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COMMON FIXED POINT THEOREMS AND C-DISTANCE IN ORDERED CONE METRIC SPACES

ТЕОРЕМИ ПРО СПІЛЬНУ НЕРУХОМУ ТОЧКУ ТА C-ВІДСТАНЬ В УПОРЯДКОВАНИХ КОНІЧНИХ МЕТРИЧНИХ ПРОСТОРАХ

We present a generalization of several fixed and common fixed point theorems on the c-distance in ordered cone metric spaces. In this way, we improve and generalize various results existing in the literature.

Наведено узагальнення деяких теорем про нерухому точку та спільну нерухому точку для c-відстані в упорядкованих конічних метричних просторах. Таким чином, покращено та узагальнено різноманітні результати, що наведені в літературі.

1. Introduction. Huang and Zhang [18] have introduced the concept of a cone metric space by replacing the set of real numbers by an ordered Banach space and have showed some fixed point theorems of contractive type mappings on cone metric spaces. Afterward, several fixed and common fixed point results in cone metric spaces with related results have been introduced in [2, 4, 5, 8, 10, 14, 16, 17, 20] and the references contained therein. Also, the existence of fixed points in partially ordered cone metric spaces has been studied in [6, 7, 24].

In 1996, Kada et al. [21] defined the concept of w-distance in complete metric spaces. Later, many authors proved some fixed point theorems in complete metric spaces (see [3, 22]). Recently, Saadati et al. [23] introduced a probabilistic version of the w-distance in a Menger probabilistic metric space. In the sequel, Cho et al. [9] and Wang and Guo [26] defined a concept of the c-distance in a cone metric space, which is a cone version of the w-distance of Kada et al. [21] and proved some fixed point theorems in ordered cone metric spaces. Then Sintunavarat et al. [25] generalized the Banach contraction theorem on c-distance of Cho et al. [9]. Also, Dordević et al. [12] proved some fixed point and common fixed point theorems under c-distance for contractive mappings in tvs-cone metric spaces.

The purpose of this work is to extend and generalize the main results of Cho et al. [9], Sintunavarat et al. [25], Huang and Zhang [18] on c-distance in ordered cone metric spaces.

2. Preliminaries.

Definition 2.1 (see [11, 18]). *Let E be a real Banach space and let 0 denote the zero element in E. A subset P of E is called a cone if the following conditions hold*:

(C_1) *P is nonempty closed and $P \neq \{0\}$*;

(C_2) *$a, b \in \mathbf{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$*;

(C_3) *if $x \in P$ and $-x \in P$, then $x = 0$.*

Given a cone $P \subset E$, we define a *partial ordering* $\preceq$ with respect to P by $x \preceq y \Longleftrightarrow y - x \in P$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in \mathrm{int}P$, where $\mathrm{int}\, P$ is interior of P. If $\mathrm{int}P \neq \varnothing$, the cone P is called *solid*. The cone P is called *normal* if there exists a number $k > 0$ such that, for all $x, y \in E$,

$$0 \preceq x \preceq y \Longrightarrow \|x\| \leq k\|y\|.$$

The least positive number satisfying the above is called the *normal constant* of P.

Definition 2.2 (see [18]). *Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that a mapping $d\colon X \times X \to E$ satisfies the following conditions:*

(CM_1) $0 \preceq d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;

(CM_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;

(CM_3) $d(x,y) \preceq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.3 (see [18]). *Let (X,d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$.*

(1) *$\{x_n\}$ is said to be convergent to x if, for any $c \in E$ with $0 \ll c$, there exists $n_0 \geq 1$ such that $d(x_n, x) \ll c$ for all $n > n_0$ and we write $\lim_{n\to\infty} d(x_n, x) = 0$.*

(2) *$\{x_n\}$ is called a Cauchy sequence if, for any $c \in E$ with $0 \ll c$, there exists $n_0 \geq 1$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$ and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.*

(3) *If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.*

Lemma 2.1 (see [18]). *Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold*:

(1) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.*

(2) *If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*

(3) *If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

(4) *If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x,y)$ as $n \to \infty$.*

(5) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Lemma 2.2 (see [6, 19]). *Let E be a real Banach space with a cone P in E. Then, for all $u, v, w, c \in E$, the following hold*:

(1) *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

(2) *If $0 \preceq u \ll c$ for all $c \in int\, P$, then $u = 0$.*

(3) *If $u \preceq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.*

(4) *Let $x_n \to 0$ in E, $0 \preceq x_n$ and $0 \ll c$. Then there exists a positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.*

(5) *If $0 \preceq u \preceq v$ and k is a nonnegative real number, then $0 \preceq ku \preceq kv$.*

(6) *If $0 \preceq u_n \preceq v_n$ for all $n \geq 1$ and $u_n \to u$, $v_n \to v$ as $n \to \infty$, then $0 \preceq u \preceq v$.*

Definition 2.4 (see [9, 26]). *Let (X,d) be a cone metric space. A mapping $q\colon X \times X \to E$ is called a c-distance on X if the following are satisfied:*

(CD_1) $0 \preceq q(x,y)$ *for all* $x, y \in X$;

(CD_2) $q(x,z) \preceq q(x,y) + q(y,z)$ *for all* $x, y, z \in X$;

(CD_3) *for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;*

(CD_4) *for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.*

***Remark* 2.1** (see [9]). Each w-distance q in a metric space (X,d) is a c-distance with $E = \mathbf{R}^+$ and $P = [0,\infty)$. But the converse does not hold. Thus the c-distance is a generalization of the w-distance.

