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ON TOPOLOGICAL FUNDAMENTAL GROUPS OF QUOTIENT SPACES
ПРО ТОПОЛОГIЧНI ФУНДАМЕНТАЛЬНI ГРУПИ ФАКТОР-ПРОСТОРIВ

Let $p\colon X \to X/A$ be a quotient map, where A is a subspace of X. We study the conditions under which $p_*(\pi_1^{\text{qtop}}(X, x_0))$ is dense in $\pi_1^{\text{qtop}}(X/A, *))$, where the fundamental groups have the natural quotient topology inherited from the loop space and p_* is a continuous homomorphism induced by the quotient map p. In addition, we present some applications to determine some properties of $\pi_1^{\text{qtop}}(X/A, *)$. In particular, we establish some conditions under which $\pi_1^{\text{qtop}}(X/A, *)$ is an indiscrete topological group.

Нехай $p\colon X \to X/A$ — фактор-відображення, де A — підпростір X. Досліджуються умови, за яких $p_*(\pi_1^{\text{qtop}}(X, x_0))$ є щільною в $\pi_1^{\text{qtop}}(X/A, *))$, де фундаментальні групи наділені природною фактор-топологією, успадкованою від простору петель, а p_* — неперервний гомоморфізм, індукований фактор-відображенням p. Крім того, наведено деякі застосування з метою визначити деякі властивості $\pi_1^{\text{qtop}}(X/A, *)$. Наприклад, встановлено умови, за яких $\pi_1^{\text{qtop}}(X/A, *)$ є недискретною топологічною групою.

1. Introduction and motivation. Let $p\colon (X, x_0) \to (Y, y_0)$ be a continuous map of pointed topological spaces. By applying the fundamental group functor on p there exists the induced homomorphism

$$p_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

It seems interesting to relate the homology and homotopy groups of X with that of Y using properties of p. Vietoris first studied the problem with his mapping theorem [18]. Also, Smale first discovered an analog of Vietoris's mapping theorem hold for homotopy groups [15]. Recently, Calcut, Gompf, and Mccarthy [6] proved a generalization of Smale's theorem as follows:

Let $p\colon (X, x_0) \to (Y, y_0)$ be a quotient map of topological spaces, where X is locally path connected and Y is semilocally simply connected. If each fiber $p^{-1}(y)$ is connected, then the induced homomorphism $p_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective.

For a pointed topological space (X, x_0) by $\pi_1^{\text{qtop}}(X, x_0)$ we mean the topological fundamental group endowed with the quotient topology inherited from the loop space under the natural map $\Omega(X, x_0) \longrightarrow \pi_1(X, x_0)$ that makes it a *quasitopological group*. A *quasitopological group* G is a group with a topology such that inversion $g \longrightarrow g^{-1}$ and all translations are continuous. For more details, see [2, 3, 5]. It is known that this construction gives rise a homotopy invariant functor $\pi_1^{\text{qtop}}\colon \text{hTop}_* \longrightarrow \text{qTopGrp}$ from the homotopy category of based spaces to the category of quasitopological groups and continuous homomorphisms [3]. Also, $\pi_1^{\tau}(X, x_0)$ is the fundamental group endowed with another topology introduced by Brazas [4]. In fact, the functor π_1^{τ} removes the smallest number of open sets from the topology of $\pi_1^{\text{qtop}}(X, x)$ so that makes it a topological group.

Let X be a topological space and $A_1, A_2, \ldots, A_n$ be a finite collection of its subsets. The quotient space $X/(A_1, \ldots, A_n)$ is obtained from X by identifying each of the sets A_i to a point. Now, let (A, a) be a pointed subspace of (X, a) and $p\colon (X, a) \longrightarrow (X/A, *)$ be the associated quotient map. In this paper, first we prove that if A is an open subset of X such that the closure of A, $\overline{A}$, is path connected, then the image of p_* is dense in $\pi_1^{\text{qtop}}(X/A, *)$. Then by this fact, we show that the image of p_* is dense in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$, where the A_i are open subsets of X with path connected closures and $p\colon X \longrightarrow X/(A_1, A_2, \ldots, A_n)$ is the associated quotient map. Second, we prove that if A is a closed subset of a locally path connected and first countable space X, then the image of p_* is also dense in $\pi_1^{\mathrm{qtop}}(X/A, *)$. By the two previous results we can show that the image of p_* is dense in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$, where X is first countable, connected, locally path connected and the A_i are open or closed subsets of X with disjoint path connected closures. Moreover, we give some conditions in which p_* is an epimorphism. Also, by some examples, we show that p_* is not necessarily onto. Finally, we give some applications of the above results to find out some properties of the topological fundamental group of the quotient space $X/(A_1, A_2, \ldots, A_n)$. In particular, we prove that with the recent assumptions on X and the A_i, $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is an indiscrete topological group when X is simply connected. It should be mentioned that since the topology of $\pi_1^{\tau}(X, x_0)$ is coarser than $\pi_1^{\mathrm{qtop}}(X, x_0)$, the above results can be obtained when we replace π_1^{qtop} with π_1^{τ}.

2. Notations and preliminaries. For a topological space X, by a path in X we mean a continuous map $\alpha\colon [0,1] \longrightarrow X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha\colon [0,1] \longrightarrow X$, α^{-1} denotes a path such that $\alpha^{-1}(t) = \alpha(1-t)$, for all $t \in [0,1]$. Denote $[0,1]$ by I, two paths $\alpha, \beta\colon I \longrightarrow X$ with the same initial and terminal points are called homotopic relative to end points if there exists a continuous map $F\colon I \times I \longrightarrow X$ such that

$$F(t,s) = \begin{cases} \alpha(t), & s = 0, \\ \beta(t), & s = 1, \\ \alpha(0) = \beta(0), & t = 0, \\ \alpha(1) = \beta(1), & t = 1. \end{cases}$$

The homotopy is an equivalent relation and the homotopy class containing a path α is denoted by $[\alpha]$. Since most of the homotopies that appear in this paper have this property and end points are the same, we drop the term “relative homotopy” for simplicity. For paths $\alpha, \beta\colon I \longrightarrow X$ with $\alpha(1) = \beta(0)$, $\alpha * \beta$ denotes the concatenation of α and β that is a path from I to X such that $(\alpha * \beta)(t) = \alpha(2t)$, for all $0 \leq t \leq 1/2$ and $(\alpha * \beta)(t) = \beta(2t-1)$, for all $1/2 \leq t \leq 1$.

