UDC 517.5

Jianglong Wu (Mudanjiang Normal Univ., China),
Qingguo Liu (Univ. Nova Gorica, Slovenia)

WEIGHTED ESTIMATES FOR MULTILINEAR COMMUTATORS OF MARCINKIEWICZ INTEGRALS WITH BOUNDED KERNEL *

ЗВАЖЕНІ ОЦІНКИ ДЛЯ МУЛЬТИЛІНІЙНИХ КОМУТАТОРІВ ІНТЕГРАЛІВ МАРЦИНКЕВИЧА З ОБМЕЖЕНИМИ ЯДРАМИ

Let $\mu_{\Omega,\vec{b}}$ be a multilinear commutator generalized by μ_Ω, the n-dimensional Marcinkiewicz integral with bounded kernel, and let $b_j \in \mathrm{Osc}_{\exp L^{r_j}}\,(1 \le j \le m)$. We prove the following weighted inequalities for $\omega \in A_\infty$ and $0 < p < \infty$:

$$\|\mu_\Omega(f)\|_{L^p(\omega)} \le C\|M(f)\|_{L^p(\omega)}, \qquad \|\mu_{\Omega,\vec{b}}(f)\|_{L^p(\omega)} \le C\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}.$$

The weighted weak $L(\log L)^{1/r}$ -type estimate is also established for $p = 1$ and $\omega \in A_1$.

Нехай $\mu_{\Omega,\vec{b}}$ — мультилінійний комутатор, що узагальнює μ_Ω, n-вимірний інтеграл Марцинкевича з обмеженим ядром, та нехай $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $1 \le j \le m$. Доведено такі зважені нерівності для $\omega \in A_\infty$ та $0 < p < \infty$:

$$\|\mu_\Omega(f)\|_{L^p(\omega)} \le C\|M(f)\|_{L^p(\omega)}, \qquad \|\mu_{\Omega,\vec{b}}(f)\|_{L^p(\omega)} \le C\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}.$$

Зважену слабку оцінку $L(\log L)^{1/r}$-типу також встановлено для $p = 1$ та $\omega \in A_1$.

1. Introduction and main results. Suppose that S^{n-1} is the unit sphere in $\mathbf{R}^n$, $n \ge 2$, equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^1(S^{n-1})$ be a homogeneous function of degree zero which satisfies the cancellation condition

$$\int\limits_{S^{n-1}} \Omega(x')\,dx' = 0, \tag{1.1}$$

where $x' = x/|x|$ $(\forall x \ne 0)$.

The n-dimensional Marcinkiewicz integral corresponding to the Littlewood – Paley g-function introduced by Stein [1] is defined by $\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \dfrac{dt}{t^3}\right)^{1/2}$, where $F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \dfrac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy$.

As usual, we denote by A_p, $1 \le p < \infty$, the Muckenhoupt's weights class. We denote $[\omega]_{A_p}$ as A_p constant (see [2], Chapter V or [3], Chapter 9 for details). Operators that map L^p to L^q are called of strong type (p,q) and operators that map L^p to $L^{q,\infty}$ are called of weak type (p,q) (see [3, p. 32]). Let

$$\log^+ t = \max(\log t, 0) = \begin{cases} \log t, & \text{when } \ t > 1, \\ 0, & \text{when } \ 0 \le t \le 1, \end{cases}$$

where $\log t = \ln t$, and we denote by $L(\log L)$ the set of all f with $\int_{\mathbf{R}^n} |f(x)| \log^+ |f(x)|\,dx < \infty$ (see [2, p. 128], [3], § 7.5.a). Here and in what follows, $\|b\|_*$ denotes the BMO-norm of b (see [3], Chapter 7 for details).

* This work was supported in part by the Fund of Heilongjiang Provincial Education Department (No. 12531720), the NSF (No. A200913) of Heilongjiang Province and NNSF (No. 11161042) of China.

In 1958, Stein [1] proved that μ_Ω is of strong type (p,p) for $1 < p \le 2$ and of weak type $(1,1)$ when $\Omega \in \mathrm{Lip}_\alpha$, $0 < \alpha \le 1$, that is, there is a constant $C > 0$ such that

$$\big|\Omega(x') - \Omega(y')\big| \le C|x' - y'|^\alpha \quad \forall\, x', y' \in S^{n-1}. \tag{1.2}$$

In 1990, Torchinsky and Wang [4] studied the weighted L^p-boundedness of μ_Ω when Ω satisfies (1.1) and (1.2). They also considered the weighted L^p-norm inequality for the commutator of the Marcinkiewicz integral, which is defined by

$$\mu^m_{\Omega,b}(f)(x) = \left(\int\limits_0^\infty \left|\int\limits_{|x-y|\le t} \frac{(b(x)-b(y))^m\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad m \in \mathbf{N}.$$

In 2004, Ding, Lu and Zhang [5] studied the weighted weak $L(\log L)$-type estimates for $\mu^m_{\Omega,b}$, precisely, if $\omega \in A_1$, $b \in \mathrm{BMO}$, Ω satisfies (1.1) and (1.2), then, for all $\lambda > 0$, there exists a constant $C > 0$, such that

$$\omega\big(\big\{x \in \mathbf{R}^n : |\mu^m_{\Omega,b}(f)(x)| > \lambda\big\}\big) \le C \int\limits_{\mathbf{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+ \frac{|f(x)|}{\lambda}\right)^m \omega(x)\,dx.$$

In 2008, Zhang [6] studied the weighted boundedness for the multilinear commutator of Marcinkiewicz integral $\mu_{\Omega,\vec{b}}$ when $\Omega \in \mathrm{Lip}_\alpha$, $0 < \alpha \le 1$, $0 < p < \infty$ and $\omega \in A_\infty$ (see [3], § 9.3), and established a weighted weak $L(\log L)^{1/r}$-type estimate when $p = 1$ and $\omega \in A_1$, where

$$\mu_{\Omega,\vec{b}}(f)(x) = \left(\int\limits_0^\infty \left|\int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left(\prod_{j=1}^m \big(b_j(x) - b_j(y)\big)\right) f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad m \in \mathbf{N}.$$

And in 2012, Zhang, Wu and Liu [7] establish the weighted weak $L(\log L)^m$-type estimate for $\mu_{\Omega,\vec{b}}$ when Ω satisfies a kind of Dini conditions.

