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$S\Phi$-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS*

$S\Phi$-ДОПОВНЮВАНІ ПІДГРУПИ СКІНЧЕННИХ ГРУП

We call H an $S\Phi$-supplemented subgroup of a finite group G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H. In this paper, we characterize the p-nilpotency and supersolubility of a finite group G under the assumption that every subgroup of a Sylow p-subgroup of G with given order is $S\Phi$-supplemented in G. Some results about formations are also obtained.

Підгрупу H називають $S\Phi$-доповнюваною підгрупою скінченної групи G, якщо існує така субнормальна підгрупа T групи G, що $G = HT$ і $H \cap T \leq \Phi(H)$, де $\Phi(H)$ є підгрупою Фраттіні підгрупи H. У цій статті охарактеризовано p-нільпотентність та надрозв'язність скінченної групи G за припущення, що кожна підгрупа силовської p-підгрупи групи G заданого порядку є $S\Phi$-доповнюваною в G. Отримано також деякі результати щодо формацій.

1. Introduction. All groups considered in this paper are finite. $\mathcal{F}$ denotes a formation, a normal subgroup N of a group G is said to be $\mathcal{F}$-hypercentral in G provided N has a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_r = N$ such that each N_{i+1}/N_i is an $\mathcal{F}$-central chief factor of G, the product of all $\mathcal{F}$-hypercentral subgroups of G is again an $\mathcal{F}$-hypercentral subgroup of G. It is denoted by $Z_{\mathcal{F}}(G)$ and called the $\mathcal{F}$-hypercenter of G. $\mathcal{U}$ and $\mathcal{N}$ denote the classes of all supersoluble groups and nilpotent groups respectively. The other terminology and notations are standard, as in [7] and [13].

We know that for every normal subgroup N of G, the minimal supplement H of N in G satisfies $H \cap N \leq \Phi(H)$. Then naturally, we consider the converse case, i.e., if for some subgroup H of G, there exists a subnormal subgroup N of G such that $HN = G$ and $H \cap N \leq \Phi(H)$, what can we say about G? To study this question, we introduce the concept of $S\Phi$-supplemented subgroups of a finite group.

Definition 1.1. *A subgroup H of a group G is said to be $S\Phi$-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H.*

From the Definition 1.1, we can easily deduce that the minimal supplement of any minimal normal subgroup and every Sylow subgroup of a nilpotent group G are $S\Phi$-supplemented in G. We can also deduce that every non-trivial subgroup of G contained in $\Phi(G)$ can not be $S\Phi$-supplemented in G. Meanwhile, a group with a non-trivial $S\Phi$-supplemented subgroup cannot be a non-abelian simple group.

Inspired by [1] and [11], for each prime p dividing the order of G, let P be a Sylow p-subgroup of G and D a subgroup of P such that $1 < |D| < |P|$, we study the structure of G under the assumption that each subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G. We get some characterizations about formation.

2. Preliminaries. In this section, we list some basic results which will be used below.

Lemma 2.1. *Let H be an $S\Phi$-supplemented subgroup and N a normal subgroup of G.*

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(1) *If* $H \leq K \leq G$, *then* H *is* $S\Phi$*-supplemented in* K.

(2) *If* $N \leq H$, *then* H/N *is* $S\Phi$*-supplemented in* G/N.

(3) *Let* π *be a set of primes,* H *a* π*-subgroup and* N *a* π'*-subgroup. Then* HN/N *is* $S\Phi$*-supplemented in* G/N.

Proof. By the hypothesis, there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$. Then

(1) $K = K \cap G = H(K \cap T)$ and $H \cap (K \cap T) = (H \cap T) \cap K \leq \Phi(H) \cap K \leq \Phi(H)$. Obviously, $K \cap T$ is a subnormal subgroup of K. Hence H is $S\Phi$-supplemented in K.

(2) $G/N = (H/N)(TN/N)$ and

$$H/N \cap TN/N = (H \cap TN)/N = (H \cap T)N/N \leq \Phi(H)N/N \leq \Phi(H/N),$$

TN/N is subnormal in G/N. Hence H/N is $S\Phi$-supplemented in G/N.

