UDC 512.5

S. Doğruöz (Adnan Menderes Univ., Aydin, Turkey),
A. Harmanci (Hacettepe Univ., Ankara, Turkey),
P. F. Smith (Univ. Glasgow, Scotland, UK)

MODULES WITH UNIQUE CLOSURE RELATIVE TO A TORSION THEORY. III
МОДУЛІ З ЄДИНИМ ЗАМИКАННЯМ ВІДНОСНО ТЕОРІЇ СКРУТУ. III

We continue the study of modules over a general ring R whose submodules have a unique closure relative to a hereditary torsion theory on Mod-R. It is proved that, for a given ring R and a hereditary torsion theory τ on Mod-R, every submodule of every right R-module has a unique closure with respect to τ if and only if τ is generated by projective simple right R-modules. In particular, a ring R is a right Kasch ring if and only if every submodule of every right R-module has a unique closure with respect to the Lambek torsion theory.

Продовжено вивчення модулів над загальним кільцем R, субмодулі якого мають єдине замикання відносно спадкової теорії скруту на Mod-R. Доведено, що для заданих кільця R та спадкової теорії скруту τ на Mod-R кожний субмодуль кожного правого R-модуля має єдине замикання відносно τ тоді і тільки тоді, коли τ породжується проективними простими правими R-модулями. Зокрема, кільце R є правим кільцем Каша тоді і тільки тоді, коли кожний субмодуль кожного правого R-модуля має єдине замикання відносно теорії скруту за Ламбеком.

1. Introduction. In this note all rings are associative with identity and all modules are unitary right modules. This paper is a continuation of [1] and [2], and any unexplained terms can be found in [1, 3, 4]. Let R be a ring. *All torsion theories on* Mod-R, *the category of all right R-modules, will be hereditary.* For the Goldie torsion theory and terminology for torsion theories in general and any unexplained terminology see also [5, 6, 9].

Let R be any ring and let τ be a torsion theory on Mod-R. Given an R-module M, $\tau(M)$ will denote the τ-torsion submodule of M. A submodule L of M is called τ-*essential* provided L is an essential submodule of M and M/L is a τ-torsion module. In addition, a submodule K of M is called τ-*closed in* M provided K has no proper τ-essential extension in M, i.e., if K is a τ-essential submodule in a submodule L of M, then $K = L$. Note that if K is a submodule of M such that either M/K is τ-torsion-free or K is a closed submodule of M then K is a τ-closed submodule of M. Given a submodule N of M, by a τ-*closure* of N in M we mean a τ-closed submodule K of M containing N such that N is τ-essential in K. The module M is called a τ-UC-*module* provided every submodule has a unique τ-closure in M.

In [1, 2] we investigate, for a general torsion theory τ, when a submodule of a general R-module M has a unique τ-closure and when the module M is τ-UC. In this paper we are interested when a ring R has the property that every (right) R-module is τ-UC, for a given torsion theory τ. One consequence of [10] (Theorem (1) $\Leftrightarrow$ (16)) is that the ring R is semiprime Artinian if and only if every (right) R-module is UC. The Goldie torsion theory will be denoted by τ_G. It is pointed out in [1, p. 232] that every τ_G-UC-module is UC, and conversely. Thus R is semiprime Artinian if and only if every module is τ_G-UC. Recall that if τ and ρ are torsion theories on Mod-R then $\tau \leq \rho$ provided every τ-torsion module is ρ-torsion. It is proved in [1] (Proposition 3.6) that if τ and ρ are torsion theories on Mod-R such that $\tau \leq \rho$ then every ρ-UC-module is τ-UC and in particular every UC-module is a τ-UC-module. We have proved our first result.