In 2004, Lee and Rim [8] proved the L^p boundedness for μ_Ω when there exist constants $C > 0$ and $\rho > 1$ such that

$$\big|\Omega(x') - \Omega(y')\big| \le \frac{C}{\left(\log \dfrac{1}{|x'-y'|}\right)^\rho} \tag{1.3}$$

holds uniformly in $x', y' \in S^{n-1}$, and $\Omega \in L^\infty(S^{n-1})$ be a homogeneous function of degree zero with cancellation property (1.1). In 2005, Ding [9] studied the weak $(1,1)$-type estimate when $\rho > 2$ and Ω satisfies (1.1) and (1.3).

In the following, we will always assume that $\Omega \in L^\infty(S^{n-1})$ and satisfies (1.1) and (1.3), where $\rho > 2$. Let m be a positive integer. For $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $r_j \ge 1$, $1 \le j \le m$, we denote

$$\frac{1}{r} = \frac{1}{r_1} + \ldots + \frac{1}{r_m}, \qquad \|\vec{b}\| = \prod_{j=1}^m \|b,j\|_{\mathrm{Osc}_{\exp L^{r_j}}}. \tag{1.4}$$

For the definitions of $\mathrm{Osc}_{\exp L^r}$, $\|\cdot\|_{\mathrm{Osc}_{\exp L^r}}$ and $M_{L(\log L)^{1/r}}$, see Section 2.

Our results can be stated as follows.

Theorem 1.1. *Let* $0 < p < \infty$ *and suppose that* $\omega \in A_\infty$. *For* $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ *is homogeneous of degree zero and satisfies* (1.1) *and* (1.3). *Then there is a positive constant* C, *such that*

$$\int\limits_{\mathbf{R}^n} |\mu_\Omega(f)(x)|^p \omega(x)\,dx \leq C[\omega]_{A_\infty}^p \int\limits_{\mathbf{R}^n} [M(f)(x)]^p \omega(x)\,dx$$

for all bounded functions f *with compact support.*

Theorem 1.2. *Let* $0 < p < \infty$, $\omega \in A_\infty$ *and* $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $r_j \geq 1$, $1 \leq j \leq m$, r *and* $\|\vec{b}\|$ *be as in* (1.4). *For* $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ *is homogeneous of degree zero and satisfies* (1.1) *and* (1.3). *Then there is a positive constant* C, *such that*

$$\int\limits_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx \leq C\|\vec{b}\|^p \int\limits_{\mathrm{R}^n} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x)\,dx \tag{1.5}$$

for all bounded functions f *with compact support.*

Since $r_j \geq 1$, $j = 1, 2, \ldots, m$, then $M_{L(\log L)^{1/r}}$ is pointwise smaller than $M_{L(\log L)^m}$. Noting that $M_{L(\log L)^m}$ is equivalent to M^{m+1}, the $m+1$ iterations of the Hardy – Littlewood maximal operator M (see (21) in [10]), by using the weighted L^p-boundedness of M again, from Theorem 1.2, we have the following result.

***Corollary* 1.1.** *Let* $1 < p < \infty$, $\omega \in A_p$, $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $r_j \geq 1$, $1 \leq j \leq m$, r *and* $\|\vec{b}\|$ *be as in* (1.4). *For* $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ *is homogeneous of degree zero and satisfies* (1.1) *and* (1.3). *Then there is a positive constant* C, *such that*

$$\int\limits_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx \leq C\|\vec{b}\|^p \int\limits_{\mathrm{R}^n} |f(x)|^p \omega(x)\,dx$$

for all bounded functions f *with compact support.*

Theorem 1.3. *Let* $\omega \in A_1$, $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $r_j \geq 1$, $1 \leq j \leq m$, r *and* $\|\vec{b}\|$ *be as in* (1.4). *For* $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ *is homogeneous of degree zero and satisfies* (1.1) *and* (1.3). *Let* $\Phi(t) = t\log^{1/r}(e + t)$. *Then there is a positive constant* C, *for all bounded functions* f *with compact support and all* $\lambda > 0$, *such that*

$$\omega(\{x \in \mathbf{R}^n \colon \mu_{\Omega,\vec{b}}(f)(x) > \lambda\}) \leq C \int\limits_{\mathbf{R}^n} \Phi\left(\frac{\|\vec{b}\||f(y)|}{\lambda}\right)\omega(y)\,dy.$$

The remainder of the paper is organized as follows. In Section 2, we will recall some notation and known results we need, and establish the basic estimates for sharp functions. In Section 3 we prove Theorems 1.1 and 1.2. In the last section, we prove Theorem 1.3.

Throughout this paper, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$. For $A \sim B$, we mean that there is a constant $C > 0$ such that $C^{-1}B \leq A \leq CB$.

2. Preliminaries and estimates for sharp functions. As usual, M stands for the Hardy–Littlewood maximal operator. For a ball B in $\mathbf{R}^n$, denote by $f_B = |B|^{-1}\int_B f(y)\,dy$. We need the following variants of M and the Fefferman–Stein's sharp function. For $\delta > 0$, define

$$M_\delta(f)(x) = \big[M(|f|^\delta)(x)\big]^{1/\delta}, \qquad M_\delta^\sharp(f)(x) = \big[M^\sharp(|f|^\delta)(x)\big]^{1/\delta},$$

where

$$M^\sharp(f)(x) = \sup_{B\ni x}\inf_c \frac{1}{|B|}\int_B |f(y)-c|\,dy \approx \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)-f_B|\,dy.$$

The following relationships between $M_\delta^\sharp$ and M_δ which will be used is a version of the classical ones due to Fefferman and Stein (see [2, p. 153]).

