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J. Liu (Zhengzhou Univ. Light Industry, China)

GLOBAL WEAK SOLUTIONS FOR THE WEAKLY DISSIPATIVE μ-HUNTER – SAXTON EQUATION

ГЛОБАЛЬНІ СЛАБКІ РОЗВ'ЯЗКИ СЛАБКОДИСИПАТИВНОГО μ-РІВНЯННЯ ХАНТЕРА – САКСТОНА

The paper deals with the global existence of weak solutions for a weakly dissipative μ-Hunter –Saxton equation by using smooth data approximate to the initial data and Helly's theorem.

Розглянуто проблему глобального існування слабких розв'язків слабкодисипативного μ–рівняння Хантера– Сакстона за допомогою гладких даних, що є наближенням до початкових даних, та теорему Хеллі.

1. Introduction. Recently, Khesin et al. [7] derived and studied the following the μ-Hunter – Saxton (also called μ-Camassa – Holm) equation:

$$\mu(u)_t - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx},$$

which describes evolution of rotators in liquid crystals with external magnetic and self-interaction. Here $u(t,x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u dx$ denotes its mean. The μ-Hunter – Saxton equation lies mid-way between the periodic Hunter – Saxton and Camassa – Holm equations. Moreover, the equation describes the geodesic flow on $\mathcal{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product [7]

$$(u,v) = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x dx.$$

The Cauchy problem of the μ-Hunter – Saxton equation has been studied extensively. It has been shown that the μ-Hunter – Saxton equation is locally well-posed [7] with the initial data $u_0 \in H^s(\mathbb{S})$, $s > \dfrac{3}{2}$. Interestingly, it has global strong solutions [7] and also blow-up solutions in finite time [3, 5, 7] with a different class of initial profiles in the Sobolev spaces $H^s(\mathbb{S})$, $s > \dfrac{3}{2}$. On the other hand, it has global dissipative weak solutions in $H^1(\mathbb{S})$ [15]. Moreover, the μ-Hunter – Saxton equation admits both periodic one-peakon solution and the multi-peakons [7, 9].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. So, it is reasonable to study the model with energy dissipation. In [4] and [13], the authors discussed the energy dissipative KdV equation from different aspects. Weakly dissipative Camassa – Holm equation and weakly dissipative Degasperis – Procesi equation have been studied in [17, 19] and [2, 6, 18, 20] respectively. Recently, Wei and Yin [16] discussed the global existence and blow-up phenomena of the weakly dissipative periodic Hunter – Saxton equation.

In this paper, we will discuss global existence of weak solutions of the following weakly dissipative μ-Hunter – Saxton equation:

$$
\begin{aligned}
& y_t + uy_x + 2u_xy + \lambda y = 0, \qquad t > 0, \quad x \in \mathbb{R}, \\
& y = \mu(u) - u_{xx}, \qquad t > 0, \quad x \in \mathbb{R}, \\
& u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\
& u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R},
\end{aligned} \tag{1.1}
$$

or in the equivalent form:

$$
\begin{aligned}
& \mu(u)_t - u_{txx} + 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} + \lambda(\mu(u) - u_{xx}) = 0, \qquad t > 0, \quad x \in \mathbb{R}, \\
& u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\
& u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}.
\end{aligned} \tag{1.2}
$$

Here the constant λ is assumed to be positive and $\lambda y = \lambda(\mu(u) - u_{xx})$ is the weakly dissipative term. The Cauchy problem (1.1) has been discussed in [10] recently. The author established the local well-posedness, derived the precise blow-up scenario for Eq. (1.1) and proved that Eq. (1.1) has global strong solutions and also finite time blow-up solutions. However, the existence of global weak solutions to Eq. (1.1) has not been studied yet. The aim of this paper is to present a global existence result of weak solutions to Eq. (1.1).

Throughout the paper, we denote by $*$ the convolution. Let $\|\cdot\|_Z$ denote the norm of Banach space Z and let $\langle\cdot,\cdot\rangle$ denote the $H^1(\mathbb{S})$, $H^{-1}(\mathbb{S})$ duality bracket. Let $M(\mathbb{S})$ be the space of Radon measures on $\mathbb{S}$ with bounded total variation and $M^+(\mathbb{S})$ ($M^-(\mathbb{S})$) be the subset of $M(\mathbb{S})$ with positive (negative) measures. Finally, we write $BV(\mathbb{S})$ for the space of functions with bounded variation, $\mathbb{V}(f)$ being the total variation of $f \in BV(\mathbb{S})$.

Before giving the precise statement of the main result, we first introduce the definition of weak solution to the problem (1.2).