***Example* 2.1** (see [9, 26]). (1) Let $E = C^1_{\mathbf{R}}[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the cone $P = \{x \in E\colon x(t) \geq 0 \text{ on } [0,1]\}$. Also, let $X = [0,\infty)$ and define a mapping $d\colon X \times X \to E$ by $d(x,y) = |x-y|\psi$ for all $x, y \in X$, where $\psi\colon [0,1] \to \mathbf{R}$ such that $\psi(t) = 2^t$.

Then (X, d) is a cone metric space. Define a mapping $q\colon X \times X \to E$ by $q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.

(2) Let (X, d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$ for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

(3) Let (X, d) be a cone metric space and P be a normal cone. Define $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

(4) Let $E = \mathbf{R}$, $P = \{x \in E\colon x \geq 0\}$ and $X = [0, \infty)$. Define a mapping $d\colon X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q\colon X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance.

***Remark* 2.2** (see [9, 26])**.** From (2) and (4) in Example 2.1, we have two important results:

(1) For any c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

(2) For any c-distance q, $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.

Lemma 2.3 (see [9, 25, 26])**.** *Let (X, d) be a cone metric space and q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to* 0*. Then the following hold*:

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \geq 1$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for each $n \geq 1$, then $\{y_n\}$ converges to z.*

(3) *If $q(x_n, x_m) \preceq u_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(4) *If $q(y, x_n) \preceq u_n$ for each $n \geq 1$, then $\{x_n\}$ is a Cauchy sequence in X.*

Definition 2.5 (see [6, 9])**.** *Let $(X, \sqsubseteq)$ be a partially ordered set. Two mappings $f, g\colon X \to X$ are said to be weakly increasing if $fx \sqsubseteq gfx$ and $gx \sqsubseteq fgx$ hold for all $x \in X$.*

3. Main results. Our first result is the following theorem of Hardy–Rogers type (see [15]) for any c-distance in a cone metric space without normality condition of cone.

Theorem 3.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i\colon X \to [0, 1)$ such that the following condition hold:*

$$\alpha_i(fx) \leq \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, \ldots, 5$. Also, let q be a c-distance on X and $f\colon X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following conditions:

$$q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) + \alpha_4(x)q(x, fy) + \alpha_5(x)q(y, fx), \tag{3.1}$$

$$q(fy, fx) \preceq \alpha_1(x)q(y, x) + \alpha_2(x)q(fx, x) + \alpha_3(x)q(fy, y) + \alpha_4(x)q(fy, x) + \alpha_5(x)q(fx, y) \tag{3.2}$$

for all comparable $x, y \in X$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1. \tag{3.3}$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. Since f is nondecreasing with respect to $\sqsubseteq$ and $x_0 \sqsubseteq fx_0$, we obtain by induction that

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq \ldots \sqsubseteq f^nx_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \ldots,$$

where $x_n = fx_{n-1} = f^n x_0$. Now, setting $x = x_n$ and $y = x_{n-1}$ in (3.1), we have

$$q(x_{n+1}, x_n) = q(fx_n, fx_{n-1}) \preceq$$

$$\preceq \alpha_1(x_n)q(x_n, x_{n-1}) + \alpha_2(x_n)q(x_n, fx_n) + \alpha_3(x_n)q(x_{n-1}, fx_{n-1}) +$$

$$+\alpha_4(x_n)q(x_n, fx_{n-1}) + \alpha_5(x_n)q(x_{n-1}, fx_n) =$$

$$= \alpha_1(fx_{n-1})q(x_n, x_{n-1}) + \alpha_2(fx_{n-1})q(x_n, x_{n+1}) + \alpha_3(fx_{n-1})q(x_{n-1}, x_n) +$$

$$+\alpha_4(fx_{n-1})q(x_n, x_n) + \alpha_5(fx_{n-1})q(x_{n-1}, x_{n+1}) \preceq$$

$$\preceq \alpha_1(x_{n-2})q(x_n, x_{n-1}) + \alpha_2(x_{n-2})q(x_n, x_{n+1}) + \alpha_3(x_{n-2})q(x_{n-1}, x_n) +$$

$$+\alpha_4(x_{n-2})[q(x_n, x_{n+1}) + q(x_{n+1}, x_n)] + \alpha_5(x_{n-2})[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \preceq \ldots$$

$$\ldots \preceq \alpha_1(x_0)q(x_n, x_{n-1}) + (\alpha_2 + \alpha_4 + \alpha_5)(x_0)q(x_n, x_{n+1} +$$

$$+(\alpha_3 + \alpha_5)(x_0)q(x_{n-1}, x_n) + \alpha_4(x_0)q(x_{n+1}, x_n). \tag{3.4}$$

