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A MATRIX APPROACH TO THE BINOMIAL THEOREM
МАТРИЧНИЙ ПІДХІД ДО БІНОМІАЛЬНОЇ ТЕОРЕМИ

Motivated by the formula $x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k$, we investigate factorizations of the lower triangular Toeplitz matrix with (i,j)th entry equal to x^{i-j} via the Pascal matrix. In this way, a new computational approach to a generalization of the binomial theorem is introduced. Numerous combinatorial identities are obtained from these matrix relations.

На основі формули $x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k$ розглянуто факторизації нижньотрикутної матриці Тепліца, (i,j)-й елемент якої дорівнює x^{i-j}, з використанням матриці Паскаля. Тим самим уведено новий обчислювальний підхід до узагальнення біноміальної теореми. Із використанням цих матричних співвідношень отримано численні комбінаторні тотожності.

1. Introduction. The Pascal matrix of order n, denoted by $\mathcal{P}_n[x] = [p_{i,j}[x]]$, $i,j = 1,\ldots,n$, is a lower triangular matrix with elements equal to

$$p_{i,j}[x] = \begin{cases} x^{i-j}\dbinom{i-1}{j-1}, & i-j \geqslant 0, \\ 0, & i-j < 0, \end{cases}$$

and its inverse $\mathcal{P}_n[x]^{-1} = [p'_{i,j}[x]]$, $i,j = 1,\ldots,n$, has the elements equal to

$$p'_{i,j}[x] = \begin{cases} (-x)^{i-j}\dbinom{i-1}{j-1}, & i-j \geqslant 0, \\ 0, & i-j < 0. \end{cases}$$

For the sake of simplicity we denote $\mathcal{P}_n[1]$ with $\mathcal{P}_n$. Many properties of the Pascal matrix have been examined in the recent literature (see for instance [1, 11, 12]). We are particulary interested on the usage of the Pascal matrix as a powerful tool for deriving combinatorial identities. Precisely, recalling that a Toeplitz matrix is matrix having constant entries along the diagonals, then the Pascal matrix can be factorized in a form $\mathcal{P}_n = T_n R_n$ or $\mathcal{P}_n = L_n T_n$, where T_n denotes the $n \times n$ lower triangular Toeplitz matrix. Usually, the Toeplitz matrix T_n is filled with the numbers from the well-known sequences. By equalizing the (i,j) th elements of the matrices in these matrix equalities, we establish correlations between binomial coefficients and the terms from the well-known sequences.

Following this idea, some combinatorial identities via Fibonacci numbers were derived in [4, 14], as well as the identities for the Catalan numbers [9], Bell [10], Bernoulli [13] and the Lucas numbers [15] were also computed. In [5] the authors derived identities by using the factorizations of the Pascal matrix via generalized second order recurrent matrix. Some combinatorial identities were also computed in [2, 3, 6 – 8] by various matrix methods.

The starting point of the present paper is one of the most beautiful formulas in mathematics, a particular case of the binomial theorem

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k. \tag{1.1}$$

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The essential observation is that the binomial coefficient in (1.1) may be considered as the element of the Pascal matrix $\mathcal{P}_n$, and the left-hand side of (1.1) may be considered as the element of the lower triangular Toeplitz matrix with (i,j) th entry equal to x^{i-j}, which is denoted by Zhang [11] with $\mathcal{S}_n[x]$. Therefore, our goal is to factorize the matrix $\mathcal{S}_n[x]$ via the Pascal matrix $\mathcal{P}_n$ and to give some combinatorial identities via this computational method. Some of our results represent generalizations of some well-known identities, such as the binomial theorem (1.1). These identities involve the binomial coefficients and the hypergeometric function ${}_2F_1$. Recall that the hypergeometric function ${}_2F_1(a,b,c;z)$ is defined by

$${}_2F_1(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where

$$(a)_n=\begin{cases}a(a+1)\ldots(a+n-1), & n>0,\\ 1, & n=0,\end{cases}$$

is the well-known rising factorial symbol.