Definition 1.1. *A function $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S}) \cap L^\infty(\mathbb{R}^+; H^1(\mathbb{S}))$ is said to be an admissible global weak solution to* (1.2) *if u satisfies the equations in* (1.2) *and $z(t,\cdot) \to z_0$ as $t \to 0^+$ in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Moreover,*

$$
\mu(u) = \mu(u_0)e^{-\lambda t} \qquad \textit{and} \qquad \|u_x(t,\cdot)\|_{L^2(\mathbb{S})} = e^{-\lambda t}\|u_{0,x}\|_{L^2(\mathbb{S})}.
$$

The main result of this paper can be stated as follows.

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{S})$. Assume that $y_0 = (\mu(u_0) - u_{0,xx}) \in M^+(\mathbb{S})$, then the equation* (1.2) *has an admissible global weak solution in the sense of Definition* 1.1. *Moreover,*

$$
u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}(\mathbb{S})) \cap H^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{S}).
$$

Furthermore, $y = \big(\mu(u) - u_{xx}(t,\cdot)\big) \in M^+(\mathbb{S})$ for a.e. $t \in \mathbb{R}^+$ is uniformly bounded on $\mathbb{S}$.

***Remark* 1.1.** If $y_0 = (\mu(u_0) - u_{0,xx}) \in M^-(\mathbb{S})$, then the conclusions in Theorem 1.1 also hold with $y = (\mu(u) - u_{xx}(t,\cdot)) \in M^-(\mathbb{S})$.

The paper is organized as follows. In Section 2, we recall some useful lemmas and derive some priori estimates on global strong solutions to (1.2). In Section 3, we obtain the global existence of approximate solutions to (1.2) with smooth approximate initial data. In Section 4, we show that the conclusions in Theorem 1.1 hold by using Helly's theorem.

2. Preliminaries. On one hand, with $y = \mu(u) - u_{xx}$, the first equation in (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type

$$u_t + uu_x = -\partial_x A^{-1}\left(2\mu(u)u + \frac{1}{2}u_x^2\right) - \lambda u, \tag{2.1}$$

where $A = \mu - \partial_x^2$ is an isomorphism between H^s and H^{s-2} with the inverse $v = A^{-1}w$ given explicitly by [1, 7]

$$v(x) = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(s)dsdy-$$

$$-\int_0^x\int_0^y w(s)dsdy + \int_0^1\int_0^y\int_0^s w(r)drdsdy.$$

Since A^{-1} and ∂_x commute, the following identities hold:

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)dx - \int_0^x w(y)dy + \int_0^1\int_0^x w(y)dydx$$

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)dx. \tag{2.2}$$

On the other hand, integrating both sides of the first equation in (1.2) with respect to x on $\mathbb{S}$, we obtain

$$\frac{d}{dt}\mu(u) = -\lambda\mu(u),$$

it follows that

$$\mu(u) = \mu(u_0)e^{-\lambda t} := \mu_0 e^{-\lambda t}, \tag{2.3}$$

where

$$\mu_0 := \mu(u_0) = \int_{\mathbb{S}} u_0(x)dx.$$

Using (2.1) and (2.3), the equation (1.2) can be rewrited as

$$\begin{aligned} &u_t + uu_x = -\partial_x A^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right) - \lambda u, \qquad t > 0, \quad x \in \mathbb{R}, \\ &u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\ &u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}. \end{aligned} \tag{2.4}$$

If we rewrite the inverse of the operator $A = \mu - \partial_x^2$ in terms of a Green's function, we find $(A^{-1}m)(x) = \int_0^1 g(x - x')m(x')dx' = (g * m)(x)$. So, we get another equivalent form

$$
\begin{aligned}
&u_t + uu_x = -\partial_x g * \left(2\mu_0 e^{-\lambda t} u + \frac{1}{2}u_x^2\right) - \lambda u, \qquad t > 0, \quad x \in \mathbb{R},\\
&u(0,x) = u_0(x), \qquad x \in \mathbb{R},\\
&u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.5}
$$

where the Green's function $g(x)$ is given [9] by

$$
g(x) = \frac{1}{2}x(x-1) + \frac{13}{12} \quad \text{for} \quad x \in \mathbb{S}, \tag{2.6}
$$

and is extended periodically to the real line. In other words,

$$
g(x - x') = \frac{(x - x')^2}{2} - \frac{|x - x'|}{2} + \frac{13}{12}, \qquad x,\ x' \in \mathbb{S}.
$$

In particular, $\mu(g) = 1$.