© S. DOĞRUÖZ, A. HARMANCI, P. F. SMITH, 2014

Proposition 1.1. *Let R be a ring and let τ be any torsion theory on* Mod-R. *Then R semiprime Artinian implies that every R-module is τ-UC. Moreover the converse holds in case $\tau_G \leq \tau$.*

Let M be an R-module. For any nonempty subset X of M, $\mathrm{ann}_R(X)$ will denote $\{r \in R\colon xr = 0$ for all $x \in X\}$ and for any nonempty subset Y of R, $\mathrm{ann}_M(Y)$ will denote $\{m \in M\colon my = 0$ for all $y \in Y\}$. In case $X = \{x\}$ and $Y = \{y\}$ we shall write $\mathrm{ann}_R(x)$ and $\mathrm{ann}_M(y)$ for $\mathrm{ann}_R(X)$ and $\mathrm{ann}_M(Y)$, respectively. Now we prove the following result.

Theorem 1.1. *Let R be any ring. Then the following statements are equivalent for a torsion theory τ on* Mod-R.

(1) *Every right R-module is τ-UC.*

(2) *For every right R-module X and τ-essential submodule Y of X, then $Y = X$.*

(3) *Every τ-torsion right R-module is projective.*

(4) *If E is an essential right ideal of R then the R-module R/E is τ-torsion-free.*

(5) *If a right ideal E of R is τ-essential in R_R, then $E = R$.*

(6) *Every singular right R-module is τ-torsion-free.*

(7) *Every proper submodule of any module is τ-closed.*

Proof. (1) $\Rightarrow$ (2). Let Y be a τ-essential submodule of X. Then $X \oplus (X/Y)$ is not τ-UC if $Y \neq X$ by [1] (Lemma 3.2). By hypothesis (1), $Y = X$.

(2) $\Rightarrow$ (3). Let M be any τ-torsion R-module. There exists a free R-module F and a submodule K of F such that $M \cong F/K$. Let L be a submodule of F maximal with respect to the property $K \cap L = 0$ (Zorn's lemma). Then $K \subseteq K \oplus L$ and $K \oplus L$ is an essential submodule of F. Since F/K is τ-torsion it follows that $K \oplus L$ is a τ-essential submodule of F and hence $F = K \oplus L$ and $M \cong L$ is projective.

(3) $\Rightarrow$ (4). Let E be an essential right ideal of R. There exists a submodule E' of R_R with $E \subseteq E'$ such that E'/E is τ-torsion and R/E' τ-torsion-free. By (3), E'/E is projective and hence $E' = E$. Thus R/E is τ-torsion-free.

(4) $\Rightarrow$ (5). Let E be a τ-essential submodule of R_R. Then E is essential in R and R/E is τ-torsion. By (4), $E = R$.

(5) $\Rightarrow$ (1). Let M be an R-module. Suppose M is not τ-UC. Then there exists a submodule X of M and a proper τ-essential submodule Y of X such that the R-module $X \oplus (X/Y)$ embeds in M (see [1], Lemma 3.2). Let $x \in X \setminus Y$. Then $xR \oplus (xR + Y)/Y$ can be embedded in M, i.e., $xR \oplus (xR/(xR \cap Y))$ embeds in M and $xR \cap Y$ is τ-essential in xR. If $A = \mathrm{ann}_R(x)$, then $xR \cong R/A$ and $xR \cap Y \cong B/A$ where B/A is τ-essential in R/A. Then B is τ-essential in R_R. By (5), $B = R$ a contradiction. Therefore M is a τ-UC-module.

(4) $\Rightarrow$ (6). Let M be a singular R-module. Let $m \in M$. Then $mR \cong R/E$ for some essential right ideal E of R. By (4), mR is a τ-torsion-free module. Hence M is τ-torsion-free.

(6) $\Rightarrow$ (7). Let N be a proper submodule of an R-module M and let K be a τ-closure of N in M. Then N is τ-essential in K and hence K/N is singular and so τ-torsion. But by hypothesis K/N is τ-torsion-free. Thus $N = K$ and hence N is τ-closed in M.