Similarly, setting $x = x_n$ and $y = x_{n-1}$ in (3.2), we get

$$q(x_n, x_{n+1}) \preceq \alpha_1(x_0)q(x_{n-1}, x_n) + (\alpha_2 + \alpha_4 + \alpha_5)q(x_{n+1}, x_n) +$$

$$+\alpha_4(x_0)q(x_n, x_{n+1}) + (\alpha_3 + \alpha_5)(x_0)q(x_n, x_{n-1}). \tag{3.5}$$

Thus, adding up (3.4) and (3.5), we obtain

$$q(x_{n+1}, x_n) + q(x_n, x_{n+1}) \preceq (\alpha_1 + \alpha_3 + \alpha_5)(x_0)[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] +$$

$$+(\alpha_2 + 2\alpha_4 + \alpha_5)(x_0)[q(x_{n+1}, x_n) + q(x_n, x_{n+1})].$$

Set $v_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$ and then we have

$$v_n \preceq (\alpha_1 + \alpha_3 + \alpha_5)(x_0)v_{n-1} + (\alpha_2 + 2\alpha_4 + \alpha_5)(x_0)v_n.$$

Thus we get $v_n \preceq \lambda v_{n-1}$, where $\lambda = \dfrac{(\alpha_1 + \alpha_3 + \alpha_5)(x_0)}{1 - (\alpha_2 + 2\alpha_4 + \alpha_5)(x_0)} < 1$ by (3.4). By repeating the procedure, we obtain $v_n \preceq \lambda^n v_0$ for all $n \geq 1$. Thus it follows that

$$q(x_n, x_{n+1}) \preceq v_n \preceq \lambda^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{3.6}$$

Let $m > n$, then it follows from (3.6) and $\lambda < 1$ that

$$q(x_n, x_m) \preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \preceq$$

$$\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})[q(x_1, x_0) + q(x_0, x_1)] \preceq$$

$$\preceq \frac{\lambda^n}{1 - \lambda}[q(x_1, x_0) + q(x_0, x_1)].$$

Lemma 2.3 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$. The continuity of f implies that $x_{n+1} = fx_n \to fx'$ as $n \to \infty$ and, since the limit of a sequence is unique, we get that $fx' = x'$. Thus x' is a fixed point of f.

Now, suppose that $fz = z$. Then, by using (3.1), we have

$$q(z,z) = q(fz,fz) \preceq$$

$$\preceq \alpha_1(z)q(z,z) + \alpha_2(z)q(z,fz) + \alpha_3(z)q(z,fz) + \alpha_4(z)q(z,fz) + \alpha_5(z)q(z,fz) \preceq$$

$$\preceq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(z)q(z,z).$$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(z) < (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(z) < 1$, we get that $q(z,z) = 0$ by Lemma 2.2.

Theorem 3.1 is proved.

***Corollary* 3.1** ([25], Theorem 3.1). *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i \colon X \to [0,1)$ such that the following condition hold*:

$$\alpha_i(fx) \le \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, 3$. Also, let q be a c-distance on X and $f \colon X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following condition:

$$q(fx,fy) \preceq \alpha_1(x)q(x,y) + \alpha_2(x)q(x,fx) + \alpha_3(x)q(y,fy)$$

for all $x, y \in X$ with $y \sqsubseteq x$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z,z) = 0$.

Theorem 3.2. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exists a continuous and nondecreasing mapping $f \colon X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold*:

$$q(fx,fy) \preceq \alpha_1 q(x,y) + \alpha_2 q(x,fx) + \alpha_3 q(y,fy) + \alpha_4 q(x,fy) + \alpha_5 q(y,fx),$$

$$q(fy,fx) \preceq \alpha_1 q(y,x) + \alpha_2 q(fx,x) + \alpha_3 q(fy,y) + \alpha_4 q(fy,x) + \alpha_5 q(fx,y)$$

for all comparable $x, y \in X$, where α_i are nonnegative coefficients for $i = 1, 2, \ldots, 5$ with

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z,z) = 0$.

Proof. We can prove this result by applying Theorem 3.1 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \ldots, 5$.