Given $u_0 \in H^s$ with $s > \frac{3}{2}$, Theorem 2.2 in [10] ensures the existence of a maximal $T > 0$ and a solution u to (1.2) such that

$$
u = u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{S})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{S})\big).
$$

Consider now the following initial value problem:

$$
\begin{aligned}
&q_t = u(t,q), \qquad t \in [0,T),\\
&q(0,x) = x, x \in \mathbb{R}.
\end{aligned}
\tag{2.7}
$$

Lemma 2.1 [10]. *Let $u_0 \in H^s$ with $s > \frac{3}{2}$, $T > 0$ be the maximal existence time. Then Eq.* (2.7) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$ and the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$
q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.
$$

Moreover, with $y = \mu(u) - u_{xx}$, we have

$$
y(t, q(t,x))q_x^2 = y_0(x)e^{-\lambda t}.
$$

Lemma 2.2. *If $y_0 = \mu_0 - u_{0,xx} \in H^1(\mathbb{S})$ does not change sign, then the corresponding solution u to* (2.5) *of the initial value u_0 exists globally in time, that is $u \in C\big(\mathbb{R}^+, H^3(\mathbb{S})\big) \cap C^1\big(\mathbb{R}^+, H^2(\mathbb{S})\big)$. Moreover, the following properties hold*:

(1) $\mu(u) = \mu_0 e^{-\lambda t}$, $t \in [0,\infty)$,

(2) $\|u_x\|^2_{L^2(\mathbb{S})} = e^{-2\lambda t}\mu_1^2$, $t \in [0,\infty)$, *with* $\mu_1 = \left(\int_{\mathbb{S}} u_{0,x}^2 dx\right)^{1/2}$,

(3) $\|u(t,\cdot)\|_{L^\infty(\mathbb{S})} \le |\mu_0| + \dfrac{\sqrt{3}}{6}\mu_1,$

(4) $y(t,x)$, $u(t,x)$ *have the same sign with* $y_0(x)$, *and* $\|u_x\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0|,$

(5) $|\mu_0|e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})}e^{-\lambda t} = \|y(t,\cdot)\|_{L^1(\mathbb{S})} = \|u(t,\cdot)\|_{L^1(\mathbb{S})}.$

Proof. Except for (4) and (5), all of the conclusions in Lemma 2.2 can be found in [10]. So we only need to prove (4) and (5) here.

Firstly, Lemma 2.1 and $u = g * y$, $g \ge 0$ imply $y(t,x)$ and $u(t,x)$ have the same sign with $y_0(x)$. Moreover, from the proof of Theorem 5.1 in [10], we have $u_x(t,x) \ge -|\mu_0|$. Now note that given $t \in [0,T)$, there is a $\xi(t) \in \mathbb{S}$ such that $u_x(t,\xi(t)) = 0$ by the periodicity of u to x-variable. If $y_0 \ge 0$, then $y \ge 0$. For $x \in [\xi(t), \xi(t)+1]$, we have

$$u_x(t,x) = \int\limits_{\xi(t)}^{x} \partial_x^2 u(t,x)dx = \int\limits_{\xi(t)}^{x} (\mu(u) - y)dx =$$

$$= \mu(u)(x - \xi(t)) - \int\limits_{\xi(t)}^{x} ydx \le \mu(u)(x-\xi(t)) \le |\mu_0|.$$

It follows that $u_x(t,x) \le |\mu_0|$. On the other hand, if $y_0 \le 0$, then $y \le 0$. Therefore, for $x \in$ $\in [\xi(t), \xi(t)+1]$, we obtain

$$u_x(t,x) = \int\limits_{\xi(t)}^{x} \partial_x^2 u(t,x)dx = \int\limits_{\xi(t)}^{x} (\mu(u) - y)dx \le \mu(u)(x-\xi(t)) - \int\limits_{\mathbb{S}} ydx =$$

$$= \mu(u)(x-\xi(t)) - \int\limits_{\mathbb{S}} (\mu(u) - u_{xx})dx = \mu(u)(x - \xi(t) - 1) \le |\mu_0|.$$

It follows that $u_x(t,x) \le |\mu_0|$. So we have $\|u_x\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0|$, this completes the proof of (4).

By the first equation of (1.1), we get

$$\int\limits_{\mathbb{S}} y(t,x)dx = \left(\int\limits_{\mathbb{S}} y_0(x)dx\right) e^{-\lambda t} = \mu_0 e^{-\lambda t}.$$

If $y_0 \ge 0$, then $y \ge 0$ and $\mu_0 \ge 0$, we have

$$\|y\|_{L^1(\mathbb{S})} = \int\limits_{\mathbb{S}} y(t,x)dx = \left(\int\limits_{\mathbb{S}} y_0(x)dx\right) e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})}e^{-\lambda t} = \mu_0 e^{-\lambda t}.$$

If $y_0 \le 0$, then $y \le 0$ and $\mu_0 \le 0$, we obtain

$$\|y\|_{L^1(\mathbb{S})} = -\int\limits_{\mathbb{S}} y(t,x)dx = \left(\int\limits_{\mathbb{S}} (-y_0(x))dx\right) e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})}e^{-\lambda t} = -\mu_0 e^{-\lambda t}.$$

It follows from this two equalities that $|\mu_0|e^{-\lambda t} = \|y_0\|_{L^1(\mathbb{S})}e^{-\lambda t} = \|y(t,\cdot)\|_{L^1(\mathbb{S})}$. A similar discussion implies $\|y(t,\cdot)\|_{L^1(\mathbb{S})} = \|u(t,\cdot)\|_{L^1(\mathbb{S})}$.

