is f_i converges uniformly to a map $f(t) = \lim_{i\to\infty} f_i(t)$, in particular $f\colon [0,1] \to [0,1]$ is continuous. This also proves that $\rho(f,g) < \varepsilon$. Furthermore, by (3) and Lemma 2.2 for each k the map $f^{n(k)}$ is 2^{-k}-crooked, so the inverse limit of $[0,1]$ with f as a single bonding map is the pseudoarc by Lemma 2.3.

Fix an open interval $U \subset [0,1]$ and fix any $a < b$ such that $[a,b] \subset U$. Fix any $\gamma > 0$. Note that $\xi_k = \sum_{j=k}^{\infty} \delta_j \leq \sum_{j=k}^{\infty} 2^{-j}\tau \leq 2^{k-1}\tau$ so there is $k > 0$ such that $[\gamma/2, 1-\gamma/2] \subset$ $\subset [\tau_k + 2\xi_k, 1-2\xi_k - \tau_k]$. We may also assume that k is sufficiently large so that $b - a > 2^{-k}$. Then, by (3) we have that

$$f_i^{n(k)}([a,b]) \supset [\tau_k + \xi, 1-\xi-\tau_k] \supset [\tau_k + 2\xi_k, 1-2\xi_k-\tau_k] \supset [\gamma/2, 1-\gamma/2]$$

for every $i \geq k$, and additionally, if i is sufficiently large then $\rho(f, f_i) < \gamma/2$. This shows that

$$f^{n(k)}(U) \supset f^{n(k)}([a,b]) \supset [\gamma, 1-\gamma].$$

Since U and γ were arbitrary, we see that f is mixing.

Finally, observe that

$$\sum_{j=k}^{\infty} \varepsilon_j \leq \sum_{j=k}^{\infty} 2^{-j+k}\varepsilon_k \leq 2\varepsilon_k \leq \tau_k/2.$$

Since $f_k([\tau_k, 1-\tau_k]) = [\tau_k, 1-\tau_k]$ we obtain that $f([\tau_k, 1-\tau_k]) \subset [\tau_k/2, 1-\tau_k/2] \subset (0,1)$. This holds for every $k = 1, 2, \ldots$ and hence

$$f((0,1)) = f\left(\bigcup_{k=1}^{\infty} [\tau_k, 1-\tau_k]\right) \subset (0,1).$$

This shows that f is not exact completing the proof.

Theorem 3.1 is proved.

4. Final comments. In Theorem 3.1 we provided a method of perturbation in the case that we start with topologically exact map $f \in C([0,1])$ such that both endpoints are fixed points. But it is clear that similar technique will work in the case that only one of endpoints is fixed, say $f(0) = 0$, however in this case its perturbation F which generates the pseudoarc will have only one inaccessible fixed point, that is $F(0) = 0$ and $F((0,1]) = (0,1]$. It may even happen that $F(1) \neq 1$.

In fact a similar approach will be successful, if G is a topological graph with an endpoint p and f is a topologically exact map on G such that $f(p) = p$. Simply, most of our techniques work locally, so it is really unimportant that G is not an interval. The only essential ingredient is an endpoint which is a fixed point [15] (see also [8]).

Finally, it also should be clear that a simple modification of techniques involved in Theorem 3.1 will work in case of topologically exact maps with periodic endpoints, that is $f(0) = 1$, $f(1) = 0$. Simply, instead of attaching identity map at each endpoint, we will attach 2-periodic intervals.

While we present here only rough idea of other cases, a careful reader should be able to make all the necessary adjustments and prove theorems announced above.

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