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ON THE GAUSS SUMS AND GENERALIZED BERNOULLI NUMBERS*

ПРО СУМИ ГАУССА ТА УЗАГАЛЬНЕНІ ЧИСЛА БЕРНУЛЛІ

Using the properties of primitive characters, Gauss sums, and the Ramanujan sum, we study two hybrid mean values of Gauss sums and generalized Bernoulli numbers and give two asymptotic formulas.

Із використанням примітивних характерів, сум Гаусса та суми Рамануджана вивчено два гібридних середніх значення сум Гаусса й узагальнених чисел Бернуллі та отримано дві асимптотичні формули.

1. Introduction. Let χ be a Dirichlet character modulo $q \geq 3$. For any integer n, the Gauss sum $G(n,\chi)$ is defined as following:

$$G(n,\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{an}{q}\right),$$

where $e(y) = e^{2\pi i y}$. Especially for $n = 1$, we write $\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right)$. The various properties and applications of $\tau(\chi)$ appear in many analytic number theory books (see reference [1]).

Maybe the most important property of $\tau(\chi)$ is that if χ is a primitive character modulo q, then $|\tau(\chi)| = \sqrt{q}$. If χ is a non-primitive character modulo q, $\tau(\chi)$ also appears many good value distribution properties in some problems of weighted mean value. For example, Y. Yi and W. Zhang [2] studied the $2k$-th power mean of inversion of L-functions with the weight of Gauss sums, and gave some interesting formulae.

Let χ be a non-principal Dirichlet character modulo q. The generalized Bernoulli numbers $B_{n,\chi}$ is defined by the following:

$$\sum_{a=1}^{q} \chi(a) \frac{te^{at}}{e^{qt} - 1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n.$$

This sequence of numbers has considerable fascination and importance. The definition and basic properties of generalized Bernoulli numbers can be found in [3]. H. Liu and W. Zhang [4] used the properties of primitive characters and the mean value theorems of Dirichlet L-functions to study the hybrid mean value

$$\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \tau^m(\overline{\chi}) B_{n,\chi}^m,$$

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and give a sharper asymptotic formula.

It might be interesting to study more mean value of the Gauss sums and generalized Bernoulli numbers. In this paper, we use the properties of primitive characters, Gauss sums and Ramanujan sum to study two hybrid mean value of Gauss sums and generalized Bernoulli numbers, and give two interesting formulae. That is, we shall prove the following.

Theorem 1. *Let $q \geq 3$ be an integer. Then for any given positive integers $n > 1$ and m we have*

$$\sum_{\substack{\chi \neq \chi_0 \\ \tau(\chi) \neq 0}} \frac{B_{n,\chi}^m}{\tau^m(\chi)} = \frac{(-1)^m 2^{m-1} (n!)^m}{(2\pi i)^{nm}} q^{m(n-1)-1} \phi^2(q) \prod_{p \| q} \left(1 + \frac{1}{p-1}\right) + O\left(q^{m(n-1)} d(q)\right),$$

where $\sum_{\substack{\chi \neq \chi_0 \\ \tau(\chi) \neq 0}}$ denotes the summation over all non-principal characters modulo q with $\tau(\chi) \neq 0$, $\prod_{p \| q}$ denotes the product over all prime divisors p of q with $p \mid q$ and $p^2 \nmid q$, $\phi(q)$ is the Euler function, $d(q)$ denotes the divisor function, and the O-constant depends on m and n.

Theorem 2. *For any fixed positive integers $m > 2$ and $n > 1$, we have*

$$\sum_{\chi_1 \bmod q} \left(\sum_{\chi_2 \neq \chi_0} \tau(\chi_1 \overline{\chi}_2) B_{n,\chi_2} \right)^m = \frac{(-1)^{(n+1)m} (n!)^m}{(2\pi i)^{nm}} q^{m(n-1)} \phi^{2m}(q) +$$

$$+ O\left(q^{m(n+1)-1} d(q)\right) + O\left(q^{m(n+1/2)+1} d^m(q)\right),$$

where the O-constant depends on m and n.

2. Some lemmas. To complete the proof of the theorems, we need the following lemmas.

Lemma 1. *For any integer $q \geq 3$, let χ be a non-primitive character modulo q, and q^* denote the conductor of χ with $\chi \Longleftrightarrow \chi^*$. If $(n,q) > 1$, we have*

$$G(n,\chi) = \begin{cases} \overline{\chi}^*\left(\frac{n}{(n,q)}\right) \chi^*\left(\frac{q}{q^*(n,q)}\right) \mu\left(\frac{q}{q^*(n,q)}\right) \phi(q) \phi^{-1}\left(\frac{q}{(n,q)}\right) \tau(\chi^*), & q^* = \frac{q_1}{(n,q_1)}, \\ 0, & q^* \neq \frac{q_1}{(n,q_1)}, \end{cases}$$

where $\mu(n)$ is the Möbius function, and q_1 is the largest divisor of q that has the same prime factors with q^*.