Lemma 2.2 is proved.

Lemma 2.3 [14]. *Assume $X \subset B \subset Y$ with compact imbedding $X \to B$ (X, B and Y are Banach spaces), $1 \le p \le \infty$ and (1) F is bounded in $L^p(0,T;X)$, (2) $\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$ uniformly for $f \in F$. Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p=\infty$), where $(\tau_h f)(t) = f(t+h)$ for $h>0$, if f is defined on $[0,T]$, then the translated function $\tau_h f$ is defined on $[-h, T-h]$.*

Lemma 2.4 (Helly's theorem [12]). *Let an infinite family F of function $f(x)$ be defined on the segment $[a,b]$. If all functions of the family and the total variation of all functions of the family are bounded by a single number $|f(x)| \le K$, $\bigvee_a^b(f) \le K$, then there exists a sequence $f_n(x)$ in the family F which converges at every point of $[a,b]$ to some function $\varphi(x)$ of finite variation.*

Lemma 2.5 [11]. *Let $T>0$. If $f, g \in L^2((0,T);H^1(\mathbb{R}))$ and $\dfrac{df}{dt}, \dfrac{dg}{dt} \in L^2((0,T);H^{-1}(\mathbb{R}))$, then f, g are a.e. equal to a function continuous from $[0,T]$ into $L^2(\mathbb{R})$ and*

$$\langle f(t), g(t)\rangle - \langle f(s), g(s)\rangle = \int_s^t \left\langle \frac{d(f(\tau))}{d\tau}, g(\tau)\right\rangle d\tau + \int_s^t \left\langle \frac{d(g(\tau))}{d\tau}, f(\tau)\right\rangle d\tau$$

for all $s, t \in [0,T]$.

3. Global approximate solutions. In the section, we will prove the existence of global approximate solutions and give some useful estimates to the approximate solutions. Now we consider the approximate equation of (2.5) as follows:

$$\begin{aligned} &u_t^n + u^n u_x^n = -\partial_x g * \left(2\mu_0^n e^{-\lambda t} u^n + \frac{1}{2}(u_x^n)^2\right) - \lambda u^n, \qquad t>0, \quad x \in \mathbb{R}, \\ &u^n(0,x) = u_0^n(x), \qquad x \in \mathbb{R}, \\ &u^n(t,x+1) = u^n(t,x), \qquad t \ge 0, \quad x \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $u_0^n(x) = \phi_n * u_0 \in H^\infty(\mathbb{S})$, $\mu_0^n = \int_{\mathbb{S}} u_0^n(x)dx$ and

$$\phi_n(x) := \left(\int_{\mathbb{R}} \phi(\xi)d\xi\right)^{-1} n\phi(nx), \qquad x \in \mathbb{R}, \quad n \ge 1,$$

where $\phi \in C_c^\infty(\mathbb{R})$ is defined by

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x|<1, \\ 0, & |x| \ge 1. \end{cases}$$

Obviously, $\|\phi_n\|_{L^1(\mathbb{R})} = 1$. Clearly, we have

$$u_0^n \to u_0 \quad \text{in} \quad H^1(\mathbb{S}), \qquad \text{as} \quad n \to \infty \tag{3.2}$$

and

$$\|u_0^n\|_{L^2(\mathbb{S})} \le \|u_0\|_{L^2(\mathbb{S})}, \qquad \|u_{0,x}^n\|_{L^2(\mathbb{S})} \le \|u_{0,x}\|_{L^2(\mathbb{S})},$$
$$\|u_0^n\|_{H^1(\mathbb{S})} \le \|u_0\|_{H^1(\mathbb{S})}, \qquad \|u_0^n\|_{L^1(\mathbb{S})} \le \|u_0\|_{L^1(\mathbb{S})} \tag{3.3}$$

in view of Young's inequality. Note that

$$\mu_0^n = \mu(u_0^n) = \int\limits_{\mathbb{S}} u_0^n(x)dx = \int\limits_{\mathbb{S}}\int\limits_{\mathbb{R}} \phi_n(y)u_0(x-y)dydx = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{S}} \phi_n(y)u_0(x-y)dxdy =$$
$$= \int\limits_{\mathbb{R}} \phi_n(y)\left(\int\limits_{\mathbb{S}} u_0(x-y)dx\right)dy = \int\limits_{\mathbb{R}} \phi_n(y)\left(\int\limits_{\mathbb{S}} u_0(z)dz\right)dy =$$
$$= \int\limits_{\mathbb{R}} \phi_n(y)\mu(u_0)(x-y)dy = \phi_n * \mu(u_0) = \mu(u_0) = \mu_0.$$