(3) Since T contains a Hall π'-subgroup of G and T is subnormal in G, it is easy to see that $N \leq T$ and $G/N = (HN/N)(T/N)$. Since $HN/N \cap T/N = (H \cap T)N/N \leq \Phi(H)N/N \leq$ $\leq \Phi(HN/N)$ and T/N is subnormal in G/N, HN/N is $S\Phi$-supplemented in G/N.

Lemma 2.1 is proved.

Lemma 2.2. *Let* P *be a Sylow* p*-subgroup of a group* G, *where* p *is a prime dividing* $|G|$. *If every subgroup of* P *with order* p *is* $S\Phi$*-supplemented in* G, *then* G *is* p*-nilpotent.*

Proof. We use induction on $|G|$. Let H be a subgroup of P of order p. By the hypothesis, there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H) = 1$. Then T is a maximal subgroup of G and so $T \trianglelefteq G$. Hence if $p \nmid |T|$, then G is p-nilpotent, the result holds. Thus we may suppose that $p||T|$ and $P = H(P \cap T)$. Clearly $P \cap T$ is a Sylow p-subgroup of T. Then every subgroup of $P \cap T$ with order p is $S\Phi$-supplemented in G and so in T by Lemma 2.1. Hence T is p-nilpotent by induction. Since T is p-nilpotent and $T \trianglelefteq G$, we have G is p-nilpotent, as required.

Lemma 2.2 is proved.

From [2] (Theorem A) or [3] (Theorem A or B), we can easily deduce that:

Lemma 2.3. *Let* P *be a normal* p*-subgroup of a group* G, *where* p *is a prime dividing* $|G|$. *If every subgroup of* P *with order* p *is* $S\Phi$*-supplemented in* G, *then* $P \leq Z_{\mathcal{U}}(G)$.

Lemma 2.4. *Let* P *be a normal* p*-subgroup of a group* G, *where* p *is a prime dividing* $|G|$. *If every maximal subgroup of* P *is* $S\Phi$*-supplemented in* G, *then* $P \leq Z_{\mathcal{U}}(G)$.

Proof. Assume that the result is false and let (G, P) be a counterexample for which $|G||P|$ is minimal. We treat with the following two cases:

Case 1. $\Phi(P) \neq 1$.

By Lemma 2.1, every maximal subgroup of $P/\Phi(P)$ is $S\Phi$-supplemented in $G/\Phi(P)$. Then $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ by the choice of G. Hence $P \leq Z_{\mathcal{U}}(G)$ by [12] (I, Theorem 7.19), a contradiction.

Case 2. $\Phi(P) = 1$.

At this time, P is an elementary abelian group. If $|P| = p$, then $P \leq Z_{\mathcal{U}}(G)$, a contradiction. Now we may assume that $|P| = p^n$, $n \geq 2$. Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq \Phi(P_1) = 1$. Clearly, $P = P_1(P \cap K)$ and $P \cap K$ is a normal subgroup of G of order p. By Lemma 2.1, every maximal subgroup of $P/(P \cap K)$ is $S\Phi$-supplemented in $G/(P \cap K)$. Then $P/(P \cap K) \leq Z_{\mathcal{U}}(G/(P \cap K))$ by induction. Since $|P \cap K| = p$, we have $P \leq Z_{\mathcal{U}}(G)$, as required.

Lemma 2.4 is proved.

Now we can prove the following lemma.

Lemma 2.5. *Let P be a normal p-subgroup of a group G, D a subgroup of P such that $1 < |D| < |P|$. Suppose that every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Proof. Assume that the result is false and let (G, P) be a counterexample for which $|G||P|$ is minimal. Then:

(1) $|P : D| > p$.

By Lemma 2.4, it is true.

(2) $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. If $|\Phi(P)| < |D|$, then every subgroup $\overline{H}$ of $P/\Phi(P)$ with $|\overline{H}| = \dfrac{|D|}{|\Phi(P)|}$ is $S\Phi$-supplemented in $\overline{G} = G/\Phi(P)$ by Lemma 2.1. Then $P/\Phi(P) \leq Z_{\mathcal{U}}(G/\Phi(P))$ by induction. Hence $P \leq Z_{\mathcal{U}}(G)$ by [12] (I, Theorem 7.19), a contradiction. Thus $|\Phi(P)| \geq |D|$. Let H be a subgroup of $\Phi(P)$ with $|H| = |D|$. By the hypothesis, H is $S\Phi$-supplemented in G and so in P, a contradiction.