2. The results. First we find a matrix $\mathcal{R}_n[x]$ which establishes the relation between matrices $\mathcal{S}_n[x]$ and $\mathcal{P}_n$.

Theorem 2.1. *The matrix* $\mathcal{R}_n[x]=[r_{i,j}[x]]$, $i,j=1,\ldots,n$, $x\in\mathbb{R}$, *whose entries are defined by*

$$r_{i,j}[x]=\begin{cases}(-1)^{i-j}\binom{i-1}{j-1}\,{}_2F_1(1,j-i;j;x), & i\geqslant j,\\ 0, & i<j,\end{cases}$$

satisfies

$$\mathcal{S}_n[x]=\mathcal{P}_n\mathcal{R}_n[x]. \tag{2.1}$$

Proof. Our goal is to prove $\mathcal{R}_n[x]=\mathcal{P}_n^{-1}\mathcal{S}_n[x]$. Let us denote the sum $\sum_{k=1}^{n}p'_{i,k}s_{k,j}[x]$ by $m_{i,j}[x]$. It is easy to show that $m_{i,j}[x]=0=r_{i,j}[x]$ for $i<j$. On the other hand, in the case $i\geqslant j$ we have

$$m_{i,j}[x]=\sum_{k=j}^{i}p'_{i,k}s_{k,j}[x]=\sum_{k=j}^{i}(-1)^{i-k}\binom{i-1}{k-1}x^{k-j}=\sum_{k=0}^{i-j}(-1)^{i-j+k}\binom{i-1}{j+k-1}x^k.$$

After applying the transformations

$$\sum_{k=0}^{i-j}(-1)^{i-j+k}\binom{i-1}{j+k-1}x^k=(-1)^{i-j}\sum_{k=0}^{i-j}\frac{(i-1)!\,(-x)^k}{(i-j-k)!\,(j+k-1)!}=$$

$$=(-1)^{i-j}\frac{(i-1)!}{(j-1)!\,(i-j)!}\sum_{k=0}^{\infty}\frac{(1)_k\,(j-i)_k}{(j)_k}\frac{x^k}{k!}=$$

$$=(-1)^{i-j}\binom{i-1}{j-1}\,{}_2F_1(1,j-i;j;x)$$

we show that $m_{i,j}[x]=r_{i,j}[x]$ in the case $i\geqslant j$, which was our original attention.

***Corollary* 2.1.** *For positive integers i and j satisfying $i \geqslant j$ and real x, the following identity is valid*

$$\binom{i-1}{j-1}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}\,{}_2F_1(1,-k;j;x)=x^{i-j}. \tag{2.2}$$

Proof. From matrix relation (2.1) we obtain $s_{i,j}[x]=\sum_{k=j}^{i}p_{i,k}\,r_{k,j}[x]$, or in an expanded form

$$x^{i-j}=\sum_{k=j}^{i}\binom{i-1}{k-1}(-1)^{k-j}\binom{k-1}{j-1}\,{}_2F_1(1,j-k;j;x).$$

Making use of the formula for the binomial coefficients $\binom{r}{m}\binom{m}{l}=\binom{r}{l}\binom{r-l}{m-l}$, together with the substitution $k\mapsto k+j$, we finish the proof.

***Remark* 2.1.** By putting $j=1$ in equality (2.2), we obtain the binomial theorem (1.1).

In the following identity we establish the relation between hypergeometric functions ${}_2F_1$.