For a pointed topological space (X, x), let $\Omega(X, x)$ be the space of based maps from I to X with the compact-open topology. A subbase for this topology consists of neighborhoods of the form $\langle K, U \rangle = \{\gamma \in \Omega(X, x) \mid \gamma(K) \subseteq U\}$, where $K \subseteq I$ is compact and U is open in X. When X is path connected and the basepoint is clear, we just write $\Omega(X)$ and we will consistently denote the constant path at x by e_x. The topological fundamental group of a pointed space (X, x) can be described as the usual fundamental group $\pi_1(X, x)$ with the quotient topology with respect to the canonical map $\Omega(X, x) \longrightarrow \pi_1(X, x)$ identifying homotopy classes of loops, denoted by $\pi_1^{\mathrm{qtop}}(X, x)$. A basic account of topological fundamental groups may be found in [2], [5] and [3]. For undefined notation, see [12].

Definition 2.1 [1]. *A quasitopological group G is a group with a topology such that inversion $G \longrightarrow G$, $g \mapsto g^{-1}$, is continuous and multiplication $G \times G \longrightarrow G$ is continuous in each variable. A morphism of quasitopological groups is a continuous homomorphism.*

Theorem 2.1 [3]. *$\pi_1^{\rm qtop}$ is a functor from the homotopy category of based topological spaces to the category of quasitopological groups.*

A space X is called *semi-locally simply connected* if for each point $x \in X$, there is an open neighborhood U of x such that the inclusion $i\colon U \hookrightarrow X$ induces the trivial homomorphism $i_*\colon \pi_1(U,x) \longrightarrow \pi_1(X,x)$ or equivalently a loop in U can be contracted inside X.

Theorem 2.2 [3]. *Let X be a path connected space. If $\pi_1^{\rm qtop}(X,x)$ is discrete for some $x \in X$, then X is semi-locally simply connected. If X is locally path connected and semi-locally simply connected, then $\pi_1^{\rm qtop}(X,x)$ is discrete for all $x \in X$.*

3. Main results. In this section, (A,a) is a pointed subspace of (X,a), $p\colon (X,a) \longrightarrow (X/A,*)$ is the canonical quotient map so that $q := p|_{X-A}\colon X - A \longrightarrow X/A - \{*\}$ is a homeomorphism. Also, by applying the functor $\pi_1^{\rm qtop}$ on p we have a continuous homomorphism $p_*\colon \pi_1^{\rm qtop}(X,a) \longrightarrow$ $\longrightarrow \pi_1^{\rm qtop}(X/A,*)$.

Lemma 3.1. *If A is an open subset of X, then any loops $\alpha\colon I \longrightarrow \overline{\{*\}} \subseteq X/A$ based at $*$ is nullhomotopic.*

Proof. Define $F\colon I \times I \longrightarrow X/A$ by

$$F(t,s) = \begin{cases} \alpha(t), & s = 0, \\ *, & s > 0. \end{cases}$$

If we prove that F is continuous, then F is a homotopy between α and e_*. For this, let U be an open set in X/A. We show that $F^{-1}(U)$ is open in $I \times I$.

Case 1. If $* \in U$, then

$$F^{-1}(U) = F^{-1}(\{*\}) \cup (\alpha^{-1}(U) \times \{0\}) =$$

$$= (I \times (0,1]) \cup (\alpha^{-1}(U) \times \{0\}) =$$

$$= (I \times (0,1]) \cup (\alpha^{-1}(U) \times I)$$

which is open in $I \times I$.

Case 2. If $* \notin U$, then $U \cap \partial\{*\} = \varnothing$ since if there exists $x \in \partial\{*\}$ such that $x \in U$, then $\{*\} \cap U \neq \varnothing$ which is a contradiction. Since $U \cap \overline{\{*\}} = (U \cap \{*\}) \cup (U \cap \partial\{*\}) = \varnothing$ and $\alpha(I) \subseteq \overline{\{*\}}$, we have $F^{-1}(U) = \varnothing$.

Lemma 3.1 is proved.

Theorem 3.1. *Let A be an open subset of X such that $\overline{A}$ is path connected, then for each $a \in A$ the image of p_* is dense in $\pi_1^{\rm qtop}(X/A,*)$, i.e.,*

$$\overline{p_*(\pi_1^{\rm qtop}(X,a))} = \pi_1^{\rm qtop}(X/A,*).$$

Proof. *Step* 1. First, we show that for every loop $\alpha\colon (I,\partial I) \longrightarrow (X/A,*)$ such that $\alpha^{-1}(\{*\}^c)$ is connected, we have $[\alpha] \in {\rm Im}(p_*)$. By assumption and openness of $\{*\}$ in X/A, there exist $s_1, s_2 \in$ $\in (0,1)$ such that $\alpha^{-1}(\{*\}^c) = [s_1,s_2]$. Since $\alpha^{-1}(\{*\}^c)$ is a compact subset of I, it suffices to let $s_1 = \inf\{\alpha^{-1}(\{*\}^c)\}$ and $s_2 = \sup\{\alpha^{-1}(\{*\}^c)\}$. Let $\overline{\alpha} : [s_1,s_2] \longrightarrow X$ by $\overline{\alpha}(t) = q^{-1}(\alpha(t))$, then $\overline{\alpha}(s_1), \overline{\alpha}(s_2) \in \overline{A}$. Since if G is an open neighborhood of $\overline{\alpha}(s_1)$ and $G \cap A = \varnothing$, then $q(G) = p(G)$ is an open neighborhood of $\alpha(s_1)$. Using continuity of α, there exists an open neighborhood J of s_1 in I such that $\alpha(J) \subseteq G$. On the other hand, by definition of s_1, for all $s < s_1$, $\alpha(s) = *$

which implies that $* \in q(G)$ which is a contradiction since $* \notin \mathrm{Im}(q)$. Similarly $\alpha(s_2) \in \overline{A}$. Since $\overline{A}$ is path connected, there exist two paths $\lambda_1\colon [0, s_1] \longrightarrow \overline{A}$ and $\lambda_2\colon [s_2, 1] \longrightarrow \overline{A}$ such that $\lambda_1(0) = \lambda_2(1) = a$, $\lambda_1(s_1) = \overline{\alpha}(s_1)$ and $\lambda_2(0) = \overline{\alpha}(s_2)$. Define $\widetilde{\alpha}\colon I \longrightarrow X$ by

$$\widetilde{\alpha}(t) = \begin{cases} \lambda_1(t), & 0 \le t \le s_1, \\ \overline{\alpha}(t), & s_1 \le t \le s_2, \\ \lambda_2(t), & s_2 \le t \le 1. \end{cases}$$

By gluing lemma $\widetilde{\alpha}$ is continuous, so it remains to show that $p \circ \widetilde{\alpha} \simeq \alpha \,\mathrm{rel}\{*\}$. Put $\alpha_1 = \alpha|_{[0,s_1]}$, $\alpha_2 = \alpha|_{[s_2,1]}$ and let $\varphi_1\colon [0, 1] \longrightarrow [0, s_1]$ and $\varphi_2\colon 0, 1] \longrightarrow [s_2, 1]$ be linear homeomorphisms such that $\varphi_1(0) = 0$ and $\varphi_2(0) = s_2$, then $p \circ \lambda_i \circ \varphi_i \simeq \alpha_i \circ \varphi_i, \mathrm{rel}\{0, 1\}$ since the $(p \circ \lambda_i \circ \varphi_i) \circ (\alpha_i \circ \varphi_i)^{-1}$ are loops in $\overline{\{*\}}$ which by Lemma 3.1 are nullhomotopic.

