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DETERMINANTAL EVALUATION OF FOUR WRONSKIAN MATRICES

ДЕТЕРМІНАНТНІ ОЦІНКИ ЧОТИРЬОХ МАТРИЦЬ ВРОНСЬКОГО

Two determinants of Wronskian matrices are evaluated when the matrix rows are partitioned into n blocks. Analogous formulae are derived for the matrices involving compositions of formal power series as entries.

Отримано оцінки двох детермінантів матриць Вронського у випадку, коли рядки матриць розбито на n блоків. Аналогічні формули запропоновано для матриць з елементами, які містять композицію формальних степеневих рядів.

For the m-times differentiable functions $\{f_k(x)\}_{0\le k<m}$, the corresponding Wronskian determinant reads as

$$\det_{0\le i,j<m}\left[\frac{d^i}{dx^i}f_j(x)\right].$$

One class of these determinants can be evaluated (cf. [2, 4 – 7]) when all the functions are powers of one fixed function.

Another class concerns formal power series and composite functions. For a fixed unitary formal power series $f(x)$, let $[x^k]f(x)$ stand for the coefficient of x^k in $f(x)$ and $f^{\langle\lambda\rangle}(x)$ for the λth composite series of $f(x)$. When matrix entries are replaced by $[x^{i+1}]f^{\langle j\rangle}(x)$, Kedlaya [3] discovered a product expression for the corresponding determinant, which has been generalized subsequently by the author [2].

In this paper, we shall investigate the determinantal evaluation when the rows of the matrices just mentioned are partitioned into n subsets. As preliminaries, a general expansion theorem that express a determinant as a multiple sum of products of its minors over set-partitions of its row labels will be proved and then be applied to the Vandermonde determinant in the first section. The second section will derive two determinant formulae for the Wronskian matrix with its rows being partitioned into n blocks. Finally, two analogous results will be obtained in the last section for the matrices when the matrix entries are replaced by coefficients of composite functions of formal power series.

1. Expansion theorem and Vandermonde determinant. For a natural number m, denote the interval of integers by

$$[1,m]=(0,m]=\{k\mid 1\le k\le m\}.$$

It can be expressed by the set partition:

$$[1,m]=\biguplus_{k=1}^{n}\left(M_{k-1},M_k\right],$$

where

$$0=M_0<M_1<M_2<\ldots<M_n=m$$

with

$$m = M_n = \sum_{k=1}^{n} m_k \qquad \text{and} \qquad m_k := M_k - M_{k-1} \quad \text{for} \quad 1 \le k \le n.$$

For $\sigma \subset [1,m]$, denote its cardinality, norm and complement to $[1,m]$, respectively, by

$$|\sigma| = \sum_{\lambda\in\sigma} 1, \qquad \|\sigma\| = \sum_{\lambda\in\sigma} \lambda \qquad \text{and} \qquad \sigma^c = [1,m]\backslash\sigma.$$

According to the Laplace formula, the determinant of a square matrix of order m can be expanded along the first ℓ rows:

$$H := \det_{1\le i,j\le m} \big[h_{i,j}\big] = \sum_{\substack{\sigma\subset[1,m]\\|\sigma|=\ell}} (-1)^{\binom{1+\ell}{2}+\|\sigma\|} H\Big[[1,\ell]|\sigma\Big] \times H\Big[(\ell,m]|\sigma^c\Big],$$

where $H\big[\tau|\sigma\big]$ stands for the determinant of the submatrix with the rows and columns being indexed, respectively, by $\tau \subset [1,m]$ and $\sigma \subset [1,m]$, where $|\sigma| = |\tau|$, of course.

Iterating the last equation for $(n-1)$-times, we derive the algebraic identity when the matrix rows are partitioned into n blocks.

Theorem 1. *The following determinant expansion formula holds*:

$$H := \det_{1\le i,j\le m} \big[h_{i,j}\big] = \sum_{\substack{\uplus_{k=1}^{n}\sigma_k=[1,m]\\|\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma) \prod_{k=1}^{n} H\Big[(M_{k-1},M_k]|\sigma_k\Big],$$

where the alternating sign $\varepsilon(\sigma) = \pm 1$ depends only on the partition function $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$.