***Corollary* 2.2.** *The following identity is valid for arbitrary nonnegative integers i and j satisfying $i \geqslant j$ and $x \in \mathbb{R}$*

$$\binom{i}{j}+x\binom{i-1}{j}\,{}_2F_1(1,j-i+1;1-i;x)=\binom{i}{j}\,{}_2F_1(1,j-i;-i;x). \tag{2.3}$$

Proof. It is straightforward to show that

$$\left(\mathcal{S}_n[x]^{-1}\right)_{i,j}=\begin{cases}1, & i=j,\\ -x, & i=j+1,\\ 0, & \text{otherwise.}\end{cases}$$

We are now in a position to write the inverse of the $\mathcal{R}_n[x]$ as $\mathcal{R}_n[x]^{-1}=\mathcal{S}_n[x]^{-1}\mathcal{P}_n$. In this way we obtain

$$\left(\mathcal{R}_n[x]^{-1}\right)_{i,j}=\begin{cases}1, & i=j,\\ \binom{i-1}{j-1}-x\binom{i-2}{j-1}, & i>j,\\ 0, & i<j.\end{cases}$$

From relation $\mathcal{P}_n=\mathcal{S}_n[x]\mathcal{R}_n[x]^{-1}$ we get identity

$$\binom{i-1}{j-1}=\sum_{k=j+1}^{i}x^{i-k}\left(\binom{k-1}{j-1}-x\binom{k-2}{j-1}\right)+x^{i-j}$$

valid for all positive integers $i \geqslant j$. After the replacement $(i,j)\mapsto(i+1,j+1)$, we get equality

$$\binom{i}{j}=\sum_{k=0}^{i-j-1}x^k\left(\binom{i-k}{j}-x\binom{i-k-1}{j}\right)+x^{i-j}$$

valid for all nonnegative integers $i \geqslant j$. Our problem now reduces to show the following two relations:

$$\sum_{k=0}^{i-j-1} x^k \binom{i-k}{j} = -x^{i-j} + \binom{i}{j} {}_2F_1(1, j-i; -i; x), \tag{2.4}$$

$$\sum_{k=0}^{i-j-1} x^k \binom{i-k-1}{j} = \binom{i-1}{j} {}_2F_1(1, j-i+1; 1-i; x). \tag{2.5}$$

In order to prove (2.4), we start from its left-hand side and transform it into

$$\sum_{k=0}^{i-j-1} x^k \binom{i-k}{j} = -x^{i-j} + \sum_{k=0}^{i-j} x^k \binom{i-k}{j} = -x^{i-j} + \frac{i!}{j!\,(i-j)!} \sum_{k=0}^{i-j} \frac{(1)_k\,(j-i)_k}{(-i)_k} \frac{x^k}{k!},$$

and (2.4) now immediately follows. The reader may establish (2.5) in a similar way, and the proof is therefore completed.

***Remark* 2.2.** Some pedestrian manipulations yields that in the case $j = 1$, relation (2.3) reduces to the well-known identity $\sum_{k=0}^{i-1} x^k = \dfrac{x^i - 1}{x - 1}$.

In the rest of this section we investigate another factorization of the matrix $\mathcal{S}_n[x]$ via the Pascal matrix, analogical to (2.1).

Theorem 2.2. *The matrix $\mathcal{L}_n[x] = [l_{i,j}[x]]$, $i, j = 1, \ldots, n$, $x \in \mathbb{R}$, whose entries are defined by*

$$l_{i,j}[x] = \begin{cases} \dfrac{x^i}{(1+x)^j} + \dfrac{(-1)^{i-j}}{x} \dbinom{i}{j-1} {}_2F_1\left(1, i+1; i-j+2; -\dfrac{1}{x}\right), & i \geqslant j, \\ 0, & i < j, \end{cases}$$

satisfies

$$\mathcal{S}_n[x] = \mathcal{L}_n[x]\mathcal{P}_n. \tag{2.6}$$

Proof. We prove $\mathcal{L}_n[x] = \mathcal{S}_n[x]\mathcal{P}_n^{-1}$. Let us denote the sum $\sum_{k=1}^{n} s_{i,k}[x]p'_{k,j}$ by $t_{i,j}[x]$. We have $t_{i,j}[x] = 0 = l_{i,j}[x]$ for $i < j$, while in the case $i \geqslant j$,

$$t_{i,j}[x] = \sum_{k=j}^{i} s_{i,k}[x]p'_{k,j} = \sum_{k=0}^{i-j} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} =$$

$$= \sum_{k=0}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} - \sum_{k=i-j+1}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1}. \tag{2.7}$$