Step 2. By continuity of α, $\alpha^{-1}(\{*\}^c)$ is a closed subset of I. Connected subsets of I are intervals or one point sets, also connected components of $\alpha^{-1}(\{*\}^c)$ are closed in $\alpha^{-1}(\{*\}^c)$ and so they are compact in I. Therefore a component of $\alpha^{-1}(\{*\}^c)$ is either closed interval or singleton. Given $[\alpha] \in \pi_1(X/A, *)$, we show that there exists a sequence of homotopy classes of loops $\{[\alpha_n]\}_{n\in N}$ in $\mathrm{Im}\,(p_*)$ such that $[\alpha_n] \longrightarrow [\alpha]$ in $\pi_1^{\mathrm{qtop}}(X/A, *)$.

We claim that the number of non-singleton components of $\alpha^{-1}(\{*\}^c)$ is countable. Let S be the union of singleton components of $\alpha^{-1}(\{*\}^c)$ and for each $n \in \mathbb{N}$, B_n be the set of non-singleton components of $\alpha^{-1}(\{*\}^c)$ with length at least $1/n$. Each B_n is finite since if B_n is infinite, then it has at least $n + 1$ members. Therefore $\bigcup_{C\in B_n} C \subseteq \alpha^{-1}(\{*\}^c) \subseteq I$ which implies that

$$(n + 1) \times 1/n \le \sum_{C\in B_n} \mathrm{diam}\,(C) \le \mathrm{diam}\,(I) = 1$$

which is a contradiction. Thus each B_n is finite which implies that $B = \bigcup_{n\in\mathbb{N}} B_n$ is countable. Rename elements of B by $I_i = [a_i, b_i]$, $i \in J = \{1, 2, \ldots, s\}$, where $s = |B|$ if B is finite and $i \in \mathbb{N} = J$ if B is infinite. For every $n \in J$ define

$$\alpha_n(t) = \begin{cases} \alpha(t), & t \in \bigcup_{i=1}^{n} [a_i, b_i], \\ *, & \text{otherwise.} \end{cases}$$

If B is finite, put $\alpha_n = \alpha_s$, for every $n > s$. We claim that the α_n are continuous. For, if $V \subseteq X/A$ is open, then

(i) If $* \in V$, then

$$\alpha_1^{-1}(V) = [0, a_1) \cup (b_1, 1] \cup \alpha|_{[a_1,b_1]}^{-1}(V)$$

which by continuity of α is open in I.

(ii) If $* \notin V$, then we show that $\alpha_1^{-1}(V) = \alpha|_{[a_1,b_1]}^{-1}(V) = \alpha|_{(a_1,b_1)}^{-1}(V)$ which guaranties $\alpha^{-1}(V)$ is open. For this it suffices to show that $\alpha_1(a_1), \alpha_1(b_1) \notin V$.

For each $n \in \mathbb{N}$, $\alpha(a_n), \alpha(b_n) \in \overline{\{*\}}$ and $\{\alpha(a) |\ a \in S\} \subseteq \overline{\{*\}}$. For, if G is an open neighborhood of $\alpha(a_n)$, then $\alpha^{-1}(G)$ is an open neighborhood of a_n, so there exists $\varepsilon > 0$ such that $(a_n - \varepsilon, a_n + \varepsilon) \subseteq \alpha^{-1}(G)$ or equivalently $\alpha((a_n - \varepsilon, a_n + \varepsilon)) \subseteq G$. If $* \notin G$, then $(a_n - \varepsilon, a_n + \varepsilon) \subseteq \alpha^{-1}(\{*\}^c)$ which is a contradiction since $[a_n, b_n]$ is a connected component of $\alpha^{-1}(\{*\}^c)$. Similarly $\alpha(b_n) \in \overline{\{*\}}$ for each $n \in \mathbb{N}$ and $\alpha(S) \subseteq \overline{\{*\}}$. Thus if $\alpha_1(b_1) = \alpha(b_1) \in V$, then V must meet $\{*\}$ which is a contradiction since $* \notin V$. Therefore α_1 is a continuous loop such that $[\alpha_1] \in \mathrm{Im}(p_*)$. Similarly, all the α_n are continuous. Also, for every $n \in \mathbb{N}$, $[\alpha_n]$ is a product of n homotopy classes of loops which are similar to loops introduced in Step 1. This implies that $[\alpha_n] \in \mathrm{Im}(p_*)$ since $\mathrm{Im}(p_*)$ is a subgroup.

Now we show that the sequence $\{\alpha_n\}$ converges to α. Let $\alpha \in \langle K, U\rangle$, where K is a compact subset of I and U is an open subset of X/A, then

(i) If $* \in U$, then for each $n \in \mathbb{N}$, $\alpha_n \in \langle K, U\rangle$ since for each $t \in K$, $\alpha_n(t) = \alpha(t)$ or $\alpha_n(t) = *$ which in both cases $\alpha(t) \in U$.

(ii) If $K \subseteq \alpha^{-1}(\{*\}^c)$ and $K \cap S \neq \varnothing$, then there exists $a \in K \cap S \subseteq \alpha^{-1}(U)$, so $\alpha(a) \in U$, but $\alpha(a) \in \overline{\{*\}}$ and U is open. Thus $* \in U$ and by (i), for each n, $\alpha_n(K) \subseteq U$.

(iii) If $K \subseteq \bigcup_{n=1}^{\infty} I_n$ and there exist $n_1, n_2, \ldots, n_s$ such that $K \subseteq \bigcup_{i=1}^{s} I_{n_i}$, then by definition of the α_n, for each $n \geq \max\{n_1, \ldots, n_s\}$ we have $\alpha_n(K) \subseteq U$.