(7) $\Rightarrow$ (6). Let N be a singular R-module. Let $u \in \tau(N)$. Then $uF = 0$ for some essential right ideal F of R. Note that $R/F \cong uR \subseteq \tau(N)$ and hence F is τ-essential in R. By (5) $F = R$ and hence $u = 0$. It follows that N is τ-torsion-free.

(6) $\Rightarrow$ (4). Clear.

Theorem 1.1 is proved.

***Corollary* 1.1.** *If every R-module is τ-UC, then every τ-torsion R-module is semisimple.*

Proof. Assume that every R-module is τ-UC. Then every singular right R-module is τ-torsion free by Theorem 1.1 (6). Let M be a torsion R-module and let N be a submodule of M. By Zorn's lemma there exists a submodule K of M such that $N \oplus K$ is essential in M. Then $M/(N \oplus K)$ is singular. By hypothesis $M/(N \oplus K)$ is τ-torsion free, and it is also τ-torsion. Thus $M = N \oplus K$. It follows that M is semisimple.

Corollary 1.1 is proved.

The converse of Corollary 1.1 need not be true in general.

***Example* 1.1.** Let R be a commutative ring and let P be a maximal ideal of R such that $P = P^2$ and $\mathrm{ann}_R(P) = 0$. Let τ be the hereditary torsion theory generated by R/P. Then every τ-torsion module is semisimple but not every R-module is τ-UC.

Proof. If R/P is a projective R-module then $R = P \oplus I$ for some ideal I of R. Then $IP = 0$ and hence $I = 0$, a contradiction. Thus R/P is not projective. It follows that P is an essential ideal of R and hence R/P is a nonzero singular module which is τ-torsion and hence not τ-torsion-free. Thus not every R-module is τ-UC by Theorem 1.1.

Let X be a τ-torsion module. Let $S = \{x \in X : Px = 0\}$. Suppose $X \neq S$. Then there exists an R-submodule T of X such that $S \subset T \subseteq X$ and $T/S \cong R/P$. Then $PT \subseteq S$ and hence $PT = P^2T = P(PT) \subseteq PS = 0$. Thus $T \subseteq S$, a contradiction. It follows that $X = S$ and so X is semisimple.

***Example* 1.2.** Let p be any prime integer, let F be a field of characteristic $p > 0$ and let G be the Prüfer p-group. Then the group algebra $R = F[G]$ has augmentation ideal

$$P = \sum_{x \in G} R(x - 1)$$

which satisfies $P = P^2$ and $\mathrm{ann}_R(P) = 0$.

Proof. By [8] (Lemma 3.1.2), $\mathrm{ann}_R(P) = 0$. Let $x \in G$. Then $x = y^p$ for some $y \in G$ and hence $x - 1 = y^p - 1 = (y - 1)^p \in P^2$. It follows that $P = P^2$.

***Corollary* 1.2.** *Let R be a ring such that every singular right R-module is τ-torsion free, for some torsion theory τ. Then $M/\operatorname{Soc}(M_R)$ is τ-torsion- free and $\tau(M) \subseteq \operatorname{Soc}(M_R)$ for every R-module M.*

Proof. Let M be any R-module and let N_i $(i \in I)$ denote the collection of essential submodules of M. For each $i \in I$, M/N_i is τ-torsion-free. Thus $\prod_{i \in I}(M/N_i)$ is τ-torsion-free. Because $M/\operatorname{Soc}(M)$ is isomorphic to a submodule of $\prod_{i \in I}(M/N_i)$, it is τ-torsion free. The last part clearly follows.

Corollary 1.2 is proved.

Let R be any ring. Let $\tau(R_R) = \bigoplus_{i \in I} U_i$ where U_i is a simple module for each $i \in I$. Choose $J \subseteq I$ such that $U_j \ncong U_k$ if $j \neq k$ in J and for each $i \in I$ there exists $j \in J$ such that $U_i \cong U_j$. Then the torsion class of a torsion theory τ is generated by $\{U_j : j \in J\}$, i.e., X belongs to the torsion class of $\tau \Leftrightarrow X = \bigoplus_{\lambda \in \Lambda} V_\lambda$ where for every $\lambda \in \Lambda$, $V_\lambda \cong U_j$ for some $j \in J$.