Now we examine the application of this theorem to the Vandermonde determinant

$$\Delta(x|[1,m]) = \Delta(x_1,x_2,\ldots,x_m) = \det_{1\le i,j\le m} \big[x_i^{j-1}\big] = \prod_{1\le i<j\le m} (x_j - x_i). \tag{1}$$

Specifying the kth part of the partition $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ by

$$\sigma_k = \{\lambda_1(k),\lambda_2(k),\ldots,\lambda_{m_k}(k)\},$$

dividing the variables $\{x_i\}_{i=1}^m$ into n subsets

$$\{x_i\}_{i=1}^m = \biguplus_{k=1}^{n} \{x_1(k),x_2(k),\ldots,x_{m_k}(k)\},$$

and then applying Theorem 1 to (1), we get the following expression.

Proposition 1 (the Vandermonde determinant)**.**

$$\prod_{1\le\imath<\jmath\le n} \prod_{i=1}^{m_\imath}\prod_{j=1}^{m_\jmath} \{x_j(\jmath) - x_i(\imath)\} \prod_{k=1}^{n}\prod_{\kappa=1}^{m_k} x_\kappa(k) \prod_{1\le i<j\le m_k} \{x_j(k) - x_i(k)\} =$$

$$= \sum_{\substack{\uplus_{k=1}^{n}\sigma_k=[1,m]\\|\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma) \prod_{k=1}^{n} \det_{1\le i,j\le m_k} \Big[x_i^{\lambda_j(k)}(k)\Big].$$

Under the substitutions $x_i(k) = q^i x_k$, the last equality becomes

$$\prod_{1\le \imath<\jmath\le n}\prod_{i=1}^{m_\imath}\prod_{j=1}^{m_\jmath}\{q^j x_\jmath - q^i x_\imath\}\prod_{k=1}^{n}(qx_k)^{\binom{m_k+1}{2}}\prod_{1\le i<j\le m_k}\{q^j - q^i\} =$$

$$= \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}(qx_k)^{\|\sigma_k\|}\Delta(q^{\sigma_k}). \tag{2}$$

Dividing both sides of the last equation by $(1-q)^M$ with $M = \sum_{k=1}^n \binom{m_k}{2}$ and then letting $q \to 1$, we find the following limiting equality.

***Corollary* 1** (algebraic identity)**.**

$$\prod_{1\le \imath<\jmath\le n}\{x_\jmath - x_\imath\}^{m_\imath m_\jmath}\prod_{k=1}^{n} x_k^{\binom{m_k+1}{2}}\prod_{j=1}^{m_k}(j-1)! = \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n} x_k^{\|\sigma_k\|}\Delta(\sigma_k).$$

2. Wronskian matrices in blocks.

Lemma 1 ([2], Corollaries 4.3 and 4.4)**.** *Let $f(x)$ and $w(x)$ be two n-times differentiable functions. The following two Wronskian determinant identities hold*:

$$\det_{1\le i,j\le n}\left[\frac{d^{i-1}}{dx^{i-1}}\{w_i(x)f^{y_j}(x)\}\right] = \left\{\frac{f'(x)}{f(x)}\right\}^{\binom{n}{2}}\Delta(y)\prod_{k=1}^{n} w_k(x)f^{y_k}(x),$$

$$\det_{1\le i,j\le n}\left[\frac{d^{i}}{dx^{i}}\{w_i(x)f^{y_j}(x)\}\right] = \left\{\frac{f'(x)}{f(x)}\right\}^{\binom{n+1}{2}}\Delta(y)\prod_{k=1}^{n} y_k w_k(x)f^{y_k}(x),$$

where there is an additional restriction for the second formula that the function $w_k(x)$ is a polynomial of degree $< k$ for $1 \le k \le n$.