(vi) If $K \subseteq \bigcup_{n=1}^{\infty} I_n$ and there exists an infinite subsequence $\{I_{n_r}\}$ such that $K \cap I_{n_r} \neq \varnothing$, then there is a sequence $\{x_{n_r} | x_{n_r} \in K \cap I_{n_r}\}$ such that it has a subsequence $\{x_{n_{r_s}}\}$ converges to an element of K, b say, by compactness of K. Since $b \in K \subseteq \alpha^{-1}(U)$, there exists $\varepsilon > 0$ such that $(b-\varepsilon, b+\varepsilon) \subseteq \alpha^{-1}(U)$. Also, there exists s_0 such that for each $s \geq s_0$, $x_{n_{r_s}} \in (b-\varepsilon/2, b+\varepsilon/2)$ since $x_{n_{r_s}} \longrightarrow b$. Since $\mathrm{diam}(I_n) \longrightarrow 0$, there is n_0 such that for each $n \geq n_0$, $\mathrm{diam}(I_n) < \varepsilon/2$. Choose $s_1 \in N$ such that $s_1 \geq s_0$ and $n_{r_{s_1}} \geq n_0$, then $x_{n_{r_{s_1}}} \in (b-\varepsilon/2, b+\varepsilon/2)$. Also $x_{n_{r_{s_1}}} \in K \cap I_{n_{r_{s_1}}}$ and $\mathrm{diam}(I_{n_{r_{s_1}}}) < \varepsilon/2$, so $a_{n_{r_{s_1}}} \in I_{n_{r_{s_1}}} \subseteq (b-\varepsilon, b+\varepsilon) \subseteq \alpha^{-1}(U)$ which implies that $\alpha(a_{n_{r_{s_1}}}) \in U$ and therefore $* \in U$ since $\alpha(a_{n_{r_{s_1}}}) \in \partial(\{*\})$. Using the last (i) we have $\alpha_n(K) \subseteq U$, for each $n \in \mathbb{N}$.

Theorem 3.1 is proved.

Definition 3.1. *Let X be a topological space and $A_1, A_2, \ldots, A_n$ be any subsets of X, $n \in \mathbb{N}$. By the quotient space $X/(A_1, \ldots, A_n)$ we mean the quotient space obtained from X by identifying each of the sets A_i to a point. Also, we denote the associated quotient map by $p\colon X \longrightarrow$ $\longrightarrow X/(A_1, A_2, \ldots, A_n)$.*

***Corollary* 3.1.** *If A_1, A_2 are open subsets of a path connected space X such that $\overline{A_1}$, $\overline{A_2}$ are path connected. Then for every $a \in A_1 \cup A_2$ the following equality holds*:

$$\overline{p_*(\pi_1^{\mathrm{qtop}}(X, a))} = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *).$$

Proof. We can assume that the A_i are disjoint. If they are not disjoint, the result follows from Theorem 3.1. Let $p_1\colon X \longrightarrow X/A_1$, $p_2\colon X/A_1 \longrightarrow X/(A_1, A_2)$ be associated quotient maps and $a_1 = a \in A_1$. By Theorem 3.1, $\overline{(p_1)_*(\pi_1^{\mathrm{qtop}}(X, a_1))} = \pi_1^{\mathrm{qtop}}(X/A_1, *_1)$, where $*_1 = p_1(a_1)$. Since X is path connected, so is X/A_1. Also, $p_1(A_2)$ is an open subset of X/A_1 and the closure of $p_1(A_2)$ in X/A_1 is path connected. Let $a_2 \in p_1(A_2)$, then $\overline{(p_2)_*(\pi_1^{\mathrm{qtop}}(X/A_1, a_2))} =$ $= \pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *_2)$, where $*_2 = p_2(a_2)$. Since X/A_1 is path connected, there exists a homeomorphism $\varphi_1\colon \pi_1^{\mathrm{qtop}}(X/A_1, *_1) \longrightarrow \pi_1^{\mathrm{qtop}}(X/A_1, a_2)$ by $\varphi_1([\alpha]) = [\gamma * \alpha * \gamma^{-1}]$, where γ is a path from a_2 to $*_1$. We have $\overline{(p_2)_* \circ \varphi_1 \circ (p_1)_*(\pi_1^{\mathrm{qtop}}(X, a))} \supseteq ((p_2)_* \circ \varphi_1)(\overline{(p_1)_*(\pi_1^{\mathrm{qtop}}(X, a))}) =$ $= ((p_2)_* \circ \varphi_1)(\pi_1^{\mathrm{qtop}}(X/A_1, *_1)) = (p_2)_*(\pi_1^{\mathrm{qtop}}(X/A_1, a_2)) = \mathrm{Im}(p_2)_*$ which implies that $\mathrm{Im}(p_2)_* \circ \varphi_1 \circ (p_1)_*$ is dense in $\pi_1(X/(A_1, A_2), *_2)$. If $\gamma' = p_2 \circ \gamma$, then $\varphi_2\colon \pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *_2) \longrightarrow$ $\longrightarrow \pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *_1)$ by $\varphi_2([\alpha]) = [\gamma'^{-1} * \alpha * \gamma']$ is a homeomorphism. Hence $\mathrm{Im}(\varphi_2 \circ (p_2)_* \circ \varphi_1 \circ (p_1)_*)$ is dense in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *_1)$. Moreover $\varphi_2 \circ (p_2)_* \circ \varphi_1 \circ (p_1)_* = p_*$ which implies that $\mathrm{Im}(p_*)$ is dense in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2), *)$, as desired.

Corollary 3.1 is proved.

By induction and Corollary 3.1, we have the following results.

***Corollary* 3.2.** *Let $A_1, A_2, \ldots, A_n$ be open subsets of a path connected space X such that the $\overline{A_i}$ are path connected for each $i = 1, 2, \ldots, n$. Then for any $a \in \bigcup_{i=1}^{n} A_i$ the following equality holds*:

$$\overline{p_*(\pi_1^{\text{qtop}}(X, a))} = \pi_1^{\text{qtop}}(X/(A_1, A_2, \ldots, A_n), *).$$

***Corollary* 3.3.** *Let $A_1, A_2, \ldots, A_n$ be open subsets of a connected, locally path connected space X such that the $\overline{A_i}$ are path connected for every $i = 1, 2, \ldots, n$. If $X/(A_1, A_2, \ldots, A_n)$ is semi-locally simply connected, then for each $a \in \bigcup_{i=1}^{n} A_i$, $p_*\colon \pi_1^{\text{qtop}}(X, a) \longrightarrow \pi_1^{\text{qtop}}(X/(A_1, A_2, \ldots \ldots, A_n), *)$ is an epimorphism.*