Lemma 2.1 [10–12]. (a) *Let $\omega \in A_\infty$ and $\phi\colon (0,\infty)\to(0,\infty)$ be doubling. Then there exists a positive constant C, depending upon the doubling condition of ϕ, such that, for all λ, $\delta > 0$*

$$\sup_{\lambda>0}\phi(\lambda)\omega(\{y\in\mathbf{R}^n\colon M_\delta(f)(y)>\lambda\}) \le C[\omega]_{A_\infty}\sup_{\lambda>0}\phi(\lambda)\omega(\{y\in\mathbf{R}^n\colon M_\delta^\sharp(f)(y)>\lambda\}),$$

for every function f such that the left-hand side is finite.

(b) *Let $\omega\in A_\infty$ and $0<p,\delta<\infty$. Then there exists a positive constant C, depending upon p, such that*

$$\int_{\mathbf{R}^n}\big[M_\delta(f)(x)\big]^p\omega(x)\,dx \le C[\omega]_{A_\infty}^p\int_{\mathbf{R}^n}\big[M_\delta^\sharp(f)(x)\big]^p\omega(x)\,dx,$$

for every function f such that the left-hand side is finite.

A function Φ defined on $[0,\infty)$ is said to be a Young function, if Φ is a continuous, nonnegative, strictly increasing and convex function with $\Phi(0)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$. Define the Φ-average of a function f on a ball B by

$$\|f\|_{\Phi,B} = \inf\left\{\lambda>0\colon \frac{1}{|B|}\int_B \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}.$$

The maximal operator M_Φ associated with the Φ-average, $\|\cdot\|_{\Phi,B}$, is defined by

$$M_\Phi(f)(x) = \sup_{B\ni x}\|f\|_{\Phi,B},$$

where the supremum is taken over all the balls B containing x.

When $\Phi(t) = t\log^r(e+t)$, we denote $\|\cdot\|_{\Phi,B}$ and M_Φ by $\|\cdot\|_{L(\log L)^r,B}$ and $M_{L(\log L)^r}$, respectively. When $\Phi(t) = e^{t^r}-1$, we denote $\|\cdot\|_{\Phi,B}$ and M_Φ by $\|\cdot\|_{\exp L^r,B}$ and $M_{\exp L^r}$. If $k\in\mathbf{N}$ then $M_{L(\log L)^m}\sim M^{m+1}$ (see (21) of [10]).

We have the generalized Hölder's inequality as follows, for details and the more general cases see Lemma 2.3 in [11].

Lemma 2.2 [11]. *Let $r_1,\ldots,r_m\ge 1$ with $1/r = 1/r_1+\ldots+1/r_m$ and B be a ball in $\mathbf{R}^n$. Then there holds the generalized Hölder's inequality*

$$\frac{1}{|B|}\int_B |f_1(x)\ldots f_m(x)g(x)|\,dx \le C\|f_1\|_{\exp L^{r_1},B}\ldots\|f_m\|_{\exp L^{r_m},B}\|g\|_{L(\log L)^{1/r},B}.$$

For $r \geq 1$, we say $f \in \mathrm{Osc}_{\exp L^r}$ if $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\|f\|_{\mathrm{Osc}_{\exp L^r}} < \infty$, where

$$\|f\|_{\mathrm{Osc}_{\exp L^r}} = \sup_B \|f - f_B\|_{\exp L^r, B},$$

and the supremum is taken over all the balls $B \subset \mathbf{R}^n$.

By John–Nirenberg theorem (see [2] or [13]), it is not difficult to see that $\mathrm{Osc}_{\exp L^1} = \mathrm{BMO}(\mathbf{R}^n)$ and $\mathrm{Osc}_{\exp L^r}$ is contained properly in $\mathrm{BMO}(\mathbf{R}^n)$ when $r > 1$ (see [14]). Furthermore, $\|b\|_* \leq \leq C\|b\|_{\mathrm{Osc}_{\exp L^r}}$ when $b \in \mathrm{Osc}_{\exp L^r}$ and $r \geq 1$ (see [6]). For more information on Orlicz space see [15].

We will take the point of view of the vector-valued singular integral of Benedek, Calderón and Panzone [16]. Let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} = \left\{ h\colon \|h\|_{\mathcal{H}} = \left(\int_0^\infty \frac{|h(t)|^2}{t^3}\, dt \right)^{1/2} < \infty \right\}.$$

For all $x \in \mathbf{R}^n$ and $t > 0$, let

$$F_{\Omega,\vec{b},t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left(\prod_{j=1}^m \big(b_j(x) - b_j(y)\big) \right) f(y)\, dy, \quad m \in \mathbf{N}.$$

Then for each fixed $x \in \mathbf{R}^n$, $F_{\Omega,t}(f)(x)$ and $F_{\Omega,\vec{b},t}(f)(x)$ can be regarded as mapping from $[0,\infty)$ to $\mathcal{H}$, and

$$\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|_{\mathcal{H}}, \qquad \mu_{\Omega,\vec{b}}(f)(x) = \|F_{\Omega,\vec{b},t}(f)(x)\|_{\mathcal{H}}.$$

The following pointwise estimates for the sharp function of μ come from [17].

Lemma 2.3 [17]. *Let $0 < \delta < 1$, f, $\mu_\Omega(f)$ be both locally integrable function. For $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies* (1.1) *and* (1.3). *Then there is a positive constant C, independent of f and x, such that*

$$M^\sharp_\delta(\mu_\Omega(f))(x) \leq CM(f)(x), \qquad a.e. \quad x \in \mathbf{R}^n.$$

Some ideas for the proof of Lemma 2.3 come from [5]. For details and the more information see Lemma 3.2.4 in [17].