We can rewrite (3.1) as follows:

$$\begin{aligned} &u_t^n + u^n u_x^n = -\partial_x g * \left(2\mu_0 e^{-\lambda t}u^n + \frac{1}{2}(u_x^n)^2\right) - \lambda u^n, \qquad t>0, \quad x\in\mathbb{R},\\ &u^n(0,x) = u_0^n(x), \qquad x\in\mathbb{R},\\ &u^n(t,x+1) = u^n(t,x), \qquad t\ge 0, \quad x\in\mathbb{R}. \end{aligned} \tag{3.4}$$

Moreover, for all $n \ge 1$, $y_0^n = \mu(u_0^n) - u_{0,xx}^n = \mu_0 - u_{0,xx}^n \in H^1(\mathbb{S})$ and

$$y_0^n = \mu(u_0^n) - u_{0,xx}^n = \phi_n * \mu(u_0) - \phi_n * u_{0,xx} = \phi_n * y_0 \ge 0.$$

Thus, by Lemma 2.2, we obtain the corresponding solution $u^n \in C(\mathbb{R}^+; H^3(\mathbb{S})) \cap C^1(\mathbb{R}^+; H^2(\mathbb{S}))$ to Eq. (3.4) with initial data $u_0^n(x)$ and $y^n = \mu(u^n) - u_{xx}^n \ge 0$, $u^n = g * y^n \ge 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{S})$. Furthermore, combining Lemma 2.2 with (3.3), we have

$$\mu(u^n) = \mu_0^n e^{-\lambda t} = \mu_0 e^{-\lambda t}, \qquad t\in[0,\infty), \tag{3.5}$$

$$\|u_x^n\|_{L^2(\mathbb{S})}^2 = e^{-2\lambda t}\|u_{0,x}^n\|_{L^2(\mathbb{S})}^2 \le \|u_{0,x}\|_{L^2(\mathbb{S})}^2 = \mu_1^2, \quad t\in[0,\infty), \tag{3.6}$$

$$\|u^n(t,\cdot)\|_{L^\infty(\mathbb{S})} \le |\mu_0^n| + \frac{\sqrt{3}}{6}\|u_{0,x}^n\|_{L^2(\mathbb{S})} \le |\mu_0| + \frac{\sqrt{3}}{6}\|u_{0,x}\|_{L^2(\mathbb{S})} = |\mu_0| + \frac{\sqrt{3}}{6}\mu_1, \tag{3.7}$$

$$\|u_x^n\|_{L^\infty(\mathbb{R}^+\times\mathbb{S})} \le |\mu_0^n| = |\mu_0|, \tag{3.8}$$

$$|\mu_0|e^{-\lambda t} = \|y^n(t,\cdot)\|_{L^1(\mathbb{S})} = \|u^n(t,\cdot)\|_{L^1(\mathbb{S})}. \tag{3.9}$$

4. Proof of Theorem 1.1. In this section, with the basic energy estimate in Section 3, we are ready to obtain the necessary compactness of the approximate solutions $u^n(t,x)$. Acquiring the precompactness of approximate solutions, we prove the existence of the global weak solutions to the equation (1.1).

Lemma 4.1. *For any fixed $T > 0$, there exist a subsequence $\{u^{n_k}(t,x)\}$ of the sequence $\{u^n(t,x)\}$ and some function $u(t,x) \in L^\infty(\mathbb{R}^+; H^1(\mathbb{S})) \cap H^1([0,T] \times \mathbb{S})$ such that*

$$u^{n_k} \rightharpoonup u \quad in \quad H^1([0,T] \times \mathbb{S}) \quad as \quad n_k \to \infty, \tag{4.1}$$

and

$$u^{n_k} \to u \quad in \quad L^\infty([0,T] \times \mathbb{S}) \quad as \quad n_k \to \infty. \tag{4.2}$$

Moreover, $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S})$.