(3) The final contradiction.

Let H be a subgroup of P with $|H| = |D|$. By the hypothesis, H is $S\Phi$-supplemented in G, then there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq \Phi(H) = 1$. Since $|G : K|$ is a power of p, there exists a normal subgroup M of G containing K such that $|G : M| = p$. Let $P_1 = P \cap M$, then P_1 is a maximal subgroup of P and it is normal in G. By (1), $|P_1| > |D|$. Then every subgroup of P_1 with order $|D|$ is $S\Phi$-supplemented in G. So $P_1 \leq Z_{\mathcal{U}}(G)$ by induction. Since $|P/P_1| = p$, we have $P \leq Z_{\mathcal{U}}(G)$, the final contradiction.

Lemma 2.5 is proved.

Lemma 2.6 ([9], Lemma 2.8)**.** *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$. Then*:

(1) *If N is normal in G and of order p, then N lies in $Z(G)$.*

(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*

(3) *If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

Lemma 2.7 ([5], X. 13)**.** *Let G be a group, then*:

(1) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.*

(2) *If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(3) $C_G(F^*(G)) \leq F(G)$.

Lemma 2.8 ([8], Lemma 2.6)**.** *Let N be a nontrivial soluble normal subgroup of a group G. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

3. Main results.

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime dividing $|G|$ such that $(|G|, p-1) = 1$. If every maximal subgroup of P is $S\Phi$-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Then we have:

(1) G has the unique minimal normal subgroup N such that G/N is p-nilpotent and $\Phi(G) = 1$.

Clearly G is not a non-abelian simple group. Let N be a minimal normal subgroup of G, consider G/N. Let M/N be a maximal subgroup of PN/N, by Lemma 2.6 we may suppose that $|PN/N| \geq \geq p^2$. Clearly $M = P_1N$ for some maximal subgroup P_1 of P and $P \cap N = P_1 \cap N$ is a Sylow p-subgroup of N. By the hypothesis, P_1 is $S\Phi$-supplemented in G, then there exists a subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq \Phi(P_1)$. Thus $G/N = (P_1N/N)(KN/N) = = (M/N)(KN/N)$. It is easy to see that $K \cap N$ contains a Hall p'-subgroup of N and then $(|N : (P_1 \cap N)|, |N : (K \cap N)|) = 1$, so $(P_1 \cap N)(K \cap N) = N = N \cap G = N \cap P_1K$. By [10] (A, Lemma 1.2), we have $P_1N \cap KN = (P_1 \cap K)N$. Thus $(P_1N)/N \cap (KN)/N = (P_1N \cap KN)/N = = (P_1 \cap K)N/N \leq \Phi(P_1)N/N \leq \Phi(P_1N/N)$, i.e., M/N is $S\Phi$-supplemented in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and G/N is p-nilpotent by (1). Hence G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$ and so N is not p-nilpotent.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since $\Phi(O_p(G)) \leq \Phi(G) = 1$, $O_p(G)$ is an elementary abelian group. It is easy to see that $O_p(G) \cap M$ is normalized by N and M, hence $O_p(G) \cap M \trianglelefteq G$. If $O_p(G) \cap M \neq 1$, by the uniqueness of N, we have $N \leq O_p(G) \cap M$, hence $G = MN = M$, a contradiction. This contradiction shows that $O_p(G) \cap M = 1$. By $N \leq O_p(G)$ and $G = MN$, we have $N = O_p(G)$. Let M_p be a Sylow p-subgroup of M. If $M_p = 1$, then N is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of N, then $\Phi(P_1) = 1$. By the hypothesis, there exists a subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq \Phi(P_1) = 1$. Since any Sylow r-subgroup of G with $r \neq p$ is a Sylow r-subgroup of K, we have $O^p(G) \leq K$. By the uniqueness of N, we obtain that $N \leq O^p(G)$. So $G = P_1K = NK = K$, then $P_1 = 1$ and $|G|_p = p$, so G is p-nilpotent by Lemma 2.6, a contradiction. Thus $M_p \neq 1$. Let P_0 be a maximal subgroup of P containing M_p, then by the hypothesis, there exists a subnormal subgroup T of G such that $G = P_0T$ and $P_0 \cap T \leq \Phi(P_0)$. By the previous argument, $N \leq O^p(G) \leq T$. Then $P_0 = P_0 \cap P = P_0 \cap NM_p = = M_p(P_0 \cap N) \leq M_p(P_0 \cap T) \leq M_p\Phi(P_0) \leq P_0$. Thus we have $M_p = P_0$ and so $|N| = p$, then $N \leq Z(G)$ by Lemma 2.6. Since G/N is p-nilpotent, G is p-nilpotent, a contradiction.