Now it suffices to prove the following two identities

$$\sum_{k=0}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} = \frac{x^i}{(1+x)^j}, \tag{2.8}$$

$$\sum_{k=i-j+1}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} = -\frac{(-1)^{i-j}}{x}\binom{i}{j-1}\,{}_2F_1\left(1, i+1; i-j+2; -\frac{1}{x}\right). \tag{2.9}$$

In order to prove (2.8), we use the binomial theorem and obtain

$$\sum_{k=0}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} = \frac{x^i}{x^j}\sum_{k=0}^{\infty}\frac{(j)_k}{k!}\left(-\frac{1}{x}\right)^k =$$

$$= \frac{x^i}{x^j}\sum_{k=0}^{\infty}\binom{-j}{k}\left(\frac{1}{x}\right)^k = \frac{x^i}{x^j}\left(\frac{1+x}{x}\right)^{-j}.$$

The identity (2.9) can be verified by applying the following transformations:

$$\sum_{k=i-j+1}^{\infty} x^{i-j-k}\,(-1)^k \binom{j-1+k}{j-1} = \sum_{k=0}^{\infty} x^{-k-1}\,(-1)^{i-j+1+k}\binom{i+k}{j-1} =$$

$$= -\frac{(-1)^{i-j}}{x}\,\frac{i!}{(j-1)!\,(i-j+1)!}\sum_{k=0}^{\infty}\frac{(1)_k\,(i+1)_k}{(i-j+2)_k}\frac{(-1/x)^k}{k!}.$$

Since formulas (2.8) and (2.9) are valid, we apply them on (2.7), and the proof is completed.

***Corollary* 2.3.** *For integers $i \geqslant j \geqslant 0$ and real x we have*

$$\sum_{k=j}^{i}(-1)^{i-k}\binom{i-j+1}{k-j}\,{}_2F_1\left(1, i+1; i-k+2; -\frac{1}{x}\right) =$$

$$= \left(\frac{x}{1+x}\right)^{i+1}\,{}_2F_1\left(1, i+1; i-j+2; \frac{1}{1+x}\right). \tag{2.10}$$

Proof. From (2.6) we obtain

$$x^{i-j} = \sum_{k=j}^{i}\frac{x^i}{(1+x)^k}\binom{k-1}{j-1} +$$

$$+ x^i\sum_{k=j}^{i}\frac{(-1)^{i-k}}{x^{i+1}}\binom{i}{k-1}\binom{k-1}{j-1}\,{}_2F_1\left(1, i+1; i-k+2; -\frac{1}{x}\right). \tag{2.11}$$

An argument similar to the one used to prove (2.8) and (2.9) can be employed to prove the following relation:

$$\sum_{k=j}^{i}\frac{1}{(1+x)^k}\binom{k-1}{j-1} = x^{-j} - \frac{1}{(1+x)^{i+1}}\binom{i}{j-1}\,{}_2F_1\left(1, i+1; i-j+2; \frac{1}{1+x}\right). \tag{2.12}$$

The proof is finished after applying (2.12), in a conjunction with the transformation formula for the binomial coefficients, on (2.11).

***Corollary* 2.4.** *The identity*

$$\sum_{k=1}^{i}(-1)^{i-k}\binom{i}{k-1}\,{}_2F_1\left(1,i+1;i-k+2;-\frac{1}{x}\right)=\left(\frac{x}{1+x}\right)^{i} \tag{2.13}$$

is valid for every positive integer i and real x.

Proof. The proof follows from the previous corollary after some calculations in the case $j=1$, since the elements in the first column of the Pascal matrix are equal to $p_{i,1}=1$.