For the m-times differentiable functions $f_k(x)$ and $w_i(x)$, define the following determinant A with its rows being divided into n blocks:

$$A := \det_{1\le i,j\le m}\left[a_{i,j}\right] \text{ with } a_{i,j} = \frac{d^{i-M_{k-1}}}{dx^{i-M_{k-1}}}\left\{w_i(x)f_k^{y_j}(x)\right\} \text{ if } i\in[M_{k-1}, M_k).$$

Applying Theorem 1 to the last determinant, we get

$$A := \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}\det_{1\le i,j\le m_k}\left[\frac{d^{i-1}}{dx^{i-1}}\left\{w_{i+M_{k-1}}(x)f_k^{y_{\lambda_j(k)}}(x)\right\}\right].$$

Evaluating, by means of the first formula of Lemma 1, the minor

$$A\Big[(M_{k-1}, M_k]|\sigma_k\Big] = \det_{1\le i,j\le m_k}\left[\frac{d^{i-1}}{dx^{i-1}}\left\{w_{i+M_{k-1}}(x)f_k^{y_{\lambda_j(k)}}(x)\right\}\right] =$$

$$= \left\{\frac{f_k'(x)}{f_k(x)}\right\}^{\binom{m_k}{2}} \Delta(y|\sigma_k) \prod_{i=1}^{m_k} w_{i+M_{k-1}}(x) f_k^{y_{\lambda_i(k)}}(x),$$

we find the equality

$$A = \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma) \prod_{k=1}^{n} \left\{\frac{f_k'(x)}{f_k(x)}\right\}^{\binom{m_k}{2}} \Delta(y|\sigma_k) \prod_{i=1}^{m_k} w_{i+M_{k-1}}(x) f_k^{y_{\lambda_i(k)}}(x) =$$

$$= \prod_{i=1}^{m} w_i(x) \prod_{k=1}^{n} \left\{\frac{f_k'(x)}{f_k(x)}\right\}^{\binom{m_k}{2}} \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma) \prod_{k=1}^{n} \Delta(y|\sigma_k) \prod_{\lambda\in\sigma_k} f_k^{y_\lambda}(x).$$

When $y_j = y + j$ for $j \in [1,m]$, the last expression results in

$$\prod_{i=1}^{m} w_i(x) \prod_{k=1}^{n} f_k^{y m_k}(x) \left\{\frac{f_k'(x)}{f_k(x)}\right\}^{\binom{m_k}{2}} \sum_{\substack{\uplus_{k=1}^n \sigma_k=[1,m] \\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma) \prod_{k=1}^{n} \Delta(\sigma_k) f_k^{\|\sigma_k\|}(x).$$

The above multiple sum can be expressed, in view of Corollary 1, as a compact product. Summing up, we have proved the following theorem.

Theorem 2. *For each $k \in [1,n]$ and $i \in [1,m]$, let $f_k(x)$ and $w_i(x)$ be the m-times differentiable functions. Define the determinant $\mathbf{A}$ by higher derivatives*

$$\mathbf{A} := \det_{1\le i,j\le m} \left[\mathbf{a}_{i,j}\right] \quad \textit{with} \quad \mathbf{a}_{i,j} = \frac{d^{i-M_{k-1}}}{dx^{i-M_{k-1}}} \left\{w_i(x) f_k^{j+y}(x)\right\},$$

where $j \in [1,m]$ and $i \in [M_{k-1}, M_k)$ for $k \in [1,n]$. Then we have the determinantal evaluation

$$\mathbf{A} = \prod_{i=1}^{m} w_i(x) \prod_{1\le\imath<\jmath\le n} \{f_\jmath(x) - f_\imath(x)\}^{m_\imath m_\jmath} \times$$

$$\times \prod_{k=1}^{n} f_k^{(1+y)m_k}(x) \left\{f_k'(x)\right\}^{\binom{m_k}{2}} \prod_{j=1}^{m_k} (j-1)!.$$

When $y = -1$ and $w_i(x) \equiv 1$ for $1 \le i \le m$, the following particular cases are worthy of mention.

Case 1: $m_1 = m_2 = \ldots = m_n = 1$. The Vandermonde determinant.