Proof. Firstly, we will prove that the sequence $\{u^n(t,x)\}$ is uniformly bounded in the space $H^1([0,T] \times \mathbb{S})$. By (3.6), (3.7), we have

$$\|u^n\|^2_{L^2([0,T]\times\mathbb{S})} = \int_0^T \int_{\mathbb{S}} (u^n)^2 dxdt = \int_0^T \|u^n\|^2_{L^2(\mathbb{S})} dx \le \left(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1\right)^2 T, \tag{4.3}$$

$$\|u^n_x\|^2_{L^2([0,T]\times\mathbb{S})} = \int_0^T \int_{\mathbb{S}} (u^n_x)^2 dxdt = \int_0^T \|u^n_x\|^2_{L^2(\mathbb{S})} dx \le \mu_1^2 T. \tag{4.4}$$

Moreover, by (3.8) and (4.3), we obtain

$$\|u^n u^n_x\|_{L^2([0,T]\times\mathbb{S})} \le \|u^n\|_{L^2([0,T]\times\mathbb{S})} \|u^n_x\|_{L^\infty([0,T]\times\mathbb{S})} \le \left(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1\right) |\mu_0|\sqrt{T}, \tag{4.5}$$

$$\begin{aligned}
&\|\partial_x g * (2\mu_0 e^{-\lambda t} u^n + \frac{1}{2}(u^n_x)^2)\|_{L^2([0,T]\times\mathbb{S})} \le \\
&\le \|\partial_x g\|_{L^2([0,T]\times\mathbb{S})} \|2\mu_0 e^{-\lambda t} u^n + \frac{1}{2}(u^n_x)^2\|_{L^1([0,T]\times\mathbb{S})} \le \\
&\le \frac{T}{12} \int_0^T \int_{\mathbb{S}} \left(2|\mu_0||u^n| + \frac{1}{2}(u^n_x)^2\right) dxdt \le \\
&\le \frac{T^2}{12}\left[\mu_0^2 + (|\mu_0| + \frac{\sqrt{3}}{6}\mu_1)^2 + \mu_1^2\right].
\end{aligned} \tag{4.6}$$

Combining (4.3), (4.5), (4.6) with Eq. (3.4), we know that $\{u^n_t(t,x)\}$ is uniformly bounded in $L^2([0,T] \times \mathbb{S})$. Thus, (4.3), (4.4) and this conclusion imply that

$$\int_0^T \int_{\mathbb{S}} ((u^n)^2 + (u^n_x)^2 + (u^n_t)^2) dxdt \le K,$$

where $K = K(|\mu_0|, \mu_1, T, \lambda) \ge 0$. It follows that $\{u^n(t,x)\}$ is uniformly bounded in the space $H^1([0,T] \times \mathbb{S})$. Thus (4.1) holds for some $u \in H^1([0,T] \times \mathbb{S})$.

Observe that, for each $0 \le s,t \le T$,

$$\|u^n(t,\cdot)-u^n(s,\cdot)\|^2_{L^2(\mathbb{S})}=\int\limits_{\mathbb{S}}\left(\int\limits_s^t\frac{\partial u^n}{\partial\tau}(\tau,x)d\tau\right)^2dx\le|t-s|\int\limits_0^T\int\limits_{\mathbb{S}}(u_t^n)^2dxdt.$$

Note that $\{u^n(t,x)\}$ is uniformly bounded in $L^\infty([0,T];H^1(\mathbb{S}))$, $\{u_t^n(t,x)\}$ is uniformly bounded in $L^2([0,T]\times\mathbb{S})$ and $H^1(\mathbb{S})\subset C(\mathbb{S})\subset L^\infty(\mathbb{S})\subset L^2(\mathbb{S})$, then (4.2) and $u(t,x)\in C(\mathbb{R}^+\times\mathbb{S})$ are a consequence of Lemma 2.3.

Lemma 4.1 is proved.

Next, we will deal with u_x^n and $\partial_x g*\left(2\mu_0e^{-\lambda t}u^n+\frac{1}{2}(u_x^n)^2\right)$. For fixed $t\in[0,T]$, it follows from (3.5), (3.7)–(3.9) that the sequence $u_x^{n_k}(t,\cdot)\in BV(\mathbb{S})$ with

$$\mathbb{V}(u_x^{n_k}(t,\cdot))=\left\|u_{xx}^{n_k}(t,\cdot)\right\|_{L^1(\mathbb{S})}=\left\|\mu(u^{n_k})-y^{n_k}\right\|_{L^1(\mathbb{S})}\le$$

$$\le\left\|\mu(u^{n_k})\right\|_{L^1(\mathbb{S})}+\left\|u^{n_k}\right\|_{L^1(\mathbb{S})}\le2|\mu_0|+\frac{\sqrt{3}}{6}\mu_1$$

and

$$\left\|u_x^{n_k}(t,\cdot)\right\|_{L^\infty(\mathbb{S})}\le|\mu_0|\le2|\mu_0|+\frac{\sqrt{3}}{6}\mu_1.$$