Now we characterize the torsion theories which satisfy the equivalent conditions in Theorem 1.1.

Theorem 1.2. *Let τ be a torsion theory on* Mod-R*. Then every right R-module is τ-UC if and only if τ is generated by a collection $\mathcal{S}$ of projective simple R-modules.*

Proof. Assume that every R-module is τ-UC. By Corollary 1.1, every τ-torsion module is semisimple. Let $\mathcal{S}$ denote the set of representatives of τ-torsion simple modules. Let $V \in \mathcal{S}$. Then $V \cong R/P$ for some maximal right ideal P of R. Hence P is not essential. Otherwise R/P is singular, so it is τ-torsion-free by Theorem 1.1. But since R/P is τ-torsion, this is a contradiction. Therefore $R = P \oplus X$ for some projective simple τ-torsion right ideal X. Then $V \cong X$ and so $\mathcal{S}$ consists of all projective simple τ-torsion R-modules.

Conversely, let $\mathcal{T}$ be the collection of modules of the form $\bigoplus_{i\in I} W_i$ where for each $i \in I$ there exists $V_i \in \mathcal{S}$ such that $W_i \cong V_i$. Then $\mathcal{T}$ is closed under submodules, homomorphic images and direct sums. To complete the proof we show $\mathcal{T}$ is closed under extensions by short exact sequences. Let X be an R-module and Y a submodule of X with $Y \in \mathcal{T}$ and $X/Y \in \mathcal{T}$. Then X/Y is projective, being a direct sum of projective simple modules. Hence Y is direct summand of X. Therefore $X \cong Y \oplus (X/Y) \in \mathcal{T}$.

Theorem 1.2 is proved.

Let I be an idempotent ideal of a ring R. Then τ_I will denote the torsion theory whose torsion modules are the R-modules X such that $XI = 0$. As an application of Theorem 1.2 we now characterize the idempotent ideals I such that every R-module is a τ_I-UC-module.

***Corollary* 1.3.** *Let I be an idempotent ideal of a ring R. Then every R-module is τ_I-UC if and only if $I = eR$ for some idempotent element e of R and the ring R/I is semiprime Artinian.*

Proof. Suppose first that every R-module is τ_I-UC. Let A be a right ideal of R maximal with respect to the property that $I \cap A = 0$. It is well known that $I \oplus A$ is an essential right ideal of R and hence $I \oplus A$ is a τ_I-essential right ideal of R because $(R/(I \oplus A))I = 0$. By Theorem 1.1 (5), $R = I \oplus A$. It follows that $I = eR$ for some $e = e^2 \in R$. Let E be a right ideal of R containing I such that E/I is an essential right ideal of R/I. Then E is an essential right ideal of R and clearly E is τ_I-essential in R. Again using Theorem 1.1, $E = R$. Thus the ring R/I does not contain a proper essential right ideal and hence R/I is a semiprime Artinian ring. Conversely, suppose that $I = eR$ for some idempotent $e \in R$ such that the ring R/I is semiprime Artinian. Let F be a right ideal of R such that F is τ_I-essential in R_R. Because R/F is τ_I-torsion, we have $(R/F)I = 0$ and hence $I \subseteq F$. Next F is essential in R_R and I is closed in R_R so that F/I is an essential right ideal of R/I by [4, p. 6]. But R/I is semiprime Artinian implies that $F/I = R/I$ and hence $F = R$. By Theorem 1.1, every R-module is τ_I-UC.

Corollary 1.3 is proved.

2. Further results. Let R be a ring and M an R-module. Given a submodule L of M and an element $m \in M$ then $(L\colon m)$ will denote the set of elements $r \in R$ such that $mr \in L$. Note that $(L\colon m)$ is a right ideal of R.