If $(n,q) = 1$, then we have

$$G(n,\chi) = \overline{\chi}^*(n) \chi^*\left(\frac{q}{q^*}\right) \mu\left(\frac{q}{q^*}\right) \tau(\chi^*).$$

Proof. See reference [5].

Lemma 2. *Let q and r be integers with $q \geq 3$ and $(r,q) = 1$, χ be a Dirichlet character modulo q. Then we have the identities*

$$\sum_{\chi \bmod q}^{*} \chi(r) = \sum_{d \mid (q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^{*}_{\chi \bmod q}$ *denotes the summation over all primitive characters modulo* q, *and* $J(q)$ *denotes the number of primitive characters modulo* q.

Proof. This is Lemma 3 of [6]. Also one can see Lemma 4 of [7].

Lemma 3. *Let* $q = uv$, *where* $(u, v) = 1$, u *be a square-full number or* $u = 1$, v *be a square-free number. Then for any given positive integers* $n > 1$ *and* m *we have*

$$\sum_{d|v} \sum^{*}_{\chi \bmod ud} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m = \frac{2^{m-1}\phi^2(q)}{q} \prod_{p\|q} \left(1 + \frac{1}{p-1}\right) + O\left(d(q)\right).$$

Proof. It is easy to show that

$$\sum_{d|v} \sum^{*}_{\chi \bmod ud} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m =$$

$$= \sum_{d|v} \sum^{*}_{\chi \bmod ud} [1 + \overline{\chi}(-1)(-1)^n]^m \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m =$$

$$= \begin{cases} 2^m \displaystyle\sum_{d|v} \sum^{*}_{\substack{\chi \bmod ud \\ \chi(-1)=1}} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m, & \text{if } 2 \mid n, \\ 2^m \displaystyle\sum_{d|v} \sum^{*}_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m, & \text{if } 2 \nmid n. \end{cases}$$

Let $\tau_m(r)$ denote the m-th divisor function (i.e., the number of positive integer solutions of the equation $r = r_1 r_2 \ldots r_m$). Note that $J(u) = \phi^2(u)/u$, if u is a square-full number. Then using the methods of Lemma 3 in [4] and Lemma 2 in this paper we have

$$\sum_{d|v} \sum^{*}_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{r=1}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m =$$

$$= \sum_{d|v} \sum_{t_1|\frac{v}{d}} \cdots \sum_{t_m|\frac{v}{d}} \frac{\mu(t_1)\ldots\mu(t_m)\phi(t_1)\ldots\phi(t_m)}{t_1^n \ldots t_m^n} \sum_{r=1}^{+\infty} \frac{\tau_m(r)}{r^n} \sum^{*}_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(t_1 \ldots t_m)\overline{\chi}(r) =$$

$$= \frac{1}{2}\sum_{d|v}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{t_1|\frac{v}{d}\\ t_1\ldots t_m r\equiv 1 \bmod s}}\ldots\sum_{t_m|\frac{v}{d}}\sum_{r=1}^{+\infty}\frac{\mu(t_1)\ldots\mu(t_m)\phi(t_1)\ldots\phi(t_m)\tau_m(r)}{(t_1\ldots t_m r)^n} -$$

$$-\frac{1}{2}\sum_{d|v}\sum_{s|ud}\mu\left(\frac{ud}{s}\right)\phi(s)\sum_{\substack{t_1|\frac{v}{d}\\ t_1\ldots t_m r\equiv -1 \bmod s}}\ldots\sum_{t_m|\frac{v}{d}}\sum_{r=1}^{+\infty}\frac{\mu(t_1)\ldots\mu(t_m)\phi(t_1)\ldots\phi(t_m)\tau_m(r)}{(t_1\ldots t_m r)^n} =$$

$$= \frac{1}{2}\sum_{d|v}J(ud) + O\left(\sum_{d|v}\sum_{s|ud}\phi(s)\sum_{l=1}^{+\infty}\frac{1}{(ls+1)^{n-1+\epsilon}}\right) +$$

$$+O\left(\sum_{d|v}\sum_{s|ud}\phi(s)\sum_{l=1}^{+\infty}\frac{1}{(ls-1)^{n-1+\epsilon}}\right) =$$

$$= \frac{1}{2}\sum_{d|v}J(ud) + O\left(d(q)\right) = \frac{\phi^2(u)}{2u}\sum_{d|v}J(d) + O\left(d(q)\right) =$$

$$= \frac{\phi^2(u)}{2u}\prod_{p|v}(p-1) + O\left(d(q)\right) = \frac{\phi^2(u)}{2u}\prod_{p|v}\left[\frac{(p-1)^2}{p}\left(1+\frac{1}{p-1}\right)\right] + O\left(d(q)\right) =$$

$$= \frac{\phi^2(q)}{2q}\prod_{p\|q}\left(1+\frac{1}{p-1}\right) + O\left(d(q)\right).$$