***Corollary* 2.5.** *For $i\geqslant j\geqslant 0$ and $x\in\mathbb{R}$ the following is satisfied*

$$\binom{i}{j}+\frac{1}{x}\binom{i}{j-1}\,{}_2F_1\left(1,1-j;i-j+2;-\frac{1}{x}\right)=\binom{i}{j}\,{}_2F_1\left(1,-j;i-j+1;-\frac{1}{x}\right). \tag{2.14}$$

Proof. From relation $\mathcal{L}_n[x]^{-1}=\mathcal{P}_n\mathcal{S}_n[x]^{-1}$ we verify that the matrix $\mathcal{L}_n[x]^{-1}$ has the elements equal to

$$\left(\mathcal{L}_n[x]^{-1}\right)_{i,j}=\begin{cases}\binom{i-1}{j-1}-x\binom{i-1}{j}, & i\geqslant j,\\[2mm] 0, & i<j.\end{cases}$$

Now we exploit the relation $\mathcal{P}_n=\mathcal{L}_n[x]^{-1}\mathcal{S}_n[x]$, and obtain

$$\binom{i}{j}=\sum_{k=0}^{i-j}\left(\binom{i}{i-k}-x\binom{i}{i-k+1}\right)x^{i-j-k}.$$

The proof is finished after verifying the following two identities:

$$\sum_{k=0}^{i-j}\binom{i}{i-k}x^{i-j-k}=\frac{(1+x)^i}{x^j}-\frac{1}{x}\binom{i}{j-1}\,{}_2F_1\left(1,1-j;i-j+2;-\frac{1}{x}\right),$$

$$\sum_{k=0}^{i-j}\binom{i}{i-k+1}x^{i-j-k+1}=\frac{(1+x)^i}{x^j}-\binom{i}{j}\,{}_2F_1\left(1,-j;i-j+1;-\frac{1}{x}\right),$$

analogously as in Corollary 2.2.

***Corollary* 2.6.** *For integer $n\geqslant 0$ and $x\in\mathbb{R}$, we have*

$$\sum_{k=1}^{n}2^{k-1}(-1)^{n-k}\binom{n}{k-1}\,{}_2F_1\left(1,n+1;n-k+2;-\frac{1}{x}\right)=\frac{x}{x-1}\left(\left(\frac{2x}{1+x}\right)^{n}-1\right). \tag{2.15}$$

Proof. Let $E_n=[1,1,\ldots,1]^T$. Since $\mathcal{S}_n[x]E_n=\mathcal{L}_n[x]\mathcal{P}_nE_n$, we have

$$\begin{bmatrix}1\\(x^2-1)/(x-1)\\(x^3-1)/(x-1)\\ \vdots\\(x^n-1)/(x-1)\end{bmatrix}=\mathcal{L}_n[x]\cdot\begin{bmatrix}1\\2\\4\\ \vdots\\2^{n-1}\end{bmatrix},$$

that is

$$\frac{x^n-1}{x-1}=\frac{x^n}{1+x}\sum_{k=1}^{n}\left(\frac{2}{1+x}\right)^{k-1}+$$

$$+\sum_{k=1}^{n}\frac{2^{k-1}}{x}(-1)^{n-k}\binom{n}{k-1}\,{}_2F_1\left(1,n+1;n-k+2;-\frac{1}{x}\right).$$

Now we apply $\sum_{k=1}^{n}\left(\frac{2}{1+x}\right)^{k-1}=-\frac{x+1}{x-1}\left(\left(\frac{2}{1+x}\right)^{n}-1\right)$, and the proof is completed.

3. Conclusion. By observing the fact that the binomial theorem $x^n=\sum_{k=0}^{n}\binom{n}{k}(x-1)^k$ can be written in the matrix mode, in this note we investigate factorizations of the matrix with (i,j) th entry equal to x^{i-j} via the Pascal matrix. Later, we use these factorizations to derive numerous combinatorial identities. Some of them are especially interesting, like (2.2) which represents a generalization of the Binomial theorem, as well as (2.3) which represents a generalization of the well-known identity $\sum_{k=0}^{i-1}x^k=\frac{x^i-1}{x-1}$. We leave for the future research deriving more combinatorial identities from various matrix factorizations.

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