Applying Lemma 2.4, we obtain that there exists a subsequence, denoted again $\left\{u_x^{n_k}(t,\cdot)\right\}$, which converges at every point to some function $v(t,x)$ of finite variation with $\mathbb{V}(v(t,\cdot))\le2|\mu_0|+\frac{\sqrt{3}}{6}\mu_1$. Since for almost all $t\in[0,T]$, $u_x^{n_k}(t,\cdot)\to u_x(t,\cdot)$ in $\mathcal{D}'(\mathbb{S})$ in view of Lemma 4.1, it follows that $v(t,\cdot)=u_x(t,\cdot)$ for a.e.$t\in[0,T]$. So we have

$$u_x^{n_k}(t,\cdot)\to u_x(t,\cdot)\quad\text{a.e.}\quad\text{on}\quad[0,T]\times\mathbb{S},\qquad\text{as}\quad n_k\to\infty,\tag{4.7}$$

and for a.e. $t\in[0,T]$,

$$\mathbb{V}(u_x(t,\cdot))=\|u_{xx}(t,\cdot)\|_{M(\mathbb{S})}\le2|\mu_0|+\frac{\sqrt{3}}{6}\mu_1.\tag{4.8}$$

Therefore,

$$\left\|\partial_xg*\left(2\mu_0e^{-\lambda t}u^{n_k}+\frac{1}{2}(u_x^{n_k})^2\right)-\partial_xg*\left(2\mu_0e^{-\lambda t}u+\frac{1}{2}(u_x)^2\right)\right\|_{L^\infty([0,T]\times\mathbb{S})}\le$$

$$\le\|\partial_xg\|_{L^1([0,T]\times\mathbb{S})}\left\|2\mu_0e^{-\lambda t}(u^{n_k}-u)+\frac{1}{2}(u_x^{n_k})^2-(u_x)^2\right\|_{L^\infty([0,T]\times\mathbb{S})}\le$$

$$\le\frac{T}{4}\left(2|\mu_0|\|u^{n_k}-u\|_{L^\infty([0,T]\times\mathbb{S})}+\frac{1}{2}\|u_x^{n_k}+u_x\|_{L^\infty([0,T]\times\mathbb{S})}\|u_x^{n_k}-u_x\|_{L^\infty([0,T]\times\mathbb{S})}\right).$$

Combining this inequality with (4.2), (4.7) and note that

$$\left\|u_x(t,\cdot)\right\|_{L^\infty(\mathbb{S})}\le\lim_{n_k\to\infty}\left\|u_x^{n_k}(t,\cdot)\right\|_{L^\infty(\mathbb{S})}\le|\mu_0|,$$

we obtain

$$\partial_x g * \left(2\mu_0 e^{-\lambda t} u^{n_k} + \frac{1}{2}(u_x^{n_k})^2\right) \to \partial_x g * \left(2\mu_0 e^{-\lambda t} u + \frac{1}{2} u_x^2\right) \tag{4.9}$$

a.e. on $[0,T]\times\mathbb{S}$. The relations (4.2), (4.7) and (4.9) imply that u satisfies Eq. (2.5) in $\mathcal{D}'([0,T]\times\mathbb{S})$. Moreover, since

$$\big\|u(t,\cdot)\big\|_{L^\infty(\mathbb{S})} \le \lim_{n_k\to\infty} \|u^{n_k}(t,\cdot)\|_{L^\infty(\mathbb{S})} \le |\mu_0| + \frac{\sqrt{3}}{6}\mu_1,$$

we get $u \in L^\infty_{\text{loc}}\big(\mathbb{R}^+, W^{1,\infty}(\mathbb{S})\big)$ in view of T in (4.2) and (4.7) being arbitrary.

Now, we prove that $\mu(u) = \mu(u_0)e^{-\lambda t}$, $\|u_x\|^2_{L^2(\mathbb{S})} = e^{-2\lambda t}\|u_{0,x}\|^2_{L^2(\mathbb{S})}$ and $(\mu(u) - u_{xx}(t,\cdot)) \in$ $\in M^+(\mathbb{S})$ is uniformly bounded on $\mathbb{S}$.

On one hand, by (4.2), we have

$$\int_{\mathbb{S}} u^{n_k}(t,x)dx \to \int_{\mathbb{S}} u(t,x)dx = \mu(u) \qquad \text{as} \quad n_k\to\infty.$$

On the other hand,

$$\int_{\mathbb{S}} u^{n_k}(t,x)dx = \mu(u^{n_k}) = \mu_0 e^{-\lambda t}.$$

We find that $\mu(u) = \mu(u_0)e^{-\lambda t}$ by the uniqueness of limit.

