We prove some general fixed-point theorems in complete G-metric space that generalize some recent results.

1. Introduction. In [3, 4] Dhage introduced a new class of generalized metric space, named D-metric space. Mustafa and Sims [7, 8] proved that most of the claims concerning the fundamental topological structures on D-metric spaces are incorrect and introduced appropriate notion of generalized metric space, named G-metric space. In fact, Mustafa, Sims and other authors [2, 9 – 11] studied many fixed-point results for self mappings in G-metric spaces under certain conditions.

Quite recently [12], Mustafa et al. obtained new results for mappings in G-metric spaces.

In [13, 14], Popa initiated the study of fixed points in metric spaces for mappings satisfying an implicit relation.

Let $T$ be a self mapping of a metric space $(X, d)$. We denote by Fix $(T)$ the set of all fixed points of $T$. $T$ is said to satisfy property $(P)$ if Fix $(T) = \text{Fix} (T^n)$ for each $n \in \mathbb{N}$. An interesting fact about mappings satisfying property $(P)$ is that they have not nontrivial periodic points. Papers dealing with property $(P)$ are, between others, [2, 13 – 15].

The purpose of this paper is to prove a general fixed-point theorem in complete G-metric space which generalize the results from [1, 10 – 12] for mappings satisfying a new form of implicit relation.

In the last part of this paper is proved a general theorem for mappings in G-metric space satisfying property $(P)$, which generalize some results from [1].

2. Preliminaries.

Definition 2.1 [8]. Let $X$ be a nonempty set and $G: X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1)$ $G(x, y, z) = 0$ if $x = y = z$;

$(G_2)$ $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

$(G_3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

$(G_4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);

$(G_5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function $G$ is called a G-metric and the pair $(X, G)$ is called a G-metric space.

Note that if $G(x, y, z) = 0$ then $x = y = z$ [8].

Lemma 2.1 [8]. $G(x, y, y) \leq 2G(x, x, y)$ for all $x, y \in X$.

Definition 2.2 [8]. Let $(X, G)$ be a metric space. A sequence $(x_n)$ in $X$ is said to be:

a) G-convergent to $x \in X$ if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$;
b) **G-Cauchy** if for \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that for all \( n,m,p \geq k \), \( G(x_n, x_m, x_p) < \varepsilon \) that is \( G(x_n, x_m, x_p) \to 0 \) as \( n,m,p \to \infty \).

A \( G \)-metric space is said to be \( G \)-complete if every \( G \)-Cauchy sequence in \( X \) is \( G \)-convergent.

**Lemma 2.2** [8]. Let \((X, G)\) be a \( G \)-metric space. Then, the following properties are equivalent:

1) \((x_n)\) is \( G \)-convergent to \( x \);
2) \( G(x, x_n, x_n) \to 0 \) as \( n \to \infty \);
3) \( G(x_n, x, x) \to 0 \) as \( n \to \infty \).

**Lemma 2.3** [8]. Let \((X, G)\) be a \( G \)-metric space. Then the following properties are equivalent:

1) The sequence \((x_n)\) is \( G \)-Cauchy.
2) For every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( G(x_n, x_m, x_m) < \varepsilon \) for \( n, m > k \).

**Definition 2.3** [8]. Let \((X, G)\) and \((X', G')\) be two \( G \)-metric spaces and \( f: (X, G) \to (X', G')\). Then, \( f \) is said to be \( G \)-continuous at \( x \in X \) if for \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x, y \in X \) and \( G(a, x, y) < \delta \), then \( G'(fa, fx, fy) < \varepsilon \). \( f \) is \( G \)-continuous if it is \( G \)-continuous at each \( a \in X \).

**Lemma 2.4** [8]. Let \((X, G)\) and \((X', G')\) be two \( G \)-metric spaces. Then, a function \( f: (X, G) \to (X', G')\) is \( G \)-continuous at a point \( x \in X \) if and only if \( f \) is sequentially continuous, that is, whenever \((x_n)\) is \( G \)-convergent to \( x \) we have that \( f(x_n) \) is \( G \)-convergent to \( fx \).