Case 2: $n = 1$. The determinant discovered by Mina [4] (see also van der Poorten [5])

$$\det_{1\le i,j\le n} \left[\frac{d^{i-1}}{dx^j} f^{j-1}(x)\right] = \left\{f'(x)\right\}^{\binom{n}{2}} \prod_{k=1}^{n} (k-1)!.$$

It has been generalized by Wilf et al. [6, 7] to the formula

$$\det_{1\le i,j\le n} \left[\frac{d^{i-1}}{dx^{i-1}} f^{y_j}(x)\right] = \left\{\frac{f'(x)}{f(x)}\right\}^{\binom{n}{2}} \Delta(y) \prod_{k=1}^{n} f^{y_k}(x).$$

Case 3: $n = 2$. The problem proposed by Aharonov and Elias [1], which may be reproduced as follows: for the square matrix of order $m + n$ defined by

$$a_{i,j}(x) = \begin{cases} \dfrac{d^{i-1}}{dx^{i-1}} f^{j-1}(x), & 1 \le i \le m, \\ \dfrac{d^{i-m-1}}{dx^{i-m-1}} g^{j-1}(x), & m < i \le m + n, \end{cases}$$

its determinant is evaluated as follows (where the product expression has been corrected by removing an extra factor):

$$\det_{1 \le i,j \le m+n} \left[a_{i,j}(x) \right] = \left\{ f'(x) \right\}^{\binom{m}{2}} \left\{ g'(x) \right\}^{\binom{n}{2}} \times$$

$$\times \left\{ g(x) - f(x) \right\}^{mn} \prod_{i=1}^{m-1} i! \prod_{j=1}^{n-1} j!.$$

Following exactly the same proving procedure, we can evaluate, by means of the second formula of Lemma 1, another Wronskian determinant with the rows being partitioned into n blocks.

Theorem 3. *For each $k \in [1, n]$ and $i \in (M_{k-1}, M_k]$, let $f_k(x)$ be a m-times differentiable function and $w_i(x)$ a polynomial of degree $< i - M_{k-1}$. Define the determinant $\mathbf{B}$ by higher derivatives*

$$\mathbf{B} := \det_{1 \le i,j \le m} \left[b_{i,j} \right] \quad \textit{with} \quad \mathbf{b}_{i,j} = \frac{d^{i-M_{k-1}}}{dx^{i-M_{k-1}}} \left\{ w_i(x) f_k^{j+y}(x) \right\},$$

where $j \in [1, m]$ and $i \in (M_{k-1}, M_k]$ for $k \in [1, n]$. Then we have the determinantal evaluation

$$\mathbf{B} = \prod_{i=1}^{m} (y + i) w_i(x) \prod_{1 \le \imath < \jmath \le n} \left\{ f_\jmath(x) - f_\imath(x) \right\}^{m_\imath m_\jmath} \times$$

$$\times \prod_{k=1}^{n} f_k^{y m_k}(x) \left\{ f_k'(x) \right\}^{\binom{m_k+1}{2}} \prod_{j=1}^{m_k} (j-1)!.$$

When $y = 0$ and $w_i(x) \equiv 1$ for $1 \le i \le m$, three cases are given below.

Case 1: $m_1 = m_2 = \ldots = m_n = 1$. It is equivalent to the Vandermonde determinant.

Case 2: $n = 1$. We get the determinant

$$\det_{1 \le i,j \le n} \left[\frac{d^i}{dx^j} f^j(x) \right] = \left\{ f'(x) \right\}^{\binom{n+1}{2}} \prod_{k=1}^{n} k!.$$

This is a variant of Mina's determinant which was extended by the author [2] (Proposition 2.4) to the following one:

$$\det_{1 \le i,j \le n} \left[\frac{d^i}{dx^i} f^{y_j}(x) \right] = \left\{ \frac{f'(x)}{f(x)} \right\}^{\binom{n+1}{2}} \Delta(y) \prod_{k=1}^{n} y_k f^{y_k}(x).$$

Case 3: $n = 2$. A variant of the determinant by Aharonov and Elias [1]. For the square matrix of order $m + n$ defined by

$$b_{i,j}(x) = \begin{cases} \dfrac{d^i}{dx^i} f^j(x), & 1 \le i \le m, \\ \dfrac{d^{i-m}}{dx^{i-m}} g^j(x), & m < i \le m+n, \end{cases}$$

there exists the determinantal evaluation

$$\det_{1 \le i,j \le m+n} \Big[b_{i,j}(x) \Big] = (m+n)! \, \{f'(x)\}^{\binom{m+1}{2}} \, \{g'(x)\}^{\binom{n+1}{2}} \times$$

$$\times \{g(x) - f(x)\}^{mn} \prod_{i=1}^{m-1} i! \prod_{j=1}^{n-1} j!.$$

***Proof of Theorem* 3.** Recalling Theorem 1, we can express the determinant in question as the multiple sum

$$\mathbf{B} = \sum_{\substack{\uplus_{k=1}^n \sigma_k = [1,m] \\ |\sigma_k| = m_k : 1 \le k \le n}} \varepsilon(\sigma) \prod_{k=1}^n \det_{1 \le i,j \le m_k} \left[\frac{d^i}{dx^i} \Big\{ w_{i+M_{k-1}}(x) f_k^{y+\lambda_j(k)}(x) \Big\} \right].$$