Similarly we can get

$$\sum_{d|v}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\sum_{r=1}^{+\infty}\frac{\overline{\chi}(r)}{r^n}\right]^m = \frac{\phi^2(q)}{2q}\prod_{p\|q}\left(1+\frac{1}{p-1}\right) + O\left(d(q)\right).$$

So we have

$$\sum_{d|v}\sideset{}{^*}\sum_{\chi \bmod ud}\left[\sum_{t|\frac{v}{d}}\frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{\overline{\chi}(r)}{r^n}\right]^m = \frac{2^{m-1}\phi^2(q)}{q}\prod_{p\|q}\left(1+\frac{1}{p-1}\right) + O\left(d(q)\right).$$

Lemma 3 is proved.

Lemma 4. *Let χ be a Dirichlet character modulo q and $n>1$ be a fixed integer. Then we have*

$$\sum_{\substack{r=-\infty\\ r\neq 1}}^{+\infty}\frac{G(r,\chi)}{(r-1)^n} = \begin{cases}(-1)^n\phi(q) + O\left(d(q)\right), & \text{if } \chi=\chi_0 \text{ is the principal character,}\\ O\left(q^{1/2}d(q)\right), & \text{otherwise.}\end{cases}$$

Proof. First we suppose that χ is a non-principal character modulo q. Noting that

$$G(r,\chi) \leq (r,q)q^{1/2},$$

then we have

$$\sum_{\substack{r=-\infty \\ r\neq 1}}^{+\infty} \frac{G(r,\chi)}{(r-1)^n} \ll q^{1/2} \sum_{r=2}^{+\infty} \frac{(r,q)}{(r-1)^n} = q^{1/2} \sum_{d|q} \sum_{2/d\leq l<+\infty} \frac{d}{(ld-1)^n} =$$

$$= q^{1/2} \sum_{d|q} \frac{1}{d^{n-1}} \sum_{2/d\leq l<+\infty} \frac{1}{(l-1/d)^n} \ll q^{1/2} d(q).$$

If $\chi = \chi_0$ is the principal character modulo q, then $G(r,\chi_0) = C_q(r)$ is the Ramanujan sum. Noting that

$$C_q(r) = \sum_{d|(q,r)} d\mu\left(\frac{q}{d}\right),$$

then we have

$$\sum_{\substack{r=-\infty \\ r\neq 1}}^{+\infty} \frac{G(r,\chi_0)}{(r-1)^n} = \sum_{\substack{r=-\infty \\ r\neq 1}}^{+\infty} \frac{1}{(r-1)^n} \sum_{d|(q,r)} d\mu\left(\frac{q}{d}\right) = \sum_{d|q} \sum_{\substack{l=-\infty \\ ld\neq 1}}^{+\infty} \frac{d\mu\,(q/d)}{(ld-1)^n} =$$

$$= (-1)^n \sum_{d|q} d\mu\left(\frac{q}{d}\right) + \sum_{d|q} \sum_{\substack{l=-\infty \\ ld\neq 1 \\ l\neq 0}}^{+\infty} \frac{d\mu\,(q/d)}{(ld-1)^n} =$$

$$= (-1)^n \sum_{d|q} d\mu\left(\frac{q}{d}\right) + O\left(d(q)\right) = (-1)^n \phi(q) + O\left(d(q)\right).$$

Lemma 4 is proved.

3. Proof of the theorems. In this section, we complete the proof of the theorems. Let $q \geq 3$ be an integer, and χ be a Dirichlet character modulo q. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_{n,\chi} = q^{n-1} \sum_{a=1}^{q} \chi(a) B_n\left(\frac{a}{q}\right).$$

From Theorem 12.19 of [1] we also have

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{e\,(rx)}{r^n}, \quad \text{if} \quad 0 < x \leq 1.$$

Therefore

$$B_{n,\chi} = q^{n-1} \sum_{a=1}^{q} \chi(a) \left[-\frac{n!}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{e\left(\frac{ar}{q}\right)}{r^n} \right] = -\frac{n! q^{n-1}}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{G(r,\chi)}{r^n}. \tag{1}$$