***Corollary* 3.2** ([9], Theorem 3.1)**.** *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exists a continuous and nondecreasing mapping $f \colon X \to X$ with respect to $\sqsubseteq$ such that the following condition hold:*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy)$$

for all $x, y \in X$ with $y \sqsubseteq x$, where α_i are nonnegative coefficients for $i = 1, 2, 3$ with

$$\alpha_1 + \alpha_2 + \alpha_3 < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Our second result is the following theorem of Hardy – Rogers type (see [15]) for any c-distance in a cone metric space with a normal cone.

Theorem 3.3. *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i \colon X \to [0, 1)$ such that the following condition hold*:

$$\alpha_i(fx) \leq \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, \ldots, 5$. Also, let q be a c-distance on X and $f \colon X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following conditions:

$$q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy) + \alpha_4(x)q(x, fy) + \alpha_5(x)q(y, fx),$$

$$q(fy, fx) \preceq \alpha_1(x)q(y, x) + \alpha_2(x)q(fx, x) + \alpha_3(x)q(fy, y) + \alpha_4(x)q(fy, x) + \alpha_5(x)q(fx, y)$$

for all comparable $x, y \in X$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$ and $\inf\{\|q(x, y)\| + \|q(x, fx)\| \colon x \in X\} > 0$ *for all $y \in X$ with $y \neq fy$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.*

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. As in the proof of Theorem 3.1, we have

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq \ldots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \ldots,$$

where $x_n = fx_{n-1} = f^n x_0$. Moreover, $\{x_n\}$ converges to a point $x' \in X$ and

$$q(x_n, x_m) \preceq \frac{\lambda^n}{1 - \lambda}[q(x_1, x_0) + q(x_0, x_1)]$$

for all positive numbers with $m > n \geq 1$, where $\lambda = \dfrac{(\alpha_1 + \alpha_3 + \alpha_5)(x_0)}{1 - (\alpha_2 + 2\alpha_4 + \alpha_5)(x_0)} < 1$. By (CD_3), it follows that

$$q(x_n, x') \preceq \frac{\lambda^n}{1 - \lambda}[q(x_1, x_0) + q(x_0, x_1)]$$

for all $n \geq 1$. Since P is a normal cone with normal constant k, we get

$$\|q(x_n, x_m)\| \leq k(\frac{\lambda^n}{1-\lambda})\|q(x_1, x_0) + q(x_0, x_1)\|$$

for all $m > n \geq 1$. In particular, we obtain

$$\|q(x_n, x_{n+1})\| \leq k\left(\frac{\lambda^n}{1-\lambda}\right)\|q(x_1, x_0) + q(x_0, x_1)\| \tag{3.7}$$

for all $n \geq 1$. Also, we get

$$\|q(x_n, x')\| \leq k\left(\frac{\lambda^n}{1-\lambda}\right)\|q(x_1, x_0) + q(x_0, x_1)\| \tag{3.8}$$

for all $n \geq 1$. Suppose that $x' \neq fx'$. Then, by the hypothesis, (3.7) and (3.8), we have

$$0 < \inf\{\|q(x, x')\| + \|q(x, fx)\| \colon x \in X\} \leq$$

$$\leq \inf\{\|q(x_n, x')\| + \|q(x_n, fx_n)\| \colon n \geq 1\} =$$

$$= \inf\{\|q(x_n, x')\| + \|q(x_n, x_{n+1})\| \colon n \geq 1\} \leq$$

$$\leq \inf\left\{k\left(\frac{\lambda^n}{1-\lambda}\right)\|q(x_1, x_0) + q(x_0, x_1)\| + k\left(\frac{\lambda^n}{1-\lambda}\right)\|q(x_1, x_0) + q(x_0, x_1)\| \colon n \geq 1\right\} = 0$$

which is a contradiction. Hence $x' = fx'$.

Moreover, suppose that $fz = z$. Then, we have $q(z, z) = 0$ by the final part of the proof of Theorem 3.1.

Theorem 3.3 is proved.

***Corollary* 3.3** ([25], Theorem 3.2). *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i \colon X \to [0, 1)$ such that the following condition hold*:

$$\alpha_i(fx) \leq \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, 3$. Also, let q be a c-distance on X and $f \colon X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following condition:

$$q(fx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, fy)$$

for all $x, y \in X$ with $y \sqsubseteq x$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$ and $\inf\{\|q(x, y)\| + \|q(x, fx)\| \colon x \in X\} > 0$ for all $y \in X$ with $y \neq fy$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Theorem 3.4. *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exists a nondecreasing mapping $f\colon X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold:*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy) + \alpha_5 q(y, fx),$$

$$q(fy, fx) \preceq \alpha_1 q(y, x) + \alpha_2 q(fx, x) + \alpha_3 q(fy, y) + \alpha_4 q(fy, x) + \alpha_5 q(fx, y)$$

for all comparable $x, y \in X$, where α_i are nonnegative coefficients for $i = 1, 2, \ldots, 5$ with

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$ and $\inf\{\|q(x, y)\| + \|q(x, fx)\|\colon x \in X\} > 0$ for all $y \in X$ with $y \neq fy$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by applying Theorem 3.3 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \ldots, 5$.