Proof. Let U be an open neighborhood of $\bar{x} \in (X/A_1) \setminus \overline{\{*\}}$. Since X is locally path connected, there is a path connected open neighborhood $\widetilde{U} \subseteq p^{-1}(U)$ of $x = q^{-1}(\bar{x})$ such that $\widetilde{U} \cap A_1 = \varnothing$. Then $V := p(\widetilde{U}) = q(\widetilde{U}) \subseteq U$ is a path connected open neighborhood of $\bar{x}$. X/A_1 is locally path connected at $*$ since $\{*\}$ is an open subset of X/A_1. Let U be an open neighborhood of $\bar{x} \in \partial(\{*\})$, then there exists a path connected open neighborhood $\widetilde{U} \subseteq p^{-1}(U)$ of x. Since $p^{-1}(p(\widetilde{U})) = \widetilde{U} \cup A_1$, $p(\widetilde{U})$ is a path connected open neighborhood of $\bar{x}$ in U. Therefore X/A_1 is locally path connected. Similarly $X/(A_1, A_2, \ldots, A_n)$ is connected, locally path connected. Since $X/(A_1, A_2, \ldots, A_n)$ is a connected, semi-locally simply connected and locally path connected space, by Theorem 2.2, $\pi_1^{\text{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is a discrete topological group which implies that $\operatorname{Im}(p_*) = \pi_1^{\text{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ by Corollary 3.2.

Corollary 3.3 is proved.

In the following example we show that with the assumptions of Theorem 3.1, p_* is not necessarily onto.

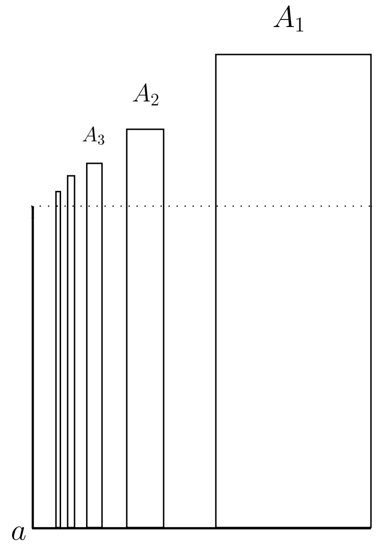
***Example* 3.1.** Let $A_n = \{1/(2n-1), 1/2n\} \times [0, 1 + 1/2n] \bigcup [1/2n, 1/2n-1] \times \{1 + 1/2n\}$ for each $n \in \mathbb{N}$. Consider $X = \left(\bigcup_{n \in \mathbb{N}} A_n\right) \bigcup \{0\} \times [0,1] \bigcup [0,1] \times \{0\}$ with $a = (0,0)$ as the base point and $A = \{(x,y) \in X \mid y < 1\}$ (see Figure). A is an open subset of X with path connected closure. Assume $I_n = (1/2 + 1/2(n+1), 1/2 + 1/2n]$ and f_n be a homeomorphism from I_n to $A_n - \{(1/2n, 0)\}$ for every $n \in \mathbb{N}$. Define $f\colon I \longrightarrow X$ by

$$f(t) = \begin{cases} \text{the point } (0, 2t), & t \in [0, 1/2], \\ f_n(t), & t \in I_n. \end{cases}$$

We claim that $\alpha = p \circ f$ is a loop in X/A at $*$. It suffices to show that α is continuous on $t = 1/2$ and boundary points of I_n since f is continuous on $[0, 1/2)$ and by gluing lemma on $\bigcup \operatorname{int}(I_n)$. Since α is locally constant at $t = 1/2 + 1/2n$ for each $n \in \mathbb{N}$, α is continuous at boundary points of I_n. For each open neighborhood G of $f(1/2) = (0,1)$ in X, there exists $n_0 \in \mathbb{N}$ such that G contains $A_n \bigcap A^c$ for $n > n_0$. Therefore continuity at $t = 1/2$ follows from $\alpha(1/2) \in \overline{\{*\}}$. Now let $B \subseteq \mathbb{N}$ and define

$$g_B(t) = \begin{cases} (p \circ f)(t), & t \in \bigcup_{m \in B} I_m, \\ *, & \text{otherwise.} \end{cases}$$

Then g_B is continuous and for $B_1, B_2 \subseteq \mathbb{N}$ such that $B_1 \neq B_2$, $[g_{B_1}] \neq [g_{B_2}]$ which implies that $\pi_1(X/A, *)$ is uncountable. But by compactness of I, a given path in X can traverse finitely many of the A_n and therefore $\pi_1(X, a)$ is a free group on countably many generators which implies that p does not induce a surjection of fundamental groups.



Let (X, x) be a pointed topological space such that $\{x\}$ is closed. If $\alpha\colon [0,1] \longrightarrow X$ is a loop in X based at x, then $\alpha^{-1}(\{x\})$ is a closed subset of $[0,1]$. Its complement, $\alpha^{-1}(\{x\}^c)$ is therefore the union of a countable collection of disjoint open intervals. We denote this collection of intervals by W_α.

Definition 3.2. *Let (X, x) be a pointed topological space. A loop α in X based at x is called semi-simple if $W_\alpha = \{(0,1)\}$ and is called geometrically simple if W_α has one element. If W_α is finite, then the loop α is called geometrically finite* [11].

Lemma 3.2. *Every geometrically simple loop is homotopic to a semi-simple loop.*

Proof. Let α be a geometrically simple loop at $x \in X$. Then there are $r, s \in [0,1]$ such that $\alpha^{-1}(\{x\}^c) = (r, s)$ and $\alpha(r) = \alpha(s) = x$. Let $\beta := \alpha|_{[r,s]}$ and $\varphi\colon [0,1] \longrightarrow [r,s]$ be a linear homeomorphism, then $\beta \circ \varphi$ is a semi-simple loop at x and $\alpha \simeq \beta \circ \varphi$.

Lemma 3.2 is proved.

In the sequel, for a semi-simple loop $\alpha\colon I \longrightarrow X/A$ denote $\widetilde{\alpha} = q^{-1} \circ \alpha|_{(0,1)}\colon (0,1) \longrightarrow (X - A)$, where A is a closed subset of a topological space X.

Lemma 3.3. *Let $A \subseteq X$ be a closed subset of X and α be a semi-simple loop at $*$ in X/A. If $\lim_{t\to 0} \widetilde{\alpha}(t)$ and $\lim_{t\to 1} \widetilde{\alpha}(t)$ do not exist, then for each $t_0 \in (0,1)$, there are $b_0, b_1 \in A$ such that b_0 is a limit point of $\widetilde{\alpha}((0, t_0))$ and b_1 is a limit point of $\widetilde{\alpha}((t_0, 1))$.*

Proof. Let $t_0 \in (0,1)$ and by contrary suppose that each $b \in A$ has an open neighborhood G_b such that $G_b \cap \widetilde{\alpha}((0, t_0)) = \varnothing$. Then $G = \bigcup_{b \in A} G_b$ is an open neighborhood of A and so $p(G)$ is an open neighborhood of $*$ (since $p^{-1}(p(G)) = G$) such that does not intersect $\alpha((0, t_0))$ which is a contradiction to continuity of α.