Let $q = uv$, where $(u,v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Let q^* denote the conductor of χ with $\chi \Longleftrightarrow \chi^*$, then

$$\tau(\chi) = \chi^*\left(\frac{q}{q^*}\right) \mu\left(\frac{q}{q^*}\right) \tau(\chi^*) \neq 0$$

if and only if $q^* = ud$, where $d \mid v$. So from Lemmas 1 and 3 we have

$$\sum_{\substack{\chi\neq\chi_0 \\ \tau(\chi)\neq 0}} \frac{B_{n,\chi}^m}{\tau^m(\chi)} = \sum_{d|v} \sideset{}{^*}\sum_{\chi \bmod ud} \frac{\left[-\frac{n! q^{n-1}}{(2\pi i)^n} \sum_{t|\frac{v}{d}} \frac{\chi\left(\frac{v}{dt}\right)\mu\left(\frac{v}{dt}\right)\phi(q)\tau(\chi)}{t^n \phi\left(\frac{q}{t}\right)} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m}{\chi^m\left(\frac{v}{d}\right)\mu^m\left(\frac{v}{d}\right)\tau^m(\chi)}.$$

Noting that

$$\chi\left(\frac{v}{d}\right) = \chi\left(\frac{v}{dt}\right)\chi(t), \quad \mu\left(\frac{v}{d}\right) = \mu\left(\frac{v}{dt}\right)\mu(t),$$

then we get

$$\frac{\chi\left(\frac{v}{dt}\right)\mu\left(\frac{v}{dt}\right)\phi(q)}{t^n \phi\left(\frac{q}{t}\right)\chi\left(\frac{v}{d}\right)\mu\left(\frac{v}{d}\right)} = \frac{\phi(t)}{t^n \chi(t)\mu(t)} = \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n}.$$

Therefore by Lemma 3 and the above we have

$$\sum_{\substack{\chi\neq\chi_0 \\ \tau(\chi)\neq 0}} \frac{B_{n,\chi}^m}{\tau^m(\chi)} = \frac{(-1)^m (n!)^m q^{m(n-1)}}{(2\pi i)^{nm}} \sum_{d|v} \sideset{}{^*}\sum_{\chi \bmod ud} \left[\sum_{t|\frac{v}{d}} \frac{\overline{\chi}(t)\mu(t)\phi(t)}{t^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{\overline{\chi}(r)}{r^n} \right]^m =$$

$$= \frac{(-1)^m 2^{m-1} (n!)^m}{(2\pi i)^{nm}} q^{m(n-1)-1} \phi^2(q) \prod_{p\|q} \left(1 + \frac{1}{p-1}\right) + O\left(q^{m(n-1)} d(q)\right).$$

Theorem 1 is proved.

From the orthogonality relations for character sums, formula (1) and Lemma 4 we can get

$$\sum_{\chi_2\neq\chi_0} \tau(\chi_1\overline{\chi}_2) B_{n,\chi_2} = -\frac{n! q^{n-1}}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{1}{r^n} \sum_{\chi_2\neq\chi_0} \tau(\chi_1\overline{\chi}_2) G(r,\chi_2) =$$

$$= -\frac{n! q^{n-1}}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{1}{r^n} \sum_{a=1}^{q} \chi_1(a) e\left(\frac{a}{q}\right) \sum_{b=1}^{q} e\left(\frac{br}{q}\right) \sum_{\chi_2\neq\chi_0} \overline{\chi}_2(a)\chi_2(b) =$$

$$= -\frac{n!q^{n-1}\phi(q)}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 0}}^{+\infty} \frac{1}{r^n} \sum_{a=1}^{q} \chi_1(a) e\left(\frac{a(1+r)}{q}\right) + O\left(q^n\right) =$$

$$= -\frac{n!q^{n-1}\phi(q)}{(2\pi i)^n} \sum_{\substack{r=-\infty \\ r\neq 1}}^{+\infty} \frac{G(r,\chi_1)}{(r-1)^n} + O\left(q^n\right) =$$

$$= \begin{cases} \dfrac{(-1)^{n+1} n! q^{n-1} \phi^2(q)}{(2\pi i)^n} + O\left(q^n d(q)\right), & \text{if } \chi_1 = \chi_0 \text{ is the principal character;} \\ O\left(q^{n+1/2} d(q)\right), & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{\chi_1 \bmod q} \left(\sum_{\chi_2 \neq \chi_0} \tau(\chi_1 \overline{\chi}_2) B_{n,\chi_2}\right)^m =$$

$$= \left(\sum_{\chi_2 \neq \chi_0} \tau(\chi_0 \overline{\chi}_2) B_{n,\chi_2}\right)^m + \sum_{\chi_1 \neq \chi_0} \left(\sum_{\chi_2 \neq \chi_0} \tau(\chi_1 \overline{\chi}_2) B_{n,\chi_2}\right)^m =$$

$$= \frac{(-1)^{(n+1)m} (n!)^m}{(2\pi i)^{nm}} q^{m(n-1)} \phi^{2m}(q) + O\left(q^{m(n+1)-1} d(q)\right) + O\left(q^{m(n+1/2)+1} d^m(q)\right),$$

which is valid for $m > 2$.

Theorem 2 is proved.

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