Corollary **3.4** ([9], Theorem 3.2)**.** *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exists a nondecreasing mapping $f\colon X \to X$ with respect to $\sqsubseteq$ such that the following condition hold*:

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy)$$

for all $x, y \in X$ with $y \sqsubseteq x$, where α_i are nonnegative coefficients for $i = 1, 2, 3$ with

$$\alpha_1 + \alpha_2 + \alpha_3 < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$ and $\inf\{\|q(x, y)\| + \|q(x, fx)\|\colon x \in X\} > 0$ for all $y \in X$ with $y \neq fy$, then f has a fixed point. Moreover, if $fz = z$, then $q(z, z) = 0$.

Our third result include two mappings and the existence of their common fixed point for any c-distance in a cone metric space without the normality condition of the cone.

Theorem 3.5. *Let $(X, \sqsubseteq)$ be a partially ordered set and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i\colon X \to [0, 1)$ such that the following conditions hold*:

$$\alpha_i(fx) \leq \alpha_i(x), \alpha_i(gx) \leq \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, \ldots, 5$. Also, let q be a c-distance on X and $f, g\colon X \to X$ be two continuous and weakly increasing mappings with respect to $\sqsubseteq$ satisfying the following conditions:

$$q(fx, gy) \preceq \alpha_1(x) q(x, y) + \alpha_2(x) q(x, fx) + \alpha_3(x) q(y, gy) + \alpha_4(x) q(x, gy) + \alpha_5(x) q(y, fx), \tag{3.9}$$

$$q(gy, fx) \preceq \alpha_1(x) q(y, x) + \alpha_2(x) q(fx, x) + \alpha_3(x) q(gy, y) + \alpha_4(x) q(gy, x) + \alpha_5(x) q(fx, y) \tag{3.10}$$

for all comparable $x, y \in X$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1. \tag{3.11}$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = fx_{2n} \quad , \quad x_{2n+2} = gx_{2n+1}$$

for all $n \geq 0$. Since f and g are weakly increasing mappings, there exist $x_1, x_2, x_3 \in X$ such that

$$x_1 = fx_0 \sqsubseteq gfx_0 = gx_1 = x_2 \quad , \quad x_2 = gx_1 \sqsubseteq fgx_1 = fx_2 = x_3.$$

Continuing in this manner, it follows that there exist $x_{2n+1} \in X$ and $x_{2n+2} \in X$ such that

$$x_{2n+1} = fx_{2n} \sqsubseteq gfx_{2n} = gx_{2n+1} = x_{2n+2},$$

$$x_{2n+2} = gx_{2n+1} \sqsubseteq fgx_{2n+1} = fx_{2n+2} = x_{2n+3}$$

for all $n \geq 0$. Thus $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \ldots$ for all $n \geq 1$, that is, $\{x_n\}$ is a nondecreasing sequence. Since $x_{2n} \sqsubseteq x_{2n+1}$ for all $n \geq 1$, by using (3.9) for $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$q(x_{2n+1}, x_{2n+2}) = q(fx_{2n}, gx_{2n+1}) \preceq$$

$$\preceq \alpha_1(x_{2n})q(x_{2n}, x_{2n+1}) + \alpha_2(x_{2n})q(x_{2n}, fx_{2n}) + \alpha_3(x_{2n})q(x_{2n+1}, gx_{2n+1}) +$$

$$+\alpha_4(x_{2n})q(x_{2n}, gx_{2n+1}) + \alpha_5(x_{2n})q(x_{2n+1}, fx_{2n}) =$$

$$= (\alpha_1 + \alpha_2)(gx_{2n-1})q(x_{2n}, x_{2n+1}) + \alpha_3(gx_{2n-1})q(x_{2n+1}, x_{2n+2}) +$$

$$+\alpha_4(gx_{2n-1})q(x_{2n}, x_{2n+2}) + \alpha_5(gx_{2n-1})q(x_{2n+1}, x_{2n+1}) \preceq$$

$$\preceq (\alpha_1 + \alpha_2)(x_{2n-1})q(x_{2n}, x_{2n+1}) + \alpha_3(x_{2n-1})q(x_{2n+1}, x_{2n+2}) +$$

$$+\alpha_4(x_{2n-1})[q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2})] +$$

$$+\alpha_5(x_{2n-1})[q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})] =$$

$$= (\alpha_1 + \alpha_2 + \alpha_4)(fx_{2n-2})q(x_{2n}, x_{2n+1}) + \alpha_5(fx_{2n-2})q(x_{2n+2}, x_{2n+1}) +$$

$$+(\alpha_3 + \alpha_4 + \alpha_5)(fx_{2n-2})q(x_{2n+1}, x_{2n+2}) \preceq \cdots$$

$$\cdots \preceq (\alpha_1 + \alpha_2 + \alpha_4)(x_0)q(x_{2n}, x_{2n+1}) +$$

$$+(\alpha_3 + \alpha_4 + \alpha_5)(x_0)q(x_{2n+1}, x_{2n+2}) + \alpha_5(x_0)q(x_{2n+2}, x_{2n+1}).$$