According to the second formula of Lemma 1, we can evaluate the minor

$$\mathbf{B}\Big[(M_{k-1}, M_k] | \sigma_k\Big] = \det_{1 \le i,j \le m_k} \left[\frac{d^i}{dx^i} \Big\{ w_{i+M_{k-1}}(x) f_k^{y+\lambda_j(k)}(x) \Big\} \right] =$$

$$= \left\{ \frac{f_k'(x)}{f_k(x)} \right\}^{\binom{m_k+1}{2}} \Delta(\sigma_k) \prod_{i=1}^{m_k} w_{i+M_{k-1}}(x) \, \{y + \lambda_i(k)\} \, f_k^{y+\lambda_i(k)}(x),$$

which leads us to the equality

$$\mathbf{B} = \sum_{\substack{\uplus_{k=1}^n \sigma_k = [1,m] \\ |\sigma_k| = m_k : 1 \le k \le n}} \varepsilon(\sigma) \prod_{k=1}^n \left\{ \frac{f_k'(x)}{f_k(x)} \right\}^{\binom{m_k+1}{2}} \Delta(\sigma_k) \times$$

$$\times \prod_{i=1}^{m_k} w_{i+M_{k-1}}(x) \, \{y + \lambda_i(k)\} \, f_k^{y+\lambda_i(k)}(x) =$$

$$= \prod_{i=1}^m (y+i) w_i(x) \prod_{k=1}^n f_k^{y m_k}(x) \left\{ \frac{f_k'(x)}{f_k(x)} \right\}^{\binom{m_k+1}{2}} \times$$

$$\times \sum_{\substack{\uplus_{k=1}^n \sigma_k = [1,m] \\ |\sigma_k| = m_k : 1 \le k \le n}} \varepsilon(\sigma) \prod_{k=1}^n \Delta(\sigma_k) f_k^{\|\sigma_k\|}(x).$$

Evaluating the last multiple sum by Corollary 1, we complete the proof of the determinant formula displayed in Theorem 3.

3. Matrices involving formal power series. In this section, two analogous determinants with matrix entries involving formal power series will be evaluated. For subsequent application, we reproduce first the following determinant identity.

Lemma 2 ([2], Theorem 3.3)**.** *Let F be a unitary formal power series defined by $F(x) :=$ $:= x + \phi x^2 + \ldots$. The following determinant identity on composition series holds:*

$$\det_{1\le i,j\le n}\Big[[x^{i+1}]F^{\langle y_j\rangle}(x)\Big] = \phi^{\binom{n+1}{2}} \prod_{1\le i<j\le n}(y_j-y_i)\prod_{k=1}^{n} y_k.$$

Let $F_k(x)$ be the unitary formal power series with the two initial terms of F_k being given explicitly by

$$[x]F_k(x)=1 \qquad \text{and} \qquad [x^2]F_k(x)=\phi_k. \tag{3}$$

Define the following determinant C with its rows being divided into n blocks:

$$C := \det_{1\le i,j\le m}\big[c_{i,j}\big] \quad \text{with} \quad c_{i,j} = [x^{1+i-M_{k-1}}]\beta_k^{y_j}F_k^{\langle y_j\rangle}(x) \text{ if } i\in(M_{k-1},M_k],$$

where $\{\beta_k\}_{k=1}^n$ are constants independent of x. Applying Theorem 1 to the last determinant, we get

$$C := \sum_{\substack{\uplus_{k=1}^n\sigma_k=[1,m]\\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}\det_{1\le i,j\le m_k}\Big[[x^{i+1}]\beta_k^{y_{\lambda_j(k)}}F_k^{\langle y_{\lambda_j(k)}\rangle}(x)\Big].$$