Lemma 2.1. *Let R be any ring. The following conditions are equivalent for torsion theories ρ, τ on* Mod-R*.*

(1) *Every ρ-essential right ideal of R is a τ-essential right ideal of R.*

(2) *For every R-module M, every ρ-essential submodule of M is a τ-essential submodule of M.*

Proof. (2) $\Rightarrow$ (1). Apply (2) in case $M = R$.

(1) $\Rightarrow$ (2). Let M be an R-module and let L be a ρ-essential submodule of M. Then L is an essential submodule of M and M/L is ρ-torsion. Let $m \in M$ and let $A = (L\colon m)$. Then A is an essential right ideal of R such that $R/A \cong (mR + L)/L$ so that R/A is ρ-torsion. Thus A is a ρ-essential right ideal of R. In particular, R/A is τ-torsion and hence so too is $(mR + L)/L$. It follows that M/L is τ-torsion and L is a τ-essential submodule of M.

Lemma 2.1 is proved.

Note the following result.

Proposition 2.1. *Let R be any ring. Consider the following conditions for torsion theories ρ, τ on* Mod-R.

(1) $\rho \leq \tau$.

(2) *Every ρ-essential right ideal of R is a τ-essential right ideal of R.*

(3) *Every τ-UC R-module is ρ-UC.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Let M be an R-module which is not ρ-UC. By [1] (Theorem 3.4) there exist an R-module X and a proper ρ-essential submodule Y of X such that $X \oplus (X/Y)$ embeds in M. Now (2) and Lemma 2.1 show that Y is a τ-essential submodule of X. Applying [1] (Theorem 3.4) again we see that M is not a τ-UC-module.

Proposition 2.1 is proved.

***Corollary* 2.1.** *Let R be any ring and let τ be a torsion theory on Mod-R. Consider the following conditions.*

(1) $\tau_G \leq \tau$.

(2) *Every essential right ideal of R is a τ-essential right ideal of R.*

(3) *Every τ-UC R-module is a UC-module.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Proposition 2.1.

(2) $\Rightarrow$ (1). Let M be a nonzero singular R-module. Let $m \in M$. Then $\mathrm{ann}_R(m)$ is an essential, and hence τ-essential, right ideal of R. Then $mR \cong R/\mathrm{ann}_R(m)$ is τ-torsion. It follows that M is τ-torsion. Thus $\tau_G \leq \tau$.

Corollary 2.1 is proved.

Next we give criteria for $\tau_G \leq \tau$.

Proposition 2.2. *Let R be any ring and let τ be a torsion theory on* Mod-R. *Then the following statements are equivalent.*

(1) $\tau_G \leq \tau$.

(2) *For any R-module M and any submodule N of M there exists a submodule K of M such that $N \cap K = 0$ and $N \oplus K$ is τ-essential in M.*

(3) *For any right ideal A of R there exists a right ideal B of R such that $A \cap B = 0$ and $A \oplus B$ is τ-essential in R_R.*

(4) *Every singular R-module is τ-torsion.*

Proof. (1) $\Rightarrow$ (2). Assume that $\tau_G \leq \tau$. Let N be a submodule of M. By Zorn's Lemma there exists a submodule K of M maximal with respect to the property $N \cap K = 0$. Then $N \oplus K$ is essential in M. Hence $M/(N \oplus K)$ is singular. By assumption $M/(N \oplus K)$ is τ-torsion and therefore $N \oplus K$ is τ-essential in M.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Let M be any singular module. For any $x \in M$, $xR \cong R/\operatorname{ann}_R(x)$ where $\operatorname{ann}_R(x)$ is an essential right ideal of R. Now $R/\operatorname{ann}_R(x)$, and hence xR, is τ-torsion by (3). Therefore M is τ-torsion. Thus every singular module is τ-torsion.