Similarly, by using (3.10) for $x = x_{2n}$ and $y = x_{2n+1}$, we get

$$q(x_{2n+2}, x_{2n+1}) \preceq (\alpha_1 + \alpha_2 + \alpha_4)(x_0)q(x_{2n+1}, x_{2n}) + \alpha_5(x_0)q(x_{2n+1}, x_{2n+2}) +$$

$$+(\alpha_3 + \alpha_4 + \alpha_5)(x_0)q(x_{2n+2}, x_{2n+1}).$$

Thus, adding up two previous relations, we obtain

$$q(x_{2n+2},x_{2n+1})+q(x_{2n+1},x_{2n+2})\preceq(\alpha_1+\alpha_2+\alpha_4)(x_0)[q(x_{2n+1},x_{2n})+q(x_{2n},x_{2n+1})]+$$

$$+(\alpha_3+\alpha_4+2\alpha_5)(x_0)[q(x_{2n+2},x_{2n+1})+q(x_{2n+1},x_{2n+2})].$$

Setting $v_n=q(x_{2n+1},x_{2n})+q(x_{2n},x_{2n+1})$ and $u_n=q(x_{2n+2},x_{2n+1})+q(x_{2n+1},x_{2n+2})$, it follows that

$$u_n\preceq(\alpha_1+\alpha_2+\alpha_4)(x_0)v_n+(\alpha_3+\alpha_4+2\alpha_5)(x_0)u_n.$$

Thus we have

$$u_n\preceq\lambda v_n, \tag{3.12}$$

where $\lambda=\dfrac{(\alpha_1+\alpha_2+\alpha_4)(x_0)}{1-(\alpha_3+\alpha_4+2\alpha_5)(x_0)}\in[0,1)$ by (3.11). By a similar procedure, starting with $x=x_{2n+2}$ and $y=x_{2n+1}$, we have

$$v_{n+1}\preceq\lambda u_n. \tag{3.13}$$

From (3.12) and (3.13), we get that

$$v_{n+1}\preceq\lambda^2 v_n \quad , \quad u_n\preceq\lambda^2 u_{n-1}$$

for all $n\geq 1$. Therefore, $\{u_n\}$ and $\{v_n\}$ are two sequences converging to 0. Also, we obtain $q(x_{2n},x_{2n+1})\preceq v_n$ and $q(x_{2n+1},x_{2n+2})\preceq u_n$ and so $q(x_n,x_{n+1})\preceq v_n+u_n$.

On the other hand, it is easy to show that, if $\{u_n\}$ and $\{v_n\}$ are two sequences in E converging to 0, then $\{u_n+v_n\}$ is a sequence converging to 0 (see [9, 12]). Lemma 2.3 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x'\in X$ such that $x_n\to x'$ as $n\to\infty$. The continuity of f and g implies that $x_{2n+1}=fx_{2n}\to fx'$ and $x_{2n+2}=gx_{2n+1}\to gx'$ as $n\to\infty$. Since the limit of a sequence is unique, we get $fx'=x'$ and $gx'=x'$. Thus x' is a common fixed point of f and g.

Suppose that $z\in X$ is another point satisfying $fz=gz=z$. Then (3.9) implies that

$$q(z,z)=q(fz,gz)\preceq$$

$$\preceq\alpha_1(z)q(z,z)+\alpha_2(z)q(z,fz)+\alpha_3(z)q(z,gz)+\alpha_4(z)q(z,gz)+\alpha_5(z)q(z,fz)\preceq$$

$$\preceq(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)(z)q(z,z).$$

Since $(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)(z)<(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5)(z)$ and $(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5)(z)<1$ for all $z\in X$, by (3.9), we get $q(z,z)=0$ by Lemma 2.2.

Theorem 3.5 is proved.