For the multilinear commutators $\mu_{\Omega,\vec{b}}$, there holds a similar pointwise estimate. To state it, we first introduce some notations. For all $1 \leq j \leq m$, we denote by $\mathcal{C}^m_j$ the family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of $\{1, 2, \ldots, m\}$ with j different elements. For any $\sigma \in \mathcal{C}^m_j$ and $\vec{b} = (b_1, \ldots, b_m)$, we define $\sigma' = \{1, 2, \ldots, m\} \setminus \sigma$, $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$, and $b_\sigma = b_{\sigma(1)} \ldots b_{\sigma(j)}$. For any vector $(r_{\sigma(1)}, \ldots, r_{\sigma(j)})$ of j positive numbers and $1/r_\sigma = 1/r_{\sigma(1)} + \ldots + 1/r_{\sigma(j)}$, we write

$$\|\vec{b}_\sigma\| = \|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} = \|b_{\sigma(1)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(1)}}}} \ldots \|b_{\sigma(j)}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma(j)}}}}. \tag{2.1}$$

For any $\sigma = \{\sigma(1), \ldots, \sigma(j)\} \in \mathcal{C}^m_j$ and $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$, we write

$$F_{\Omega,\vec{b}_\sigma,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left(\prod_{i=1}^j \big(b_{\sigma(i)}(x) - b_{\sigma(i)}(y)\big) \right) f(y)\, dy$$

and

$$\mu_{\Omega,\vec{b}_\sigma}(f)(x) = \left\| F_{\Omega,\vec{b}_\sigma,t}(f)(x) \right\|_{\mathcal{H}}.$$

If $\sigma = \{1, \ldots, m\}$, then $\sigma' = \varnothing$. We understand $\mu_{\Omega,\vec{b}_\sigma} = \mu_{\Omega,\vec{b}}$ and $\mu_{\Omega,\vec{b}_{\sigma'}} = \mu_\Omega$.

Lemma 2.4 [17]. *Let $r_j \geq 1$, $b_j \in \mathrm{Osc}_{\exp L^{r_j}}$, $1 \leq j \leq m$, r and $\|\vec{b}\|$ be as in* (1.4). *For $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero satisfying* (1.1) *and* (1.3)*, then for any δ and ε with $0 < \delta < \varepsilon < 1$, there is a constant $C > 0$, depending only on δ and ε, such that, for any bounded function f with compact support,*

$$M_\delta^\sharp(\mu_{\Omega,\vec{b}}(f))(x) \leq C \left(\|\vec{b}\| M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} M_\varepsilon\big(\mu_{\Omega,\vec{b}_{\sigma'}}(f)\big)(x) \right).$$

Some ideas for the proof of Lemma 2.4 come from [5, 6, 10, 11]. For details and the more information see Lemma 3.2.5 in [17].

Remark **2.1.** Noting that (1.3) is weaker than Lip_α, $0 < \alpha \leq 1$, condition, the main results in this paper improve the main results in [6]. And the Theorem 1.3 is equivalent to the Theorem 4.1.1 in [18] when $b_1 = b_2 = \ldots = b_m$.

3. Proof of Theorems 1.1 and 1.2. The proof of Theorem 1.1 is similar as Theorem 1.1 in [6]. So, we omit the details and only give the proof of Theorem 1.2 here. For brevity, we write

$$\|h(x)\|_{L^p(\omega)} = \left(\int\limits_{\mathbf{R}^n} |h(x)|^p \omega(x)\, dx \right)^{1/p} \quad \text{for} \quad 0 < p < \infty.$$

Proof of Theorem **1.2.** Without loss of generality, we assume

$$\int\limits_{\mathbf{R}^n} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x)\, dx < \infty, \tag{3.1}$$

since otherwise there is nothing to be proven. We divide the proof into two cases.

Case I. Suppose that ω and $b_j, 1 \leq j \leq m$, are all bounded. Firstly, we take it for granted that, for all bounded functions f with compact supports,

$$\int\limits_{\mathbf{R}^n} [M_\delta(\mu_{\Omega,\vec{b}}(f))(x)]^p \omega(x)\, dx < \infty \tag{3.2}$$

holds for $0 < p < \infty$ and appropriate δ with $0 < \delta < 1$.

Under the assumption of (3.2), we will proceed the proof by induction on m. For $m = 1$, $\vec{b} = b_1$, $\mu_{\Omega,\vec{b}} = \mu_{\Omega,b_1}$. By Lemma 2.1(b) and Lemma 2.4, for $0 < \delta < \varepsilon < 1$, we have

$$\|\mu_{\Omega,b_1}(f)\|_{L^p(\omega)} \leq \|M_\delta(\mu_{\Omega,b_1}(f))\|_{L^p(\omega)} \leq C\|M_\delta^\sharp(\mu_{\Omega,b_1}(f))\|_{L^p(\omega)} \leq$$

$$\leq C\|b_1\|_{\mathrm{Osc}_{\exp L^{r_1}}} \Big(\|M_{L(\log L)^{1/r_1}}(f)\|_{L^p(\omega)} + \|M_\varepsilon(\mu_\Omega(f))\|_{L^p(\omega)} \Big). \tag{3.3}$$

Since $\omega \in A_\infty$, there is a $p_0 > 1$, such that $\omega \in A_{p_0}$. We can choose $\delta > 0$ small enough, so that $p/\delta > p_0$. So $\omega \in A_{p/\delta}$. Then by the definition of M_δ and the weighted $L^{p/\delta}$-boundedness of the Hardy – Littlewood maximal operator M, we get

$$\int\limits_{\mathbf{R}^n} [M_\delta(\mu_\Omega(f))(x)]^p \omega(x)\, dx = \int\limits_{\mathbf{R}^n} [M(|\mu_\Omega(f)|^\delta)(x)]^{p/\delta} \omega(x)\, dx \le$$

$$\le \int\limits_{\mathbf{R}^n} |\mu_\Omega(f)(x)|^p \omega(x)\, dx. \tag{3.4}$$