By u satisfies (2.5) in the sense of distribution, we obtain

$$\phi_n * u_t + \phi_n * (uu_x) = -\phi_n * \left(\partial_x g * (2\mu(u)u + \frac{1}{2}u_x^2)\right) - \lambda\phi_n * u. \tag{4.10}$$

Differentiating (4.10) with respect to x, we get

$$(\phi_n * u_x)_t + \phi_n * (uu_{xx}) = -\phi_n * \left(2(\mu(u))^2 + \frac{1}{2}\mu(u_x^2) - 2\mu(u)u + \frac{1}{2}u_x^2\right) - \lambda\phi_n * u_x,$$

here we used the formula (2.2). Multiplying the equality above with $\phi_n * u_x$ and integrating the result with respect to x on $\mathbb{S}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} (\phi_n * u_x)^2 dx + \int_{\mathbb{S}} (\phi_n * u_x)(\phi_n * (uu_{xx}))dx =$$

$$= -\int_{\mathbb{S}} (\phi_n * u_x)\left(\phi_n * (2(\mu(u))^2 + \frac{1}{2}\mu(u_x^2) - 2\mu(u)u + \frac{1}{2}u_x^2)\right)dx - \lambda\int_{\mathbb{S}} (\phi_n * u_x)^2 dx.$$

Note that

$$\int_{\mathbb{S}} (\phi_n * u_x)(\phi_n * (2(\mu(u))^2 + \frac{1}{2}\mu(u_x^2))dx = 0$$

and

$$\int_{\mathbb{S}} (\phi_n * u_x)\big(\phi_n * (-2\mu(u)u)\big)dx = -2\mu(u)\int_{\mathbb{S}} (\phi_n * u_x)(\phi_n * u)dx = 0,$$

we have

$$\frac{d}{dt}\int\limits_{\mathbb{S}}(\phi_n * u_x)^2dx + 2\lambda\int\limits_{\mathbb{S}}(\phi_n * u_x)^2dx =$$

$$= -2\int\limits_{\mathbb{S}}(\phi_n * u_x)(\phi_n * (uu_{xx}))dx - \int\limits_{\mathbb{S}}(\phi_n * u_x)(\phi_n * u_x^2)dx.$$

Let

$$f_n(t) = \int\limits_{\mathbb{S}}(\phi_n * u_x)^2dx,$$

$$g_n(t) = -2\int\limits_{\mathbb{S}}(\phi_n * u_x)\big(\phi_n * (uu_{xx})\big)dx - \int\limits_{\mathbb{S}}(\phi_n * u_x)(\phi_n * u_x^2)dx,$$

then we obtain

$$\frac{df_n(t)}{dt} + 2\lambda f_n(t) = g_n(t), \quad \text{for} \quad \text{a.e.} \quad t \in \mathbb{R}^+. \tag{4.11}$$

Applying Lemma 2.5 to $\phi_n * u_x$, it follows from (4.11) that

$$f_n(t) - e^{-2\lambda t}f_n(0) = \int\limits_0^t e^{-2\lambda(t-s)}g_n(s)ds. \tag{4.12}$$

Note that $g_n(t) \to 0$ as $n \to \infty$ for a.e. $t \in \mathbb{R}^+$. For any $T > 0$, there exists a constant $K(T) > 0$ such that $|g_n(t)| \le K(T)$, $t \in [0, T]$, $n \ge 1$. An application of Lebesgue's dominated convergence theorem to (4.12), we get

$$\lim_{n\to\infty}\Big[f_n(t) - e^{-2\lambda t}f_n(0)\Big] = 0.$$

Let $t \in \mathbb{R}^+$ be given. We have $\|u_x\|^2_{L^2(\mathbb{S})} = e^{-2\lambda t}\|u_{0,x}\|^2_{L^2(\mathbb{S})}$.

Note that $L^1(\mathbb{S}) \subset M(\mathbb{S})$. By (4.8) and $\mu(u) = \mu_0 e^{-\lambda t}$, we obtain

$$\big\|\mu(u) - u_{xx}(t,\cdot)\big\|_{M(\mathbb{S})} \le \|\mu(u)\|_{L^1(\mathbb{S})} + \|u_{xx}(t,\cdot)\|_{M(\mathbb{S})} \le 3|\mu_0| + \frac{\sqrt{3}}{6}\mu_1.$$

It follows that for all $t \in \mathbb{R}^+$, $(\mu(u) - u_{xx}(t,\cdot)) \in M(\mathbb{S})$ is uniformly bounded on $\mathbb{S}$. In view of (4.2) and (4.7), we have

$$\big[\mu(u^{n_k}) - u^{n_k}_{xx}(t,\cdot)\big] \to [\mu(u) - u_{xx}(t,\cdot)] \quad \text{in} \quad \mathcal{D}'(\mathbb{S}) \quad \text{for} \quad n_k \to \infty, \quad t \in [0, T].$$

Since $\mu(u^{n_k}) - u^{n_k}_{xx}(t,\cdot) = y^{n_k}(t,\cdot) \ge 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{S}$, we have $(\mu(u) - u_{xx}(t,\cdot)) \in$ $\in L^\infty_{\text{loc}}(\mathbb{R}^+, M^+(\mathbb{S}))$.

Theorem 1.1 is proved.

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