**Lemma 2.5** [8]. Let \((X, G)\) be a \( G \)-metric space. Then, the function \( G(x, y, z) \) is continuous in all three of its variables.

Quite recently, the following theorem is proved in [12].

**Theorem 2.1.** Let \((X, G)\) be a complete \( G \)-metric space and \( T: X \to X \) be a mapping which satisfies the following condition, for all \( x, y \in X \)

\[
G(Tx, Ty, Ty) \leq \max\{aG(x, y, y), b[G(x, Tx, Tx) + 2G(y, Ty, Ty)],
\]

\[
b[G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Ty, Tx)]\},
\]

where \( a \in [0, 1) \) and \( b \in \left[0, \frac{1}{3}\right)\). Then \( T \) has a unique fixed point.

The purpose of this paper is to prove a general fixed point theorem in \( G \)-metric space for mappings satisfying a new type of implicit relation which generalize Theorem 2.1 and other results from [1, 2, 10 – 12].

3. **Implicit relations.**

**Definition 3.1.** Let \( \mathcal{F}_a \) be the set of all continuous functions \( F(t_1, \ldots, t_6): \mathbb{R}_{+}^6 \to \mathbb{R} \) such that

\( F_1 \) \( F \) is nonincreasing in variables \( t_5 \) and \( t_6 \);

\( F_2 \) there exists \( h \in [0,1) \) such that for each \( u, v \geq 0 \) and \( F(u, v, u, u + v, 0) \leq 0 \), then \( u \leq hv \);

\( F_3 \) \( F(t, t, 0, 0, t, 2t) > 0 \) \( \forall t > 0 \).

**Example 3.1.** \( F(t_1, \ldots, t_6) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_4 + t_5 + t_6)\} \), where \( a \in [0, 1) \) and \( b \in \left[0, \frac{1}{3}\right)\).

\( F_1 \) Obviously.

\( F_2 \) Let \( u, v \geq 0 \) be and \( F(u, v, u, u + v, 0) = u - \max\{av, b(v + 2u)\} \leq 0 \). If \( u > v \), then \( u[1 - \max\{a, 3b\}] \leq 0 \), a contradiction. Hence \( u \leq v \), which implies \( u \leq hv \), where \( h = \max\{a, 3b\} < 1 \).

\( F_3 \) \( F(t, t, 0, 0, t, 2t) = t(1 - \max\{a, 3b\}) > 0 \) \( \forall t > 0 \).
**Example 3.2.** $F(t_1, \ldots, t_6) = t_1 - at_2 - b(t_3 + 2t_4) - c(t_5 + t_6)$, where $a, b, c \geq 0$, $a + 3b + 2c < 1$ and $a + 3c < 1$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - av - b(u + 2u) - c(u + v) \leq 0$. Then $u \leq hv$, where $h = \frac{a + b + c}{1 - 2b - c} < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 3c)] > 0 \ \forall t > 0$.

**Example 3.3.** $F(t_1, \ldots, t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_5, t_6\}$, where $a, b, c \geq 0$, $a + b + 2c < 1$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - av - b\max\{u, v\} - c(u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + 2c)] \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = \frac{a + b + c}{1 - c} < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 2c)] > 0 \ \forall t > 0$.

**Example 3.4.** $F(t_1, \ldots, t_6) = t_1 - k\max\{t_2, t_3, \ldots, t_6\}$, where $k \in \left[0, \frac{1}{2}\right]$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - k(u + v) \leq 0$ which implies $u \leq hv$, where $h = \frac{k}{k - 1} < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t(1 - 2k) > 0 \ \forall t > 0$.