(4) $\Rightarrow$ (1). Let X be any τ_G-torsion module. There exists a submodule Y of X such that both Y and X/Y are singular. By (4) both Y and X/Y are τ-torsion. It follows that X is τ-torsion. Thus $\tau_G \leq \tau$.

Proposition 2.2 is proved.

Next note that in Proposition 2.1, in general (2) and (3) do not imply (1) as the following example shows.

***Example* 2.1.** There are rings R and torsion theories ρ and τ on Mod-R such that every ρ-essential right ideal of R is a τ-essential right ideal of R (and hence every τ-UC-module is ρ-UC) but it is not the case that $\rho \leq \tau$.

Proof. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be the ring of 2×2 upper triangular matrices over a field F. The right ideals of R are 0, $I_1 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$, $I_{(x,y)} = \left\{ \begin{bmatrix} 0 & xc \\ 0 & yc \end{bmatrix} \,\middle|\, c \in F \right\}$, for some $x, y \in F$ and R. Let $\tau_{I_1} = \{N \in \text{Mod-}R \mid NI_1 = 0\}$ and $\tau_{I_2} = \{N \in \text{Mod-}R \mid NI_2 = 0\}$ denote the torsion theories determined by the idempotent ideals I_1 and I_2, respectively. The ring R has I_2 as its only essential right ideal. Since $(R/I_2)I_2 = 0$, I_2 is a τ_{I_2}-essential right ideal of R. On the other hand R does not have any τ_{I_1}-essential right ideal except R itself. Hence every τ_{I_1}-essential right ideal of R is a τ_{I_2}-essential right ideal of R. Since $I_{(0,1)}I_1 = 0$ and $I_{(0,1)}I_2 = I_{(0,1)} \neq 0$, $I_{(0,1)}$ is a τ_{I_1}-torsion but not τ_{I_2}-torsion module. Hence $\tau_{I_1} \not\leq \tau_{I_2}$.

Example 2.1 is completed.

Combining Proposition 2.1 and Example 2.1 we see that in general (3) does not imply (1) in Proposition 2.1. We do not know whether (3) implies (2) in Corollary 2.1. In some situations (2) does imply (1) in Proposition 2.1. For example, we have the following result.

Proposition 2.3. *Let R be a ring and let ρ, τ be torsion theories on* Mod-R *such that $\rho \leq \tau_G$. Then $\rho \leq \tau$ if and only if every ρ-essential right ideal of R is a τ-essential right ideal of R.*

Proof. The necessity follows by Proposition 2.1. Conversely, suppose that every ρ-essential right ideal of R is a τ-essential right ideal of R. Let M be a ρ-torsion module. There exists a submodule N of M such that both N and M/N are singular. Let $m \in N$. Because N is singular, $\operatorname{ann}_R(m)$ is an essential right ideal of R. Since ρ is a hereditary torsion theory mR is ρ-torsion and thus $R/\operatorname{ann}_R(m)$ is also ρ-torsion since $mR \cong R/\operatorname{ann}_R(m)$. This implies that $\operatorname{ann}_R(m)$ is a ρ-essential right ideal of R and therefore a τ-essential right ideal of R. Hence $R/\operatorname{ann}_R(m)$ is τ-torsion, i.e., mR is τ-torsion for all $m \in N$, and hence N is τ-torsion. Similarly, M/N is τ-torsion. Thus M is τ-torsion. It follows that $\rho \leq \tau$.

Proposition 2.3 is proved.

Let τ be any torsion theory on Mod-R. For any R-module M, $Z_\tau(M)$ will denote the set of elements m in M such that $mE = 0$ for some τ-essential right ideal E of R. Note that $Z_\tau(M)$ is a submodule of the singular submodule $Z(M)$ of M.