***Corollary* 3.5** ([25], Theorem 3.3)**.** *Let* $(X,\sqsubseteq)$ *be a partially ordered set and* (X,d) *be a complete cone metric space. Suppose that there exist mappings* $\alpha_i\colon X\to[0,1)$ *such that the following conditions hold*:

$$\alpha_i(fx)\leq\alpha_i(x),\alpha_i(gx)\leq\alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, 3$. Also, let q be a c-distance on X and $f, g\colon X \to X$ be two continuous and weakly increasing mappings with respect to $\sqsubseteq$ satisfying the following conditions:

$$q(fx, gy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, gy),$$

$$q(gx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, gx) + \alpha_3(x)q(y, fy)$$

for all $x, y \in X$ with $y \sqsubseteq x$ such that

$$(\alpha_1 + \alpha_2 + \alpha_3)(x) < 1.$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Theorem 3.6. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exist two continuous and weakly increasing mappings $f, g\colon X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold*:

$$q(fx, gy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, gy) + \alpha_4 q(x, gy) + \alpha_5 q(y, fx),$$

$$q(gy, fx) \preceq \alpha_1 q(y, x) + \alpha_2 q(fx, x) + \alpha_3 q(gy, y) + \alpha_4 q(gy, x) + \alpha_5 q(fx, y)$$

for all comparable $x, y \in X$, where α_i are nonnegative coefficients for $i = 1, 2, \ldots, 5$ with

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Proof. We can prove this result by applying Theorem 3.5 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \ldots, 5$.

***Corollary* 3.6** ([9], Theorem 3.3)**.** *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exist two continuous and weakly increasing mappings $f, g\colon X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold*:

$$q(fx, gy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, gy),$$

$$q(gx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, gx) + \alpha_3 q(y, fy)$$

for all comparable $x, y \in X$, where α_i are nonnegative coefficients for $i = 1, 2, 3$ with

$$\alpha_1 + \alpha_2 + \alpha_3 < 1.$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.

Our next result include two mappings and the existence of their common fixed point for any c-distance in a cone metric space with the normal cone.

Theorem 3.7. *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone and (X, d) be a complete cone metric space. Suppose that there exist mappings $\alpha_i\colon X \to [0, 1)$ such that the following conditions hold*:

$$\alpha_i(fx) \le \alpha_i(x), \alpha_i(gx) \le \alpha_i(x)$$

for all $x \in X$ and $i = 1, 2, \ldots, 5$. Also, let q be a c-distance on X and $f, g\colon X \to X$ be two weakly increasing mappings with respect to $\sqsubseteq$ satisfying the following conditions:

$$q(fx, gy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, gy) + \alpha_4(x)q(x, gy) + \alpha_5(x)q(y, fx),$$

$$q(gy, fx) \preceq \alpha_1(x)q(y, x) + \alpha_2(x)q(fx, x) + \alpha_3(x)q(gy, y) + \alpha_4(x)q(gy, x) + \alpha_5(x)q(fx, y)$$

for all comparable $x, y \in X$ *such that*

$$(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) < 1.$$

If $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *and* $\inf\{\|q(x, y)\| + \|q(x, gx)\| : x \in X\} > 0$ *for all* $y \in X$ *with* $y \neq fy$ *and* $y \neq gy$, *respectively, then* f *and* g *have a common fixed point. Moreover, if* $fz = gz = z$, *then* $q(z, z) = 0$.

Proof. The proof is similar to Theorem 3.3. One can prove this theorem by using the proof of Theorems 3.3 and 3.6.

***Corollary* 3.7** ([25], Theorem 3.4)**.** *Let* $(X, \sqsubseteq)$ *be a partially ordered set,* P *be a normal cone and* (X, d) *be a complete cone metric space. Suppose that there exist mappings* $\alpha_i \colon X \to [0, 1)$ *such that the following conditions hold*:

$$\alpha_i(fx) \leq \alpha_i(x), \alpha_i(gx) \leq \alpha_i(x)$$

for all $x \in X$ *and* $i = 1, 2, 3$. *Also, let* q *be a c-distance on* X *and* $f, g \colon X \to X$ *be two weakly increasing mappings with respect to* $\sqsubseteq$ *satisfying the following conditions*:

$$q(fx, gy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, fx) + \alpha_3(x)q(y, gy),$$

$$q(gx, fy) \preceq \alpha_1(x)q(x, y) + \alpha_2(x)q(x, gx) + \alpha_3(x)q(y, fy)$$

for all comparable $x, y \in X$ *such that*

$$(\alpha_1 + \alpha_2 + \alpha_3)(x) < 1.$$

If $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *and* $\inf\{\|q(x, y)\| + \|q(x, gx)\| : x \in X\} > 0$ *for all* $y \in X$ *with* $y \neq fy$ *and* $y \neq gy$, *respectively, then* f *and* g *have a common fixed point. Moreover, if* $fz = gz = z$, *then* $q(z, z) = 0$.