**Example 3.5.** $F(t_1, \ldots, t_6) = t_1 - at_2 - b(t_3 + 2t_4) - c\max\{t_4 + t_5, 2t_6\}$, where $a, b, c \geq 0$, $a + b + 3c < 1$, $a + 4c < 1$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - av - bv - c(2u + v) \leq 0$. Then $u \leq hv$, where $h = \frac{a + b + c}{1 - 2c} < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t[1 - (a + 4c)] > 0 \ \forall t > 0$.

**Example 3.6.** $F(t_1, \ldots, t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{2t_4 + t_6}{3}, \frac{2t_4 + t_3}{3}, \frac{t_5 + t_6}{3}\right\} \leq 0$, where $k \in [0, 1)$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - k\max\left\{u, v, \frac{2u + v}{3}, \frac{u + v}{3}\right\}$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = k < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t(1 - k) > 0 \ \forall t > 0$.

**Example 3.7.** $F(t_1, \ldots, t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}$, where $k \in \left[0, \frac{2}{3}\right]$.

(F1) Obviously.

(F2) Let $u, v \geq 0$ be and $F(u, v, v, u + v, 0) = u - k\max\left\{u, v, \frac{u + v}{2}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$ which implies $u \leq hv$, where $h = k < 1$.

(F3) $F(t, t, 0, 0, t, 2t) = t - k\max\left\{t, \frac{3t}{2}\right\} = t\left[1 - \frac{3k}{2}\right] > 0 \ \forall t > 0$. 

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Example 3.8. \( F(t_1, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6, \) where \( a, b, c \geq 0, a + b + c < 1, a + 2d < 1. \)

\( F_1 \) Obviously.

\( F_2 \) Let \( u, v \geq 0 \) be and \( F(u, v, v, u, u + v, 0) = u^2 - u(av + bv + cu) \leq 0. \) If \( u > 0, \) then \( u - av - bv - cu \leq 0 \) which implies \( u \leq hv, \) where \( h = \frac{a + b}{1 - c} < 1. \) If \( u = 0, \) then \( u \leq hv. \)

\( F_3 \) \( F(t, t, 0, 0, t, 2t) = t^2[1 - (a + 2d)] > 0 \quad \forall t > 0. \)

Example 3.9. \( F(t_1, \ldots, t_6) = t_1 - k \max \left\{ \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2} \right\}, \) where \( k \in \left[ 0, \frac{2}{3} \right]. \)

\( F_1 \) Obviously.

\( F_2 \) Let \( u, v \geq 0 \) be and \( F(u, v, v, u, u + v, 0) = u - k \max \left\{ \frac{v}{2} \right\} \leq 0. \) If \( u > 0, \) then \( u(1 - k) \leq 0, \) a contradiction. Hence \( u \leq v \) which implies \( u \leq hv, \) where \( h = k < 1. \)

\( F_3 \) \( F(t, t, 0, 0, t, 2t) = t \left[ 1 - \frac{3k}{2} \right] > 0 \quad \forall t > 0. \)

Example 3.10. \( F(t_1, \ldots, t_6) = t_1 - k \max \left\{ t_2, \sqrt{t_3t_4}, \sqrt{t_5t_6} \right\}, \) where \( k \in \left[ 0, \frac{2}{3} \right]. \)

\( F_1 \) Obviously.

\( F_2 \) Let \( u, v \geq 0 \) be and \( F(u, v, v, u, u + v, 0) = u - k \max \left\{ \frac{v}{2} \right\} \leq 0. \) If \( u > v, \) then \( u(1 - k) \leq 0, \) a contradiction. Hence, \( u \leq v \) which implies \( u \leq hv, \) where \( 0 \leq h = k < 1. \)

\( F_3 \) \( F(t, t, 0, 0, t, 2t) = t(1 - \sqrt{2k}) > 0 \quad \forall t > 0. \)

4. Main results.

Theorem 4.1. Let \( (X, G) \) be a \( G \)-metric space and \( T: (X, G) \to (X, G) \) be a mapping such that

\[
F(G(Tx, Ty, Ty), G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Tx, Tx)) \leq 0
\]

(4.1)

for all \( x, y \in X, \) where \( F \) satisfies property \( (F_3). \) Then \( T \) has at most a fixed point.