Lemma 3.3 is proved.

Theorem 3.2. *If A is a closed path connected subset of a locally path connected space X such that every point of A has a countable local base in X, then for each $a \in A$ we have*

$$\overline{p_*(\pi_1^{\text{qtop}}(X, a))} = \pi_1^{\text{qtop}}(X/A, *).$$

Proof. *Step* 1. Let $[\alpha] \in \pi_1(X/A, *)$, where α is a semi-simple loop in X/A at $*$.

Case 1. Assume $a_0 = \lim_{t\to 0}\widetilde{\alpha}(t)$ and $a_1 = \lim_{t\to 1}\widetilde{\alpha}(t)$ exist, so a_0,$a_1 \in A$ and we can define a path $\overline{\alpha}\colon I \longrightarrow X$ such that $\overline{\alpha}|_{(0,1)} = \widetilde{\alpha}$, $\overline{\alpha}(0) = a_0$, $\overline{\alpha}(1) = a_1$. Since A is path connected, there exist paths $\lambda_0, \lambda_1\colon I \longrightarrow A$ such that λ_0 is a path from a to a_0 and λ_1 is a path from a_1 to a. Therefore $\lambda_0 * \overline{\alpha} * \lambda_1$ is a loop at a such that $p_*([\lambda_0 * \overline{\alpha} * \lambda_1]) = [\alpha]$.

Case 2. If at least one of the above limits does not exist, then we make a sequence $\{[\alpha_n]\}_{n\in\mathbb{N}}$ in $\mathrm{Im}(p_*)$ so that converges to $[\alpha]$. Without lost of generality, we can assume that $a_0 = \lim_{t\to 0}\widetilde{\alpha}(t)$ exists and $a_1 \in A$ is a limit point of $\widetilde{\alpha}((1/2,1))$ by Lemma 3.3. We can define a continuous map $\overline{\alpha} : [0,1) \longrightarrow X$ such that $\overline{\alpha}|_{(0,1)} = \widetilde{\alpha}$, $\overline{\alpha}(0) = a_0$. By hypothesis, there is a countable local base $\{O_i\}_{i\in\mathbb{N}}$ at a_1. Let $\{G_i\}_{i\in\mathbb{N}}$ be a sequence of open neighborhoods of a_1 such that $G_i = O_1\cap\ldots\cap O_i$. Since X is locally path connected and the G_i are open neighborhoods of a_1, there exist path connected open neighborhoods $G'_i \subseteq G_i$ of a_1. Since the point a_1 is a limit point, there are $t_i \in (1/2,1)$ such that $\widetilde{\alpha}(t_i) \in G'_i$, $t_i < t_{i+1}$, $t_n \longrightarrow 1$ and there are paths $\gamma_i\colon [t_i,1] \longrightarrow G'_i$ from $\overline{\alpha}(t_i)$ to a_1, for all $i \in \mathbb{N}$. Since A is path connected, there exist paths $\lambda_0, \lambda_1\colon I \longrightarrow A$ such that λ_0 is a path from a to a_0 and λ_1 is a path from a_1 to a. Let $\overline{\alpha}_n := \lambda_0 * \overline{\alpha}|_{[0,t_n]} \circ \xi_n * \gamma_n \circ \zeta_n * \lambda_1$, where $\xi_n : [0,1] \longrightarrow [0,t_n]$ and $\zeta_n\colon [0,1] \longrightarrow [t_n,1]$ are increasing linear homeomorphisms. Note that every $\overline{\alpha}_n$ is a loop in X at a and if $\beta_n := p\circ(\gamma_n\circ\zeta_n)$, then

$$\alpha'_n := p\circ\overline{\alpha}_n = e_* * \alpha|_{[0,t_n]}\circ\xi_n * \beta_n * e_*$$

is a loop in X at $*$ and $p_*([\overline{\alpha}_n]) = [\alpha'_n]$. Define $\alpha_n\colon I \longrightarrow X/A$ by

$$\alpha_n(t) = \begin{cases} \alpha(t), & 0 \le t \le t_n, \\ p\circ\gamma_n(t), & t_n \le t \le 1, \end{cases}$$

which is a loop at $*$ and $[\alpha'_n] = [\alpha_n]$. Thus it suffices to prove that $\alpha_n \longrightarrow \alpha$. If $\alpha \in \langle K, U\rangle$, where K is a compact subset of $[0,1]$ and U is an open subset of X/A, then

(i) If $* \notin U$, then $K \cap \alpha^{-1}(*) = \varnothing$. Let $m \in \mathbb{N}$ such that $t_m \ge \max K$. Since for each $t \in K$, $t \le t_m$, we have $\alpha_n(t) = \alpha(t) \in U$, for all $n > m$ which implies that $\alpha_n(K) = \alpha(K) \subseteq U$, for each $n > m$.

(ii) If $* \in U$, then $p^{-1}(U)$ is an open neighborhood of A, thus there exists $m \in \mathbb{N}$ such that $G'_n \subseteq p^{-1}(U)$, for each $n \ge m$. Therefore $\mathrm{Im}(\gamma_n) \subseteq G'_n$ which implies that $\mathrm{Im}(p\circ\lambda_n) \subseteq U$. Thus for all $t \in K$ and $n \ge m$ we have

$$\alpha_n(t) = \begin{cases} \alpha(t) \in U, & t \in [0,t_n], \\ (p\circ\gamma_n)(t) \in U, & t \in [t_n,1]. \end{cases}$$

Therefore for a semi-simple loop α in X/A we have $[\alpha] \in \overline{\mathrm{Im}(p_*)}$. Similarly, for every loop α such that $\alpha^{-1}(\{*\})$ is finite, $[\alpha] \in \overline{\mathrm{Im}(p_*)}$ which implies that the homotopy class of every geometrically finite loop belongs to $\overline{\mathrm{Im}(p_*)}$ by Lemma 3.2.

Step 2. If α is not geometrically finite, W_α is countable since every open subset of I is a countable union of open intervals. Let $I_j = \overline{L_j}$ where $W_\alpha = \{L_j | j \in \mathbb{N}\}$ and let

$$\alpha_j(t) = \begin{cases} \alpha(t), & t \in I_1 \cup \ldots \cup I_j, \\ *, & \text{otherwise}, \end{cases}$$

then $[\alpha_j] \in \overline{\mathrm{Im}(p_*)}$ since the α_j are geometrically finite.