This, together with (3.3), Theorem 1.1 and the fact $M(f) \le CM_{L(\log L)^{1/s}}(f)$ for any $s > 0$, gives

$$\|\mu_{\Omega,b_1}(f)\|_{L^p(\omega)} \le C\|b_1\|_{\mathrm{Osc}_{\exp L^{r_1}}} \Big(\|M_{L(\log L)^{1/r_1}}(f)\|_{L^p(\omega)} + \|\mu_\Omega(f)\|_{L^p(\omega)} \Big) \le$$

$$\le C\|b_1\|_{\mathrm{Osc}_{\exp L^{r_1}}} \Big(\|M_{L(\log L)^{1/r_1}}(f)\|_{L^p(\omega)} + \|M(f)\|_{L^p(\omega)} \Big) \le$$

$$\le C\|b_1\|_{\mathrm{Osc}_{\exp L^{r_1}}} \|M_{L(\log L)^{1/r_1}}(f)\|_{L^p(\omega)}.$$

Now, suppose that the theorem is true for $1, 2, \ldots, m-1$ and let us prove it for m. Recall that, if $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$, $1 \le j \le m$, and the corresponding satisfies $1/r_\sigma = 1/r_{\sigma(1)} + \ldots + 1/r_{\sigma(j)}$, then $\sigma' = \{1, \ldots, m\} \setminus \sigma$ and the corresponding $r_{\sigma'}$ satisfying $1/r_{\sigma'} = 1/r - 1/r_\sigma$. Reasoning as in (3.4), for $\theta > 0$ small enough, we obtain

$$\int\limits_{\mathbf{R}^n} [M_\theta(\mu_{\Omega,\vec{b}_{\sigma'}}(f))(x)]^p \omega(x)\, dx \le C \int\limits_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}_{\sigma'}}(f)(x)|^p \omega(x)\, dx. \tag{3.5}$$

The same argument as used above and the induction hypothesis give us that

$$\|\mu_{\Omega,\vec{b}}(f)\|_{L^p(\omega)} \le \|M_\delta(\mu_{\Omega,\vec{b}}(f))\|_{L^p(\omega)} \le C\|M_\delta^\sharp(\mu_{\Omega,\vec{b}}(f))\|_{L^p(\omega)} \le$$

$$\le C\|\vec{b}\|\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + C\sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^m} \|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} \|M_\varepsilon(\mu_{\Omega,\vec{b}_{\sigma'}}(f))\|_{L^p(\omega)} \le$$

$$\le C\|\vec{b}\|\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + C\sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^m} \|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} \|\mu_{\Omega,\vec{b}_{\sigma'}}(f)\|_{L^p(\omega)} \le$$

$$\le C\|\vec{b}\|\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} +$$

$$+ C\sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^m} \|\vec{b}_\sigma\|_{\mathrm{Osc}_{\exp L^{r_\sigma}}} \|\vec{b}_{\sigma'}\|_{\mathrm{Osc}_{\exp L^{r_{\sigma'}}}} \|M_{L(\log L)^{1/r_{\sigma'}}}(f)\|_{L^p(\omega)} \le$$

$$\le C\|\vec{b}\|\|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)},$$

where the fourth inequality follows from (3.5) and the last one follows from the fact that $M_{L(\log L)^{1/r_{\sigma'}}}(f) \le M_{L(\log L)^{1/r}}(f)$.

To finish the proof of this special case of Theorem 1.2, we need to check (3.2). From (3.5), it suffices to prove

$$\int_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx < \infty, \quad 0 < p < \infty, \tag{3.6}$$

whenever the weight ω and the functions b_j, $1 \le j \le m$, are all bounded.

Assume that supp$f \subset B = B(0,R)$ for some $R > 0$ and write

$$\int_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx = \int_{2B} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx + \int_{(2B)^c} |\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)\,dx = I + II.$$

Noting that ω and b_j are all bounded, by the Hölder inequality, the induction hypothesis and the fact $M_{L(\log L)^m} \sim M^{m+1}$, $L^{p/\delta}$-boundedness of M, there is

$$\int_{2B} |b_\sigma(x)|^p |\mu_{\Omega,b_{\sigma'}}(f)(x)|^p \omega(x)\,dx \le C_\omega \|b_\sigma\|^p_{L^\infty(\mathbf{R}^n)} |B|^{1-\delta} \|b_{\sigma'} f\|^p_{L^{p/\delta}(\mathbf{R}^n)} < \infty.$$

This and the definition of $\mu_{\Omega,\vec{b}}(f)$ give us that

$$I \le C \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \int_{2B} |b_\sigma(x)|^p |\mu_{\Omega,b_{\sigma'}}(f)(x)|^p \omega(x)\,dx < \infty. \tag{3.7}$$

To deal with II , we first estimate $\mu_{\Omega,\vec{b}}(f)(x)$ for $x \in (2B)^c$. $|x|/2 \le |x-y| \le 3|x|/2$ when $x \in (2B)^c$ and $y \in B$. Noting that $\Omega \in L^\infty(S^{n-1})$, ω and b_j are bounded functions and $|x| \sim |x-y|$ when $x \in (2B)^c$ and $y \in B$, there is a constant $C_{\Omega,\vec{b},\omega}$, depending on the L^∞-norm of Ω, b_j and ω, such that

$$\mu_{\Omega,\vec{b}}(f)(x) \le C\|\Omega\|_{L^\infty(S^{n-1})} \|\vec{b}\|_{L^\infty(\mathbf{R}^n)} \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \le$$

$$\le C_{\Omega,\vec{b},\omega} \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\le t} \frac{dt}{t^3} \right)^{1/2} dy \le$$

$$\le C_{\Omega,\vec{b},\omega} \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy \le C_{\Omega,\vec{b},\omega} \frac{1}{|2B|} \int_{\mathbf{R}^n} |f(y)|\,dy \le C_{\Omega,\vec{b},\omega} M(f)(x). \tag{3.8}$$

By (3.8) and the fact that $M(f)(x) \le CM_{L(\log L)^{1/r}}(f)(x)$, it follows from (3.1) that

$$II \le C_{\Omega,\vec{b},\omega} \int_{(2B)^c} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x)\,dx < \infty.$$

This together with (3.7) shows that (3.6) is true when ω and b_j are bounded functions, so does (3.2). And then Theorem 1.2 is proved for this special case.