Proof. Suppose that \( T \) has two distinct fixed points \( u \) and \( v. \) Then by (4.1) we have successively

\[
F(G(Tu, Tv, Tv), G(u, v, v), G(u, Tu, Tu), G(v, Tv, Tv), G(u, Tv, Tv), G(v, Tu, Tu))) \leq 0,
\]

\[
F(G(u, v, v), G(u, v, v), 0, 0, G(u, v, v), G(u, u, u)) \leq 0.
\]

By Lemma 2.1 \( G(v, u, u) \leq 2G(u, v, v). \) Since \( F \) is nonincreasing in variable \( t_6 \) we obtain

\[
F(G(u, v, v), G(u, v, v), 0, 0, G(u, v, v), 2G(u, v, v)) \leq 0,
\]

a contradiction of \( (F_3). \) Hence \( u = v. \)

Theorem 4.1 is proved.

Theorem 4.2. Let \( (X, G) \) be a complete \( G \)-metric space and \( T: (X, G) \to (X, G) \) satisfying inequality (4.1) for all \( x, y \in X, \) where \( F \in \mathbb{F}_u. \) Then \( T \) has a unique fixed point.

Proof. Let \( x_0 \in X \) be an arbitrary point in \( X. \) We define \( x_n = Tx_{n-1}, n = 1, 2, \ldots. \) Then by (4.1) we have successively

\[
F(G(Tx_{n-1}, Tx_n, Tx_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}),
\]

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\[ G(x_n, Tx_n, Tx_n), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_{n-1}, Tx_{n-1}) \leq 0, \]
\[ F(G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x, x), G(x_{n-1}, x_n, x_n), \]
\[ G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), 0) \leq 0. \]

By \((G_5), G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x, x) + G(x_n, x_{n+1}, x_{n+1}).\) Since \(F\) is nonincreasing in variable \(t_5\) we obtain
\[ F(G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x, x), G(x_{n-1}, x_n, x_n), \]
\[ G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}, 0) \leq 0 \]
which implies by \((F_2)\) that
\[ G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x, x_n). \]

Then
\[ G(x_n, x_{n+1}, x_{n+1}) \leq hG(x_{n-1}, x, x_n) \leq \ldots \leq h^nG(x_0, x_1, x_1). \]

Moreover, for all \(m, n \in \mathbb{N}, m > n,\) we have repeated use the rectangle inequality
\[ G(x_n, x_m, x_m) \leq G(x_{n-1}, x, x_n) + G(x_{n+1}, x, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m) \leq \]
\[ (h^n + h^{n+1} + \ldots + h^{m-1})G(x_0, x_1, x_1) \leq \frac{h^n}{1 - h}G(x_0, x_1, x_1), \]
which implies \(\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0.\) Hence, \((x_n)\) is a \(G\)-Cauchy sequence. Since \((X, G)\) is \(G\)-complete, there exists \(u \in X\) such that \(\lim_{n \to \infty} x_n = u.\)

We prove that \(u = Tu.\) By \((F_1)\) we have successively
\[ F(G(Tx_n, Tu, Tu), G(x_{n-1}, u, u), G(x_{n-1}, Tx_n, Tx_{n-1}), \]
\[ G(u, Tu, Tu), G(x_{n-1}, Tu, Tu), G(u, Tx_{n-1}, Tx_{n-1}), 0) \leq 0, \]
\[ F(G(x_n, Tu, Tu), G(x_{n-1}, u, u), G(x_{n-1}, x, x_n), \]
\[ G(u, Tu, Tu), G(x_{n-1}, Tu, Tu), G(u, x, x_n) \leq 0. \]

By continuity of \(F\) and \(G,\) letting \(n\) tend to infinity, we obtain
\[ F(G(u, Tu, Tu), 0, 0, G(u, Tu, Tu), G(u, Tu, Tu), 0) \leq 0. \]

By \((F_2)\) we obtain \(G(u, Tu, Tu) = 0,\) hence \(u = Tu\) and \(u\) is a fixed point of \(T.\) By Theorem 4.1 \(u\) is the unique fixed point of \(T.\)

Theorem 4.2 is proved.