Since $\overline{(\overline{\mathrm{Im}(p_*)})} = \overline{\mathrm{Im}(p_*)}$, it suffices to show that $\alpha_j \longrightarrow \alpha$. For, if $\alpha \in \langle K, U \rangle$ for a compact subset K of $[0,1]$ and an open subset U of X/A, then

(i) If $* \in U$, then for each $t \in K$ and $j \in \mathbb{N}$, $\alpha_j(t)$ takes value $\alpha(t)$ or $*$ which in both cases belongs to U, so $\alpha_j(K) \subseteq U$, for all $j \in \mathbb{N}$.

(ii) If $* \notin U$, then $K \cap \alpha^{-1}(\{*\}) = \varnothing$, so $K \subseteq \cup_j L_j$. By compactness of K we have $K \subseteq \cup_s L_{j_s}$, for $s = 1, 2, \ldots, n_K$. Let $M = \max\{j_s | s = 1, 2, \ldots, n_K\}$, then $\alpha_j(K) = \alpha(K) \subseteq U$, for each $j \geq M$.

Theorem 3.2 is proved.

***Corollary* 3.4.** *Let $A_1, A_2, \ldots, A_n$ be disjoint path connected, closed subsets of a first countable, connected, locally path connected space X. Then for every $a \in \bigcup_{i=1}^n A_i$ the following equality holds:*

$$\overline{p_*(\pi_1^{\mathrm{qtop}}(X, a))} = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *).$$

Proof. Let $p_i \colon X/(A_1, A_2, \ldots, A_{i-1}) \longrightarrow X/(A_1, A_2, \ldots, A_i)$. Since every point of each $p_{i-1}(A_i)$ has a countable local base in the connected, locally path connected space $X/(A_1, A_2, \ldots$ $\ldots, A_{i-1})$, by Theorem 3.2 the result holds.

***Corollary* 3.5.** *Let $A_1, A_2, \ldots, A_n$ be disjoint path connected, closed subsets of a first countable, connected, locally path connected space X such that $X/(A_1, A_2, \ldots, A_n)$ is semi-locally simply connected. Then for each $a \in \bigcup_{i=1}^n A_i$, $p_* \colon \pi_1(X, a) \longrightarrow \pi_1(X/(A_1, A_2, \ldots, A_n, *)$ is an epimorphism.*

Proof. Since $X/(A_1, A_2, \ldots, A_n)$ is connected, locally path connected and semi-locally simply connected space, $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is a discrete topological group which implies that $\mathrm{Im}(p_*) = \pi_1(X/(A_1, A_2, \ldots, A_n), *)$ by Corollary 3.4.

In the following example, we show that the condition "path connectedness for A"is necessary in Theorem 3.2.

***Example* 3.2.** Let $A = \{(1,0), (0,1)\} \subset X = S^1$. Clearly X/A is homeomorphic to the Figure 8 space, $S^1 \bigvee S^1$. Since X and X/A are locally path connected and semi-locally simply connected

$$p_* \colon \pi_1^{\mathrm{qtop}}(X, 0) \cong \mathbb{Z} \longrightarrow \pi_1^{\mathrm{qtop}}(X/A, *) \cong \mathbb{Z} * \mathbb{Z}$$

is a continuous homomorphism of discrete topological spaces. Since the free product $\mathbb{Z} * \mathbb{Z}$ is not abelian, p_* is not onto and since $\pi_1^{\mathrm{qtop}}(X/A, *)$ is discrete, $\mathrm{Im}(p_*)$ is not dense in $\pi_1^{\mathrm{qtop}}(X/A, *)$.

In the following example, we show that the condition "locally path connectedness for X"is necessary in Theorem 3.2.

***Example* 3.3.** Let $X_1 = \{(x, \sin(2\pi/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$, $X_2 = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + \dfrac{y^2}{4} = \right.$ $\left. = 1,\ y \leq 0\right\}$, $X_3 = \{(x, 0) \in \mathbb{R}^2 \mid -1 \leq x \leq 0\}$ and $A = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$. If $X = X_1 \cup X_2 \cup X_3 \cup A$, then $\pi_1(X, x_0) = 0$ and $\pi_1(X/A, *) \cong Z$. Since X/A is a locally path connected and semi-locally simply connected space, $\pi_1^{\mathrm{qtop}}(X/A, *)$ is discrete which implies that $\overline{p_*(\pi_1^{\mathrm{qtop}}(X, x_0))} \neq \pi_1^{\mathrm{qtop}}(X/A, *)$.

In the next example, we show that with the assumptions of Theorem 3.2, p_* is not necessarily an epimorphism and hence the hypothesis semi-locally simply connectedness in Corollary 3.5 is essential.

***Example* 3.4.** Let $C_n = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \left(x - \dfrac{1}{n}\right)^2 + y^2 = \dfrac{1}{n^2}\right\}$, for $n \in \mathbb{N}$, $HE_o = \bigcup_{n\in\mathbb{N}} C_{2n-1}$, $HE_e = \bigcup_{n\in\mathbb{N}} C_{2n}$ and $X = \big(HE_0 \times \{0\}\big) \cup \big(HE_e \times \{1\}\big) \cup A$, where $A = \big(\{(0,0)\} \times I\big)$. One can easily see that X/A is the Hawaiian Earring space. Let α be the loop in X/A that traverse $p(C_1),\ p(C_2),\ldots$ in ascending order. By the structure of the fundamental group of the Hawaiian Earring [7] we have $[\alpha] \notin \mathrm{Im}(p_*)$ since if $p_*([\beta]) = [\alpha]$, then the loop β must traverse infinitely many times A which is a contradiction to the continuity of β.

***Corollary* 3.6.** *Let $A_1, A_2, \ldots, A_n$ be subsets of a first countable, connected, locally path connected space X with disjoint path connected closure such that each A_i is closed or open. Then for any $a \in \bigcup_{i=1}^n A_i$ we have*

$$\overline{p_*(\pi_1^{\mathrm{qtop}}(X,a))} = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *).$$

Proof. By changing the order, we can assume that $A_1, \ldots, A_k$ are closed and $A_{k+1}, \ldots, A_n$ are open, for a $1 \le k \le n$. By applying Corollary 3.4 we have

$$\overline{q_*(\pi_1^{\mathrm{qtop}}(X,a))} = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_k), *),$$

where $q\colon X \to X/(A_1, A_2, \ldots, A_k)$ is the natural quotient map. Consider the natural quotient map $r\colon X/(A_1, A_2, \ldots, A_k) \to X/(A_1, \ldots, A_k, \ldots, A_n)$. Note that $p = r \circ q$ and since the A_j have disjoint path connected closures, $\overline{A_j}$ is also path connected in $X/(A_1, \ldots, A_k)$, for all $j > k$. Now, using Corollary 3.2 the result holds.