Theorem 2.1. *Let R be a ring and let τ be a hereditary torsion theory on* Mod-R *such that $Z_\tau(R_R) = 0$. Then $\tau_G \leq \tau$ if and only if every τ-UC-module is UC.*

Proof. The necessity follows by Proposition 2.1. Conversely, suppose that every τ-UC-module is a UC-module. Let E be any essential right ideal of R. Suppose that R/E is not τ-torsion. Then there exists a proper right ideal F of R such that $E \subseteq F$ and R/F is a τ-torsion-free R-module. Let M denote the R-module $R \oplus (R/F)$. Note that $Z_\tau(R_R) = 0$ and $Z_\tau(R/F) = 0$ so that $Z_\tau(M) = 0$. By [1] (Corollary 3.5), M is a τ-UC-module. However, since F is a proper essential right ideal of R, M is not a UC-module by [10] (Theorem), a contradiction. Thus the R-module R/E is τ-torsion for every essential right ideal E of R. It follows that every singular module is τ-torsion. By Proposition 2.2, $\tau_G \leq \tau$.

Theorem 2.1 is proved.

3. The Lambek torsion theory. Let R be a ring. For any R-module M, $E(M)$ will denote the injective hull of M. Let τ_L denote the Lambek torsion theory on Mod-R. Recall that τ_L is the (hereditary) torsion theory on Mod-R whose torsion class consists of all R-modules M such that $\mathrm{Hom}(M, E(R_R)) = 0$. For basic facts about the Lambek torsion theory see [5, 9]. Recall that the ring R is called *right nonsingular* provided R_R is τ_G-torsion-free.

Lemma 3.1. *Let R be any ring. Then*:

(1) $\tau_L \leq \tau_G$;

(2) $\tau_L = \tau_G$ *if and only if R is right nonsingular.*

Proof. (1) By [9] (Ch. VI, Corollary 6.5).

(2) By [9] (Ch. VI, Proposition 6.7 and Corollary 6.8).

Lemma 3.1 is proved.

Lemma 3.2. *Let $E = E(R_R)$. Then an R-module M is τ_L-torsion if and only if $\mathrm{ann}_E(\mathrm{ann}_R(m)) = 0$ for all $m \in M$.*

Proof. Let $m \in M$ and let $e \in \mathrm{ann}_E(\mathrm{ann}_R(m))$. Define a mapping $\varphi\colon mR \to E$ by $\varphi(mr) = er$ $(r \in R)$. It is easy to check that φ is well-defined and an R-homomorphism. Moreover, for every homomorphism $\theta\colon mR \to E$ we have $\theta(m) \in \mathrm{ann}_E(\mathrm{ann}_R(m))$. Because E is injective, the result follows.

Lemma 3.2 is proved.

***Corollary* 3.1.** *A submodule N of an R-module M is a τ_L-essential submodule of M if and only if*

(a) *N is an essential submodule of M, and*

(b) $\mathrm{ann}_E(N\colon m) = 0$ *for all $m \in M$, where $E = E(R_R)$.*

Proof. The submodule N is a τ_L-essential submodule of M if and only if (a) holds and M/N is τ_L-torsion. Apply Lemma 3.2.

Corollary 3.1 is proved.

Theorem 3.1. *A ring R is right nonsingular if and only if every τ_L-UC-module is UC.*

Proof. The necessity follows by Lemma 3.1 (2). Conversely, suppose that every τ_L-UC-module is UC. We know that $\tau_L \leq \tau_G$. Now we show that $Z_{\tau_L}(R_R) = 0$. Suppose that $x \in Z_{\tau_L}(R_R)$. Then there exists a τ_L-essential right ideal I of R such that $xI = 0$. Thus I is essential in R_R and R/I is τ_L-torsion. Therefore $\mathrm{Hom}_R(R/I, E(R_R)) = 0$. Define a mapping $\theta\colon R/I \to E(R_R)$ by $\theta(r + I) = xr$ for all $r \in R$. Since $xI = 0$, θ is well-defined and is clearly an R-homomorphism. Thus $x = 0$. It follows that $Z_{\tau_L}(R_R) = 0$. By Theorem 2.1 $\tau_G \leq \tau_L$. We have proved that $\tau_G = \tau_L$. Finally Lemma 3.1 gives that R is right nonsingular.