Theorem 3.8. *Let* $(X, \sqsubseteq)$ *be a partially ordered set,* P *be a normal cone,* (X, d) *be a complete cone metric space and* q *be a c-distance on* X. *Suppose that there exist two weakly increasing mappings* $f, g \colon X \to X$ *with respect to* $\sqsubseteq$ *such that the following conditions hold*:

$$q(fx, gy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, gy) + \alpha_4 q(x, gy) + \alpha_5 q(y, fx),$$

$$q(gy, fx) \preceq \alpha_1 q(y, x) + \alpha_2 q(x, gx) + \alpha_3 q(y, fy) + \alpha_4 q(gy, x) + \alpha_5 q(fx, y)$$

for all comparable $x, y \in X$, *where* α_i *are nonnegative coefficients for* $i = 1, 2, \ldots, 5$ *with*

$$\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 1.$$

If $\inf\{\|q(x, y)\| + \|q(x, fx)\| : x \in X\} > 0$ *and* $\inf\{\|q(x, y)\| + \|q(x, gx)\| : x \in X\} > 0$ *for all* $y \in X$ *with* $y \neq fy$ *and with* $y \neq gy$, *respectively, then* f *and* g *have a common fixed point. Moreover, if* $fz = gz = z$, *then* $q(z, z) = 0$.

Proof. We can prove this result by applying Theorem 3.7 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \ldots, 5$.

***Corollary* 3.8** ([9], Theorem 3.4). *Let $(X, \sqsubseteq)$ be a partially ordered set, P be a normal cone, (X, d) be a complete cone metric space and q be a c-distance on X. Suppose that there exist two weakly increasing mappings $f, g\colon X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold*:

$$q(fx, gy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, gy),$$

$$q(gx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, gx) + \alpha_3 q(y, fy)$$

for all comparable $x, y \in X$, where α_i are nonnegative coefficients for $i = 1, 2, 3$ with

$$\alpha_1 + \alpha_2 + \alpha_3 < 1.$$

If $\inf\{\|q(x, y)\| + \|q(x, fx)\|\colon x \in X\} > 0$ *and* $\inf\{\|q(x, y)\| + \|q(x, gx)\|\colon x \in X\} > 0$ *for all $y \in X$ with $y \neq fy$ and $y \neq gy$, respectively, then f and g have a common fixed point. Moreover, if $fz = gz = z$, then $q(z, z) = 0$.*

***Example* 3.1.** Let $E = \mathbf{R}$ and $P = \{x \in E\colon x \geq 0\}$. Let $X = [0, 1]$ and define a mapping $d\colon X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a function $q\colon X \times X \to E$ by $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance (by Example 2.1). Let an order relation $\sqsubseteq$ be defined by $x \sqsubseteq y \iff x \leq y$. Also, let a mapping $f\colon X \to X$ be defined by $f(x) = \dfrac{x^2}{4}$ for all $x \in X$. Define the mappings $\alpha_1(x) = \dfrac{x+1}{4}$, $\alpha_4(x) = \dfrac{x}{8}$ and $\alpha_2 = \alpha_3 = \alpha_5 = 0$ for all $x \in X$. Observe that:

(1) $\alpha_1(fx) = \dfrac{1}{4}\left(\dfrac{x^2}{4} + 1\right) \leq \dfrac{1}{4}\left(x^2 + 1\right) \leq \dfrac{x+1}{4} = \alpha(x)$ for all $x \in X$.

(2) $\alpha_4(fx) = \dfrac{x^2}{32} \leq \dfrac{x^2}{8} \leq \dfrac{x}{8} = \alpha_4(x)$ for all $x \in X$.

(3) $\alpha_i(fx) = 0 \leq 0 = \alpha_i(x)$ for all $x \in X$ and $i = 2, 3, 5$.

(4) $(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(x) = \dfrac{x+1}{4} + \dfrac{2x}{8} = \dfrac{2x+1}{4} < 1$ for all $x \in X$.

(5) For all comparable $x, y \in X$, we get

$$q(fx, fy) = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| \leq \frac{|x+y||x-y|}{4} = \left(\frac{x+y}{4}\right)|x-y| \leq \left(\frac{x+1}{4}\right)|x-y| \leq$$

$$\leq \alpha_1(x) q(x, y) + \alpha_2(x) q(x, fx) + \alpha_3(x) q(y, fy) +$$

$$+ \alpha_4(x) q(x, fy) + \alpha_5(x) q(y, fx).$$

(6) Similarly, we have

$$q(fy, fx) \leq \alpha_1(x) q(y, x) + \alpha_2(x) q(fx, x) + \alpha_3(x) q(fy, y) +$$

$$+ \alpha_4(x) q(fy, x) + \alpha_5(x) q(fx, y)$$

for all comparable $x, y \in X$.

Moreover, f is a nondecreasing and continuous mapping with respect to $\sqsubseteq$. Hence all the conditions of Theorem 3.1 are satisfied. Thus f has a fixed point $x = 0$ and $q(0, 0) = 0$.

***Remark* 3.1.** There exist many examples on fixed point results under c-distance in cone metric spaces (see, for example, [9, 12, 25, 26]). Also, most of the examples in [1, 6, 24] can be easily translated into the c-distance on ordered cone metric spaces with $q(x, y) = d(x, y)$.

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