If N is p-nilpotent, then $N_{p'} \, char \, N \trianglelefteq G$, so $N_{p'} \leq O_{p'}(G) = 1$ by (2). Thus N is a p-group and then $N \leq O_p(G) = 1$, a contradiction. Thus (3) holds.

(4) The final contradiction.

By Lemma 2.1, we know that every maximal subgroup of P is $S\Phi$-supplemented in PN. Thus if $PN < G$, then PN is p-nilpotent and N is p-nilpotent, contradicts to (3), so we have $PN = G$. Since G/N is a p-group, $N = O^p(G)$. It is easy to see that G is not a non-abelian simple group, so we have $G \neq N$. Hence there exists a maximal normal subgroup M/N of G/N such that $|G : M| = p$. Since $P \cap M$ is a maximal subgroup of P, by the hypothesis, there exists a subnormal subgroup T of G such that $G = (P \cap M)T$ and $P \cap M \cap T \leq \Phi(P \cap M)$. In this case, we still have $T \geq O^p(G) = N$. $P \cap M \trianglelefteq P$ implies that $\Phi(P \cap M) \leq \Phi(P)$, so $P \cap N \leq P \cap M \cap T \leq \Phi(P \cap M) \leq \Phi(P)$. Thus N is p-nilpotent by Tate's theorem [4] (IV, Theorem 4.7), contrary to (3). This contradiction completes the proof.

Theorem 3.1 is proved.

Remark. The hypothesis that $(|G|, p-1) = 1$ in Theorem 3.1 cannot be removed. For example, S_3, the symmetry group of degree 3 is a counter-example.

Theorem 3.2. *Let P a Sylow p-subgroup of a group G, where p is a prime dividing $|G|$ such that $(|G|, p-1) = 1$. Let D be a subgroup of P such that $1 < |D| < |P|$. If every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Then we have:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.1 shows that the hypothesis still holds for $G/O_{p'}(G)$. Then $G/O_{p'}(G)$ is p-nilpotent by our minimal choice of G and so is G, a contradiction.

(2) $|P : D| > p$.

If $|P : D| = p$, then by Theorem 3.1, G is p-nilpotent.

(3) The final contradiction.

Let H be a subgroup of P such that $|H| = |D|$, then by the hypothesis H is $S\Phi$-supplemented in G. So there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq \Phi(H)$. Since $|G : K|$ is a power of p, there exists a normal subgroup M of G containing K such that $|G : M| = p$. Let $P_1 = P \cap M$ be a Sylow p-subgroup of M, then P_1 is a maximal subgroup of P. By (2), $|P_1| > |D|$. Lemma 2.1 shows that every subgroup of P_1 with order $|D|$ is $S\Phi$-supplemented in M. Then M is p-nilpotent by our minimal choice of G and so is G, the final contradiction. This contradiction completes the proof.

Theorem 3.2 is proved.

Obviously, Theorem 3.2 is true when p is the smallest prime divisor of $|G|$. Then we have the following corollary.

***Corollary* 3.1.** *Let G be a finite group. If for every prime p dividing $|G|$, there exists a Sylow p-subgroup P of G such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then G has a Sylow tower of supersoluble type.*

If we drop the assumption that $(|G|, p-1) = 1$ and add $N_G(P)$ is p-nilpotent, we still have the similar results.

Theorem 3.3. *Let p be a prime dividing $|G|$, P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. By Theorem 3.2, we may suppose that p is not the smallest prime divisor of $|G|$ and so p is an odd prime. Moreover, we have:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then by Lemma 2.1 it is easy to see that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Thus the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction.