Theorem 3.1 is proved.

Let R be a ring. Recall that R is called a *right Kasch ring* if every simple right module embeds in R. Among examples of right Kasch rings are quasi-Frobenius rings (see, for example, [7], Theorem 3.4) and, more generally, right pseudo-Frobenius rings (see, for example, [7], Lemma 1.42 and Theorem 1.57). This brings us to our final theorem.

Theorem 3.2. *The following statements are equivalent for a ring R.*

(1) *R is a right Kasch ring.*

(2) *Every τ_L-torsion module is zero.*

(3) *Every module is a τ_L-UC-module.*

Proof. (1) $\Rightarrow$ (2). Let R be a right Kasch ring and let $M_R \neq 0$. For $0 \neq m \in M$, there exists a maximal submodule K of mR. Then there is an embedding $\varphi\colon mR/K \to R$ and an embedding $\iota\colon R \to E(R_R)$. By the injectivity $E(R_R)$, the homomorphism $\iota\varphi$ extends to a nonzero homomorphism $\theta\colon M/K \to E(R_R)$. If $\pi\colon M \to M/K$ denotes the natural epimorphism, then $\theta\pi$ is a nonzero homomorphism from M to $E(R_R)$. Hence $\mathrm{Hom}_R(M, E(R_R)) \neq 0$.

(2) $\Rightarrow$ (3). By Theorem 1.1.

(3) $\Rightarrow$ (1). Let V be any simple R-module. Then $V \cong R/P$ for some maximal right ideal P of R. If P is not an essential right ideal of R then there exists a nonzero right ideal U of R such that $P \cap U = 0$. In this case, $R = P \oplus U$ and hence $V \cong R/P \cong U$. Now suppose that P is an essential right ideal of R. Because $P \neq R$, (3) combined with Theorem 1.1 gives that R/P is not τ_L-torsion. Thus there exists a nonzero homomorphism $\alpha\colon V \to E(R_R)$. It follows that $V \cong \varphi(V) \subseteq R$, because R is essential in $E(R_R)$. We have proved that R is a right Kasch ring.

Theorem 3.2 is proved.

1. *Doğruöz S., Harmanci A., Smith P. F.* Modules with unique closure relative to a torsion theory // Can. Math. Bull. – 2010. – **53**, № 2. – P. 230 – 238.
2. *Doğruöz S., Harmanci A., Smith P. F.* Modules with unique closure relative to a torsion theory. II // Turk. J. Math. – 2009. – **33**. – P. 111 – 116.
3. *Doğruöz S.* Classes of extending modules associated with a torsion theory // East-West J. Math. – 2006. – **8**, № 2. – P. 163 – 180.
4. *Dung N. V., Huynh D. V., Smith P. F., Wisbauer R.* Extending modules // Pitman Res. Notes Math. Ser. 313. – Harlow: Longman, 1994.
5. *Golan J. S.* Localization of noncommutative rings. – New York: Marcel Dekker, 1975.
6. *Goodearl K. R., Warfield R. B.* An introduction to noncommutative Noetherian rings // London Math. Soc. Stud. Texts 16. – Cambridge: Cambridge Univ. Press, 1989.
7. *Nicholson W. K., Yousif M. F.* Quasi-Frobenius rings // Cambridge Tracts Math. – Cambridge: Cambridge Univ. Press, 2003. – **158**.
8. *Passman D. S.* The algebraic structure of group rings. – New York: Wiley-Intersci., 1977.
9. *Stenström B.* Rings of quotients. An introduction to methods of ring theory. – Berlin: Springer-Verlag, 1975.
10. *Smith P. F.* Modules for which every submodule has a unique closure // Ring Theory: Proc. Biennial Ohio-Denison Conf. (May 1992). – Singapore: World Sci., 1992. – P. 302 – 313.

Received 18.07.12