**Corollary 4.1.** Theorem 2.1.

**Proof.** The proof follows from Theorem 4.2 and Example 3.1.
**Corollary 4.2** (Theorem 2.2 [11]). Let \((X, G)\) be a \(G\)-complete metric space and \(T: (X, G) \to (X, G)\) be a mapping satisfying the following condition:

\[
G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta [G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)],
\]

for all \(x, y, z \in X\) and \(0 \leq \alpha + 3\beta < 1\). Then \(T\) has a unique fixed point.

**Proof.** By (4.2) for \(z = y\) we obtain

\[
G(Tx, Ty, Ty) \leq \alpha G(x, y, y) + \beta [G(x, Tx, Tx) + 2G(y, Ty, Ty)],
\]

for all \(x, y \in X\). By Theorem 4.2 and Example 3.2 for \(\alpha = a, \beta = b\) and \(c = 0\) it follows that \(T\) has a unique fixed point.

**Corollary 4.3** (Theorem 2.3 [11]). Let \((X, G)\) be a \(G\)-complete metric space and \(T: (X, G) \to (X, G)\) be a mapping satisfying the condition

\[
G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta \max\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\},
\]

for all \(x, y, z \in X\) and \(0 \leq \alpha + \beta < 1\). Then \(T\) has a unique fixed point.

**Proof.** By (4.3) for \(z = y\) we obtain

\[
G(Tx, Ty, Ty) \leq \alpha G(x, y, y) + \beta \max\{G(x, Tx, Tx), G(y, Ty, Ty)\},
\]

for all \(x, y \in X\). By Theorem 4.2 and Example 3.3 for \(\alpha = a, \beta = b\) and \(c = 0\) it follows that \(T\) has a unique fixed point.

**Corollary 4.4** (Theorem 2.1 [10]). Let \((X, G)\) be a \(G\)-complete metric space and \(T: (X, G) \to (X, G)\) be a mapping satisfying the condition

\[
G(Tx, Ty, Tz) \leq k \max\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(y, Ty, Ty), G(z, Tz, Tz), G(z, Tx, Tz)\},
\]

for all \(x, y, z \in X\), where \(k \in \left[0, \frac{1}{2}\right)\). Then \(T\) has a unique fixed point.

**Proof.** By (4.4) for \(z = y\) we obtain

\[
G(Tx, Ty, Ty) \leq k \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Ty, Ty), G(y, Tx, Tx)\}.
\]

By Theorem 4.2 and Example 3.4, \(T\) has a unique fixed point.

**Corollary 4.5.** Let \((X, G)\) be a \(G\)-complete metric space and \(T: (X, G) \to (X, G)\) be a mapping which satisfy the following inequality for all \(x, y \in X\),

\[
G(Tx, Ty, Ty) \leq k \max\{G(y, Ty, Ty) + G(x, Ty, Ty), 2G(y, Tx, Tx)\},
\]

where \(k \in \left[0, \frac{1}{3}\right)\). Then \(T\) has a unique fixed point.