(2) If L is a proper subgroup of G containing P, then L is p-nilpotent.

It is easy to see that $P \in \mathrm{Syl}_p(L)$ and $N_L(P) \leq N_G(P)$ is p-nilpotent. Furthermore, by Lemma 2.1 we know every subgroup of P with order $|D|$ is $S\Phi$-supplemented in G and thus $S\Phi$-supplemented in L. So L is p-nilpotent by the minimal choice of G.

(3) $O_p(G) \neq 1$.

Let $J(P)$ be the Thompson subgroup of P, then $N_G(P) \leq N_G(Z(J(P)))$ and by Lemma 2.1, we know every subgroup of P with order $|D|$ is $S\Phi$-supplemented in $N_G(Z(J(P)))$. Thus if $N_G(Z(J(P))) < G$, then $N_G(Z(J(P)))$ is p-nilpotent by (2). It follows from [6] (VIII, Theorem 3.1) that G is p-nilpotent, a contradiction. Thus we may suppose that $N_G(Z(J(P))) = G$ and hence $O_p(G) \neq 1$.

Next, we let N be a minimal normal subgroup of G contained in $O_p(G)$. Then we have:

(4) $|N| < |D|$ and G/N is p-nilpotent.

If $|N| > |D|$, pick a subgroup H of N with order $|D|$. By the hypothesis, there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq \Phi(H) \leq \Phi(N) = 1$. Clearly, we have $G = NK$, $N \cap K \neq 1$ and $N \cap K \trianglelefteq G$. Thus $N \cap K = N$ by the minimality of N and so $K = G$ and $H = 1$, a contradiction. If $|N| = |D|$, then N is $S\Phi$-supplemented in G by the hypothesis, so there exists a subnormal subgroup T of G such that $G = NT$ and $N \cap T \leq \Phi(N) = 1$. Since T is subnormal in G and $|G : T|$ is a power of p, there exists a normal subgroup M of G containing T such that $|G : M| = p$. Clearly, $G = NM$ and $N \cap M \trianglelefteq G$; then the minimality of N implies that $N \cap M = 1$. So $|N| = p$ and in this case, every minimal subgroup of P is $S\Phi$-supplemented in G. Then G is p-nilpotent by Lemma 2.2, a contradiction. Thus we have $|N| < |D|$. It is easy to see that G/N satisfies the hypothesis of the theorem, therefore G/N is p-nilpotent by the minimal choice of G.

(5) $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$. Moreover, $N = O_p(G) = F(G)$.

Since G/N is p-nilpotent and N is a p-group, G is p-soluble. By [6] (VI, Theorem 3.5), there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G for any $q \in \pi(G)$ with $q \neq p$. If $PQ < G$, then PQ is p-nilpotent by (2). Thus $O_p(G)Q = O_p(G) \times Q$ and $Q \leq C_G(O_p(G)) \leq$ $\leq O_p(G)$ by [6] (VI, Theorem 3.2), a contradiction. Hence we may assume that $G = PQ$. Since the class of all p-nilpotent subgroups formed a saturated formation, we may assume that N is the unique minimal normal subgroup of G contained in $O_p(G)$. By (1) and the fact that G is p-soluble, we can conclude that N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. By Lemma 2.8, we have $N = O_p(G) = F(G)$.

(6) $|P : D| > p$.

Now we assume that $|P : D| = p$. Since $N \nleq \Phi(G)$, there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Obviously, $P \cap M$ is a Sylow p-subgroup of M and

$P = (P \cap M)N$. Pick a maximal subgroup P_1 of P containing $P \cap M$, then by hypothesis P_1 is $S\Phi$-supplemented in G, so there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \le \Phi(P_1)$. Since $|G : T| = |P_1 \colon (P_1 \cap T)|$ is a power of p and T is subnormal in G, $O^p(G) \le T$ and thus $N \le T$. Then $P_1 = P_1 \cap P = P_1 \cap (P \cap M)N = (P \cap M)(P_1 \cap N) \le$ $\le (P \cap M)(P_1 \cap T) \le (P \cap M)\Phi(P_1)$, therefore $P \cap M = P_1$ and $|N| = p$. Since G is soluble by (5), $C_G(F(G)) \le F(G) = N$, so $C_G(N) = N$. Then we have $M \cong G/N = N_G(N)/C_G(N) \lesssim$ $\lesssim \mathrm{Aut}(N)$. Since $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$, M and in particularly Q is cyclic and hence G is q-nilpotent by Burnside's Theorem [4] (IV, Theorem 2.8). It follows that $G = N_G(P)$ is p-nilpotent by the hypothesis, a contradiction.