Case II. For unbounded ω and b_j, we will truncate the weight ω and the functions b_j, $j = 1, \ldots, m$, as follows. Let N be a positive integer, denote by $\omega_N = \inf\{\omega, N\}$ and by $\vec{b}^N = (b_1^N, \ldots, b_m^N)$, where b_j^N is defined by

$$b_j^N(x) = \begin{cases} N, & \text{when } b_j(x) > N, \\ b_j(x), & \text{when } |b_j(x)| \le N, \\ -N, & \text{when } b_j(x) < -N. \end{cases}$$

By Lemma 2.4 in [11], there is a positive constant C independent of N such that

$$\|b_j^N\|_{\mathrm{Osc}_{\exp L^{r_j}}} \le \|b_j\|_{\mathrm{Osc}_{\exp L^{r_j}}}. \tag{3.9}$$

Applying (1.5) for $\vec{b}^N$ and ω_N, and using (3.9), we have

$$\int\limits_{\mathbf{R}^n} |\mu_{\Omega,\vec{b}^N}(f)(x)|^p \omega_N(x)\,dx \le C\|\vec{b}\|^p \int\limits_{\mathbf{R}^n} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x)\,dx. \tag{3.10}$$

Next, taking into account the fact that f has compact support, we deduce that b_j^N converges to b_j and $b_{\sigma(1)}^N \ldots b_{\sigma(j)}^N f$ converges to $b_{\sigma(1)} \ldots b_{\sigma(j)} f$ in any space L^p for $p > 1$ as $N \to \infty$. Recalling the L^p-boundedness of μ_Ω, we claim that, at least for a subsequence, $\{|\mu_{\Omega,\vec{b}^N}(f)(x)|^p \omega_N(x)\}_{N=1}^\infty$ converges pointwise almost everywhere to $|\mu_{\Omega,\vec{b}}(f)(x)|^p \omega(x)$ as $N \to \infty$.

This fact, together with (3.10) and Fatou's lemma, finishes the proof of Theorem 1.2.

4. Proof of Theorem 1.3. The idea of the proof of Theorem 1.3 follows that of Theorem 1.5 in [11]. We first prove the following lemma.

Lemma 4.1. *Let $\omega \in A_\infty$, $\Phi(t) = t \log^{1/r}(e+t)$, $\vec{b}$, r, and r_j be the same as in Theorem* 1.3. *Then for $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies* (1.1) *and* (1.3), *there exists a positive constant C such that*

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\delta^\sharp(\mu_{\Omega,\vec{b}}(f))(y) > t\})}{\Phi(1/t)} \le C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\Phi(\|\vec{b}\| f)(y) > t\})}{\Phi(t)} \tag{4.1}$$

for all bounded functions f with compact support and all $0 < \delta < 1$.

Proof. To use Lemma 2.1(a), we first check that

$$\sup_{t>0} \frac{1}{\Phi(1/t)} \omega\big(\{x \in \mathbf{R}^n \colon M_\varepsilon(\mu_{\Omega,\vec{b}}(f))(x) > t\}\big) < \infty \tag{4.2}$$

for all bounded functions f with compact support and all δ with $0 < \delta < 1$.

We only prove (4.2) for the special case where ω and b_j are bounded functions. For the general case, we consider the truncations of ω and $\vec{b}$ as in the proof of Theorem 1.2, by a limit discussion, this time, we take into account the weak (1,1) boundedness of μ_Ω that gives the convergence in measure. Then we can obtain (4.2) for all ω and $\vec{b}$ with the hypotheses of Lemma 4.1, we omit the details.

Assume that $\operatorname{supp} f \subset B = B(0, R)$. Then, for any $0 < \varepsilon < 1$

$$\sup_{t>0} \frac{\omega\big(\{x \in \mathbf{R}^n \colon M_\varepsilon(\mu_{\Omega,\vec{b}}(f))(x) > t\}\big)}{\Phi(1/t)} \le$$

$$\leq C_\varepsilon \sup_{t>0} \frac{\omega\big(\{x \in \mathbf{R}^n \colon M_\varepsilon(\chi_{2B}\mu_{\Omega,\vec{b}}(f))(x) > t/2\}\big)}{\Phi(1/t)} +$$

$$+ C_\varepsilon \sup_{t>0} \frac{1}{\Phi(1/t)} \omega\big(\{x \in \mathbf{R}^n \colon M_\varepsilon(\chi_{(2B)^c}\mu_{\Omega,\vec{b}}(f))(x) > t/2\}\big) = C_\varepsilon(I + II), \tag{4.3}$$

where C_ε is a positive constant depending on ε.

For I, making use of the weak (1,1) boundedness of M and $[\Phi(1/t)]^{-1} \leq Ct$, and noting that ω and b_j are all bounded. Then there is a positive constant C_ω, depending on ω, such that

$$I \leq C_\omega \sup_{t>0} t \big|\{x \in \mathbf{R}^n \colon M_\varepsilon(\chi_{2B}\mu_{\Omega,\vec{b}}(f))(x) > t/2\}\big| \leq$$

$$\leq C_\omega \int\limits_{2B} |\mu_{\Omega,\vec{b}}(f)(x)|\, dx \leq C_\omega |B|^{1/2} \left(\int\limits_{2B} |\mu_{\Omega,\vec{b}}(f)(x)|^2\, dx \right)^{1/2} < \infty,$$

where the last step follows as (3.7).