**Proof.** By Theorem 4.2 and Example 3.5 for \(a = b = 0\) and \(c = k\), \(T\) has a unique fixed point.

**Remark 4.1.** In Theorem 2.8 [10], \(k \in \left[0, \frac{1}{2}\right)\).
**Corollary 4.6.** Let \((X, G)\) be a \(G\)-metric space and \(T: (X, G) \rightarrow (X, G)\) be a mapping satisfying the following inequality for all \(x, y, z \in X\),

\[
G(Tx, Ty, Tz) \leq h \max \left\{ \frac{G(x, y, z)}{3}, \frac{G(y, Tx, Tx)}{3}, \frac{G(y, Ty, Ty)}{3}, \frac{G(z, Tz, Tz)}{3} \right\},
\]

(4.6)

where \(k \in [0, 1)\). Then \(T\) has a unique fixed point.

**Proof.** If \(y = z\), by (4.6) we obtain that

\[
G(Tx, Ty, Ty) \leq \frac{G(x, y, y)}{3}, \frac{G(y, Tx, Tx)}{3}, \frac{G(y, Ty, Ty)}{3}, \frac{G(z, Tz, Tz)}{3} \leq 0.
\]

for all \(x, y \in X\).

By Theorem 4.2 and Example 3.6, \(T\) has a unique fixed point.

**Remark 4.2.** Corollary 4.6 is a generalization of Theorem 2.6 [1], where \(k \in \left[0, \frac{1}{2}\right)\).

**Remark 4.3.** By Theorem 4.2 and Examples 3.7–3.10 we obtain new results.

5. **Property \((P)\) in \(G\)-metric spaces.**

**Theorem 5.1.** Under the conditions of Theorem 4.2, \(T\) has property \((P)\).

**Proof.** By Theorem 4.2, \(T\) has a fixed point. Therefore, \(\text{Fix}(T^n) \neq \emptyset\) for each \(n \in \mathbb{N}\). Fix \(n > 1\) and assume that \(p \in \text{Fix}(T^n)\). We prove that \(p \in \text{Fix}(T)\). Using (4.1) we have

\[
F(G(T^n p, T^{n+1} p, T^{n+1} p), G(T^{n-1} p, T^n p, T^n p), G(T^{n-1} p, T^n p, T^n p), G(T^n p, T^{n+1} p, T^{n+1} p), G(T^{n-1} p, T^{n+1} p, T^{n+1} p), G(T^n p, T^n p, T^n p)) \leq 0.
\]

By rectangle inequality

\[
G(T^{n-1} p, T^{n+1} p, T^{n+1} p) \leq G(T^{n-1} p, T^n p, T^n p) + G(T^n p, T^{n+1} p, T^{n+1} p).
\]

By \((F_1)\) we obtain

\[
F(G(T^n p, T^{n+1} p, T^{n+1} p), G(T^{n-1} p, T^n p, T^n p), G(T^{n-1} p, T^n p, T^n p), G(T^n p, T^{n+1} p, T^{n+1} p), G(T^{n-1} p, T^n p, T^n p) + G(T^n p, T^{n+1} p, T^{n+1} p), 0) \leq 0.
\]

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By \((F_2)\) we obtain
\[
G(T^n p, T^{n+1} p, T^{n+1} p) \leq h G(T^{n-1} p, T^n p, T^n p) \leq \ldots \leq h^n G(p, Tp, Tp).
\]
Since \(p \in T^n p\), then
\[
G(p, Tp, Tp) = G(T^n p, T^{n+1} p, T^{n+1} p).
\]
Therefore
\[
G(p, Tp, Tp) \leq h^n G(p, Tp, Tp)
\]
which implies \(G(p, Tp, Tp) = 0\), i.e., \(p = Tp\) and \(T\) has property \((P)\).
Theorem 5.1 is proved.

**Corollary 5.1.** In the condition of Corollary 4.6, \(T\) has property \((P)\).

**Remark 5.1.** Corollary 5.1 is a generalization of the results from Theorem 2.6 [1].

**Corollary 5.2.** In the condition of Corollary 4.4 with \(k \in \left[0, \frac{1}{2}\right]\), instead \(k \in [0, 1)\), \(T\) has property \((P)\).

**Remark 5.2.** We obtain other new results from Examples 3.1 – 3.10.


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