(7) The final contradiction.

Since G is soluble, there is a normal maximal subgroup M of G such that $|G : M|$ is a prime. If $|G : M| = q$, then M is p-nilpotent by (2) and therefore $P = M \trianglelefteq G$ by (1), a contradiction. Thus we may assume that $|G : M| = p$, then it follows that $P \cap M \in \mathrm{Syl}_p(M)$ is a maximal subgroup of P. If $N_G(P \cap M) < G$, then $N_G(P \cap M) \ge P$ is p-nilpotent by (2) and so is $N_M(P \cap M)$. Since $|P : D| > p$ by (6), every subgroup of $P \cap M$ with order $|D|$ is $S\Phi$-supplemented in M by Lemma 2.1. Consequently, M satisfies the hypothesis of the theorem and therefore M is p-nilpotent by the minimal choice of G. The normal p-complement of M is also the normal p-complement of G, a contradiction. Hence we may suppose that $P \cap M \trianglelefteq G$ and then $N = O_p(G) = P \cap M$ is a maximal subgroup of P. This leads to $|D| < |N|$, contradicts to (4), the final contradiction.

Theorem 3.3 is proved.

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and E a normal subgroup of a group G such that $G/E \in \mathcal{F}$. If for every prime p dividing $|E|$, there exists a Sylow p-subgroup P of E such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then $G \in \mathcal{F}$.*

Proof. By Lemma 2.1, we know that for every prime p dividing $|E|$, there exists a Sylow p-subgroup P of E such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in E, then E is a Sylow tower group of supersoluble type by Corollary 3.1. Let p be the largest prime dividing $|E|$ and P a Sylow p-subgroup of E, then P is normal in G. Since $(G/P)/(E/P) \cong G/E \in \mathcal{F}$ and the hypothesis still holds for $(G/P, E/P)$ by Lemma 2.1, we have $G/P \in \mathcal{F}$ by induction on $|G|$. Since $P \le Z_{\mathcal{U}}(G)$ by Lemma 2.5 and $Z_{\mathcal{U}}(G) \le Z_{\mathcal{F}}(G)$ by [10] (IV, Proposition 3.11), we have $P \le Z_{\mathcal{F}}(G)$ and so $G \in \mathcal{F}$, as required.

Theorem 3.4 is proved.

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and E a normal subgroup of a group G such that $G/E \in \mathcal{F}$. If for every prime p dividing $|F^*(E)|$, there exists a Sylow p-subgroup P of $F^*(E)$ such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in G, then $G \in \mathcal{F}$.*

Proof. We use induction on $|G|$. By Lemma 2.1, we know that for every prime p dividing $|F^*(E)|$, there exists a Sylow p-subgroup P of $F^*(E)$ such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is $S\Phi$-supplemented in $F^*(E)$. By Corollary 3.1, $F^*(E)$ possesses an ordered Sylow tower of supersoluble type. In particular, $F^*(E)$ is soluble and so $F^*(E) = F(E)$ by Lemma 2.7. Lemma 2.5 shows that $F(E) \le Z_{\mathcal{U}}(G)$.

Since $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by [10] (IV, Proposition 3.11), we have $F(E) \leq Z_{\mathcal{F}}(G)$. By [10] (IV, Theorem 6.10), $G/C_G(Z_{\mathcal{F}}(G)) \in \mathcal{F}$ and since $F(E) \leq Z_{\mathcal{F}}(G)$, we have $G/C_G(F(E)) \in \mathcal{F}$. By the hypothesis $G/E \in \mathcal{F}$, so $G/C_E(F(E)) \in \mathcal{F}$. But $C_E(F(E)) = C_E(F^*(E)) \leq F(E)$ by Lemma 2.7, then we have $G/F(E) \in \mathcal{F}$. Hence $G \in \mathcal{F}$ by Theorem 3.4, as required.

Theorem 3.5 is proved.

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