Recall the fact that $(M(f))^\varepsilon \in A_1$ for $0 < \varepsilon < 1$ and f locally integrable, then

$$M_\varepsilon(M(f))(x) = \big[M(|M(f)|^\varepsilon)(x)\big]^{1/\varepsilon} \leq CM(f)(x).$$

Noting that ω is bounded, it follows from (3.8) and the weak (1,1) boundedness of M that

$$II \leq C_\omega \sup_{t>0} t \cdot \omega\big(\{x \in \mathbf{R}^n \colon M_\varepsilon(M(f))(x) > Ct\}\big) \leq$$

$$\leq C_\omega \sup_{t>0} t \cdot \omega\big(\{x \in \mathbf{R}^n \colon M(f)(x) > Ct\}\big) \leq$$

$$\leq C_\omega \int\limits_{\mathbf{R}^n} |f(x)|\, dx < \infty.$$

Combining (4.3) and the estimates for I and II, we have (4.2).

Now, let us turn to proving (4.1) by induction. For $\vec{b} \in \mathrm{Osc}_{\exp L^r}$, write $\tilde{b} = \vec{b}/\|\vec{b}\|$, then $\|\tilde{b}\| = 1$, and $\mu_{\Omega,\vec{b}}(f)/\|\vec{b}\| = \mu_{\Omega,\vec{b}/\|\vec{b}\|}(f) = \mu_{\Omega,\tilde{b}}(f)$. So we can assume that $\|\vec{b}\| = 1$. For $m = 1$, we understand $\vec{b} = b$, $\|\vec{b}\| = \|b\|_{\mathrm{Osc}_{\exp L^r}} = 1$, $\mu_{\Omega,\vec{b}}(f) = \mu_{\Omega,b}(f)$. Therefore, to prove (4.1), it suffices to prove

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\delta^\sharp(\mu_{\Omega,b}(f))(y) > t\})}{\Phi(1/t)} \leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_{L(\log L)^{1/r}}(f)(y) > t\})}{\Phi(1/t)} \tag{4.4}$$

for all bounded functions f with compact support.

Applying Lemma 2.4 for $m = 1$ and any ε with $0 < \delta < \varepsilon < 1$, it is easy to see that the left-hand side of (4.4) is dominated by

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\delta^\sharp(\mu_{\Omega,b}(f))(y) > t\})}{\Phi(1/t)} \leq$$

$$\leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_{L(\log L)^{1/r}}(f)(y) > t/2\})}{\Phi(1/t)} +$$

$$+ C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n : M_\varepsilon(\mu_\Omega(f))(y) > t/2\})}{\Phi(1/t)}.$$

Recall that (4.2) is valid and since $[\Phi(1/t)]^{-1}$ is doubling, then by Lemma 2.1(a), Lemma 2.3 and noting that $M(f) \leq M_{L(\log L)^{1/r}}(f)$, we have

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\delta^\sharp(\mu_{\Omega,b}(f))(y) > t\})}{\Phi(1/t)} \leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_{L(\log L)^{1/r}}(f)(y) > t\})}{\Phi(1/t)} +$$

$$+ C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\varepsilon^\sharp(\mu_\Omega(f))(y) > t\})}{\Phi(1/t)} \leq$$

$$\leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_{L(\log L)^{1/r}}(f)(y) > t\})}{\Phi(1/t)} + C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M(f)(y) > t\})}{\Phi(1/t)} \leq$$

$$\leq C \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_{L(\log L)^{1/r}}(f)(y) > t\}).$$

This is (4.4), thus, we have proved (4.1) for $m = 1$.

Now, let us check (4.1) for the general case $m \geq 2$. Suppose that (4.1) holds for $m - 1$, let us prove it for m. Noting that (4.2) is true and recalling the fact that $[\Phi(1/t)]^{-1}$ is doubling, then by Lemmas 2.3 and 2.4 for ε with $0 < \delta < \varepsilon$, Lemma 2.1(a) and the induction hypothesis on (4.1), we obtain

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\delta^\sharp(\mu_{\Omega,\vec{b}}(f))(y) > t\})}{\Phi(1/t)} \leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > t/C_m\})}{\Phi(1/t)} +$$

$$+ C \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\varepsilon(\mu_{\Omega,\vec{b}_{\sigma'}}(\|\vec{b}_\sigma\| f))(y) > t/C_m\}) \leq$$

$$\leq C_m \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > t\}) +$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\varepsilon^\sharp(\mu_{\Omega,\vec{b}_{\sigma'}}(\|\vec{b}_\sigma\| f))(y) > t\}) \leq$$

$$\leq C_m \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > t\}) +$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\Phi(\|\vec{b}_{\sigma'}\| \|\vec{b}_\sigma\| f)(y) > t\}) \leq$$

$$\leq C_m \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n : M_\Phi(f)(y) > t\}) +$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in \mathcal{C}_j^m} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > t\}),$$

where $\|\vec{b}_\sigma\|$ and $\|\vec{b}_{\sigma'}\|$ are as in (2.1), and in the last step, we make use of the fact that $\|\vec{b}_{\sigma'}\|\|\vec{b}_\sigma\| = $ $= \|\vec{b}\| = 1$.

This concludes (4.1) for all m, so the proof of Lemma 4.1 is completed.

Lemma 4.2. *Let* $\omega \in A_\infty$, $\Phi(t) = t \log^{1/r}(e+t)$, $\vec{b}$, r, *and* r_j *be the same as in Theorem* 1.3. *For* $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ *is homogeneous of degree zero and satisfies* (1.1) *and* (1.3), *there exists a positive constant* C *such that*

$$\sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon \mu_{\Omega,\vec{b}}(f)(y) > t\})}{\Phi(1/t)} \leq C \sup_{t>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\Phi(\|\vec{b}\| f)(y) > t\})}{\Phi(1/t)}$$

for all bounded functions f *with compact support.*

The proof is similar as the proof of Lemma 4.2 in [6], we omit the details here.