***Remark* 3.1.** Note that since the topology of $\pi_1^{\tau}(X,x)$ is coarser than $\pi_1^{\mathrm{qtop}}(X,x)$, the results of this section can be restated for π_1^{τ} when we replace π_1^{qtop} with π_1^{τ}.

4. Some applications. It seems interesting to investigate on the topology of quasitopological fundamental groups and some people have found some properties of this topology (see [2–5, 10, 13, 14, 17]). In this section, we intend to give some applications of the results of the previous section to find out some properties of the topological fundamental group of the quotient space $X/(A_1, A_2, \ldots, A_n)$. By $(X, A_1, A_2, \ldots, A_n)$ we mean an $(n+1)$-*tuple* of spaces with one of the following conditions (♣):

(i) The A_i are open subsets of X with path connected closures.

(ii) X is a connected, locally path connected, first countable space and the A_i are closed subsets of X with disjoint path connected closures.

Theorem 4.1. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, \ldots, A_n)$ with the assumption (♣), if X is simply connected, then $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is an indiscrete topological group.*

Proof. Since X is simply connected, $p_*(\pi_1^{\mathrm{qtop}}(X,a))=\{[e_*]\}$, where e_* is the constant loop at $*$ in $X/(A_1, A_2, \ldots, A_n)$. Then by Corollaries 3.2 and 3.6 $\{[e_*]\}$is a dense subset of $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$. Since $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is a quasitopological group, for every $[\alpha] \in \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$, the left multiplication $L_{[\alpha]}\colon \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *) \longrightarrow \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ given by $L_{[\alpha]}([\beta]) = [\alpha * \beta]$ is a homeomorphism which implies that $\{[\alpha]\}$ is also dense in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$. Hence every nonempty open subset of $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ contains every element $[\alpha]$ of $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ which implies that $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is an indiscrete topological group.

Theorem 4.1 is proved.

Theorem 4.2. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, \ldots, A_n)$ with the assumption ($\clubsuit$), if $\pi_1^{\mathrm{qtop}}(X,a)$ is compact and $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is Hausdorff, then the quasitopological fundamental group $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is either a discrete topological group or uncountable.*

Proof. If $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ has at least one isolated point, then every singleton is open since left translations

$$L_{[\alpha]}\colon \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *) \longrightarrow \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$$

are homeomorphisms, for every $[\alpha] \in \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$. Thus $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is a discrete topological group. It is a well-known result that a nonempty compact Hausdorff space without isolated points is uncountable [12] (Theorem 27.7). Hence if $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ has no isolated points, then in order to show that $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is uncountable it is enough to show that $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is compact. By Corollaries 3.2 and 3.6,

$$\overline{p_*(\pi_1^{\mathrm{qtop}}(X,a))} = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *).$$

Since $\pi_1^{\mathrm{qtop}}(X,a)$ is compact and p_* is continuous $p_*\pi_1^{\mathrm{qtop}}(X,a)$ is compact in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$. Since $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is Hausdorff, $p_*(\pi_1^{\mathrm{qtop}}(X,a))$ is closed in $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ and so $p_*(\pi_1^{\mathrm{qtop}}(X,a)) = \pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$. Hence $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is compact and so it is uncountable.

Theorem 4.2 is proved.

***Corollary* 4.1.** *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, \ldots, A_n)$ with the assumption ($\clubsuit$), if $\pi_1^{\mathrm{qtop}}(X,a)$ is a compact, countable quasitopological group, then either $X/(A_1, A_2, \ldots, A_n)$ is semi-locally simply connected or $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is not Hausdorff.*

Proof. Let $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ be Hausdorff, then by a similar proof of Theorem 4.2 p_* is onto. Therefore $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is countable since $\pi_1^{\mathrm{qtop}}(X,a)$ is countable. Theorem 4.2 implies that $\pi_1^{\mathrm{qtop}}(X/(A_1, A_2, \ldots, A_n), *)$ is a discrete topological groups. Hence by Theorem 2.2 $X/(A_1, A_2, \ldots, A_n)$ is semi-locally simply connected.

If $\mathcal{U}$ is an open cover of a connected and locally path connected space X, then the subgroup of $\pi_1(X,x)$ consisting of all homotopy classes of loops that can be represented by a product of the following type:

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

where the u_j are arbitrary paths (starting at the base point x) and each v_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$, is called the Spanier group with respect to $\mathcal{U}$, denoted by $\pi(\mathcal{U}, x)$ [8, 16].

Definition 4.1 [8, 16]**.** *The Spanier group of the space X which we denote it by $\pi_1^{sp}(X,x)$, is defined as follows*:

$$\pi_1^{sp}(X,x) = \bigcap_{\text{open covers } \mathcal{U}} \pi(\mathcal{U}, x).$$

The authors [10] introduce Spanier spaces which are spaces such that their Spanier groups are equal to their fundamental groups. Also, the authors prove that for a connected and locally path connected space X, $\overline{\{[e_x]\}} \subseteq \pi_1^{sp}(X,x)$. Hence, for an $(n+1)$-*tuple* of spaces $(X, A_1, A_2, \ldots, A_n)$ with the assumption ($\clubsuit$), where X is simply connected, we have

$$\pi_1^{\mathrm{qtop}}(X/(A_1,\ldots,A_n),*) = \overline{p_*(\pi_1(X,x))} = \overline{\{[e_x]\}} \subseteq \pi_1^{sp}(X/(A_1,\ldots,A_n),*).$$

Clearly simply connected spaces are Spanier spaces which we can call them trivial Spanier spaces. It is interesting for the authors to obtain some ways to construct nontrivial Spanier spaces. The following result which is an immediate consequence of the above argument gives a way to construct some Spanier spaces from simply connected spaces.

Theorem 4.3. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, \ldots, A_n)$ with the assumption (♣), if X is simply connected, then $X/(A_1, A_2, \ldots, A_n)$ is a Spanier space.*

In the following example, we show that there exists a simply connected, locally path connected metric space X with a closed path connected subspace A such that X/A is not simply connected and by Theorem 4.1 $\pi_1^{\mathrm{qtop}}(X/A,*)$ is an indiscrete topological group. Hence X/A is a nontrivial Spanier space.

***Example* 4.1.** Using the definitions of Example 3.4, let CHE_o and CHE_e be cones over HE_o and HE_e with height $\frac{1}{2}$ and let $X = CHE_o \cup CHE_e \cup A$. By the van Kampen theorem, X is simply connected, but X/A is not simply connected (see [9]). Hence X/A is a nontrivial Spanier space.

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