To prove Theorem 1.3, we need the following weighted weak-type inequality due to Pérez and Trujillo – González [11].

Lemma 4.3 [11]. *Let* $\omega \in A_1$, $\Phi(t) = t \log^{1/r}(e+t)$. *Then there is a positive constant* C, *for any* $\lambda > 0$ *and any locally integrable function* f, *such that*

$$\omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > \lambda\}) \leq C \int\limits_{\mathbf{R}^n} \Phi\left(\frac{|f(y)|}{\lambda}\right) \omega(y)\, dy.$$

***Proof of Theorem* 1.3.** By homogeneity of $\vec{b}$, we can assume that $\lambda = \|\vec{b}\| = 1$. Then we only need to prove that

$$\omega\big(\{y \in \mathbf{R}^n \colon \mu_{\Omega,\vec{b}}(f)(y) > 1\}\big) \leq C \int\limits_{\mathbf{R}^n} \Phi\big(|f(y)|\big) \omega(y)\, dy.$$

By $\Phi(ab) \leq 2\Phi(a)\Phi(b)$, $a, b \geq 0$ and Lemmas 4.2 and 4.3, we have

$$\omega(\{y \in \mathbf{R}^n \colon \mu_{\Omega,\vec{b}}(f)(y) > 1\}) \leq C \sup_{\lambda>0} \frac{1}{\Phi(1/\lambda)} \omega(\{y \in \mathbf{R}^n \colon \mu_{\Omega,\vec{b}}(f)(y) > \lambda\}) \leq$$

$$\leq C \sup_{\lambda>0} \frac{\omega(\{y \in \mathbf{R}^n \colon M_\Phi(f)(y) > \lambda\})}{\Phi(1/\lambda)} \leq C \sup_{\lambda>0} \frac{1}{\Phi(1/\lambda)} \int\limits_{\mathbf{R}^n} \Phi\left(\frac{|f(y)|}{\lambda}\right) \omega(y)\, dy \leq$$

$$\leq C \sup_{\lambda>0} \frac{1}{\Phi(1/\lambda)} \int\limits_{\mathbf{R}^n} \Phi\big(|f(y)|\big) \Phi(1/\lambda) \omega(y)\, dy \leq C \int : \mathbf{R}^n \Phi\big(|f(y)|\big) \omega(y)\, dy.$$

Theorem 1.3 is proved.

1. *Stein E. M.* On the functions of Littlewood – Paley, Lusin and Marcinkiewicz // Trans. Amer. Math. Soc. – 1958. – **88**, № 2. – P. 430 – 466.

2. *Stein E. M.* Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. – Princeton, NJ: Princeton Univ. Press, 1993.
3. *Grafakos L.* Classical and modern Fourier analysis. – New Jersey: Pearson Education, 2004.
4. *Torchinsky A., Wang S.* A note on the Marcinkiewicz integral // Colloq. Math. – 1990. – **60/61**. – P. 235 – 243.
5. *Ding Y., Lu S., Zhang P.* Weighted weak type estimates for commutators of the Marcinkiewicz integrals // Sci. China (Ser. A). – 2004. – **47**, № 1. – P. 83 – 95.
6. *Zhang P.* Weighted estimates for multilinear commutators of Marcinkiewicz integrals // Acta math. sinica. – 2008. – **24**, № 8. – P. 1387 – 1400.
7. *Zhang P., Wu J. L., Liu Q. G.* Weighted endpoint estimates for multilinear commutators of Marcinkiewicz integrals // Acta Math. Sci. – 2012. – **32A**, № 5. – P. 892 – 903.
8. *Lee J., Rim K. S.* Estimates of Marcinkiewicz integrals with bounded homogeneous kernel of degree zero // Integral Equat. and Operator Theory. – 2004. – **48**. – P. 213 – 223.
9. *Ding Y.* A note on end properties of Marcinkiewicz integral // J. Korean Math. Soc. – 2005. – **42**, № 5. – P. 1087 – 1100.
10. *Pérez C.* Endpoint estimates for commutators of singular integral operators // J. Funct. Anal. – 1995. – **128**, № 1. – P. 163 – 185.
11. *Pérez C., Trujillo-González R.* Sharp weighted estimates for multilinear commutators // J. London Math. Soc. – 2002. – **65**, № 3. – P. 672 – 692.
12. *Pérez C.* Sharp estimates for commutators of singular integrals via iterations of the Hardy – Littlewood maximal function // J. Fourier Anal. and Appl. – 1997. – **3**, № 6. – P. 743 – 756.
13. *John F., Nirenberg L.* On functions of bounded mean oscillation // Communs Pure and Appl. Math. – 1961. – **14**. – P. 415 – 426.
14. *Hu G. E., Meng Y., Yang D. C.* Multilinear commutators of singular integrals with non doubling measures // Integral Equat. and Operator Theory. – 2005. – **51**, № 2. – P. 235 – 255.
15. *Rao M. M., Ren Z. D.* Theory of Orlicz spaces // Pure and Appl. Math. – New York: Marcel Dekker, Inc., 1991.
16. *Benedek A., Calderón A. P., Panzone R.* Convolution operators on Banach space valued functions // Proc. Nat. Acad. Sci. USA. – 1962. – **48**, № 3. – P. 356 – 365.
17. *Liu Q. G.* Several problems for the commutator of Marcinkiewicz integral: MD Thesis. – Heilongjiang Univ., 2010.
18. *Shi X. F.* CBMO estimates for the commutator of Marcinkiewicz integral: MD Thesis. – Xinjiang Univ., 2009.

Received 30.11.11,
after revision — 06.01.13