Evaluating, by means of Lemma 2, the minor of order m_k

$$C\Big[(M_{k-1},M_k]|\sigma_k\Big] = \det_{1\le i,j\le m_k}\Big[[x^{i+1}]F_k^{\langle y_{\lambda_j(k)}\rangle}(x)\Big]\prod_{\lambda\in\sigma_k}\beta_k^{y_\lambda} =$$

$$= \phi_k^{\binom{m_k+1}{2}}\Delta(y|\sigma_k)\prod_{\lambda\in\sigma_k} y_\lambda\beta_k^{y_\lambda},$$

we find the equality

$$C = \sum_{\substack{\uplus_{k=1}^n\sigma_k=[1,m]\\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}\phi_k^{\binom{m_k+1}{2}}\Delta(y|\sigma_k)\prod_{\lambda\in\sigma_k} y_\lambda\beta_k^{y_\lambda} =$$

$$= \prod_{i=1}^{m} y_i\prod_{k=1}^{n}\phi_k^{\binom{m_k+1}{2}} \sum_{\substack{\uplus_{k=1}^n\sigma_k=[1,m]\\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}\Delta(y|\sigma_k)\prod_{\lambda\in\sigma_k}\beta_k^{y_\lambda}.$$

When $y_j = y+j$ for $j\in[1,m]$, the expression displayed above reads as

$$\prod_{i=1}^{m}(y+i)\prod_{k=1}^{n}\beta_k^{ym_k}\phi_k^{\binom{m_k+1}{2}} \sum_{\substack{\uplus_{k=1}^n\sigma_k=[1,m]\\ |\sigma_k|=m_k:1\le k\le n}} \varepsilon(\sigma)\prod_{k=1}^{n}\Delta(\sigma_k)\beta_k^{\|\sigma_k\|}.$$

In view of Corollary 1, the last multiple sum admits a closed expression. Therefore, we establish the following theorem.

Theorem 4. *For the unitary formal power series* $F_k(x)$ *subject to condition* (3), *define the determinant* $\mathbf{C}$ *by coefficients of composite series*

$$\mathbf{C} := \det_{1\le i,j\le m}\left[\mathbf{c}_{i,j}\right] \text{ with } \mathbf{c}_{i,j} = [x^{1+i-M_{k-1}}]\left\{\beta_k^{y+j}F_k^{\langle y+j\rangle}(x)\right\},$$

where $j\in[1,m]$ *and* $i\in(M_{k-1},M_k]$ *for* $k\in[1,n]$. *Then we have the determinantal evaluation*

$$\mathbf{C} = \prod_{i=1}^{m}(y+i)\prod_{1\le\imath<\jmath\le n}\{\beta_\jmath-\beta_\imath\}^{m_\imath m_\jmath}\prod_{k=1}^{n}\beta_k^{ym_k}\big(\phi_k\beta_k\big)^{\binom{m_k+1}{2}}\prod_{j=1}^{m_k}(j-1)!.$$

It is curious to notice that the last determinant vanishes for $n>1$ if all the $\{\beta_k\}_{k=1}^n$ are identical. When $n=1$ and $y=0$, Theorem 4 recovers the following determinant identity discovered by Kedlaya [3]:

$$\det_{1\le i,j\le n}\Big[[x^{i+1}]F^{\langle j\rangle}(x)\Big] = \phi^{\binom{n+1}{2}}\prod_{k=1}^{n}k!,$$

where $F(x)$ is a unitary formal power series $F(x):=x+\phi x^2+\ldots$.

If the ith row of the matrix in Lemma 2 is replaced by the extraction of ith (instead of $(i+1)$th) coefficients of composition series, we have the following stronger result.

Lemma 3 ([2], Theorem 4.6)**.** *Let* $F(x)$, $G(x)$ *and* $W_i(x)$ *be three formal power series with the initial coefficients of* F *and* G *being given explicitly by*

$$[x]F(x)=1, \qquad [x^2]F(x)=\phi, \qquad \text{and} \qquad [x]G(x)=\psi.$$

Then the following determinant identity holds:

$$\det_{1\le i,j\le n}\Big[[x^i]\Big\{W_i(x)F^{\langle y_j\rangle}[G(x)]\Big\}\Big] = \phi^{\binom{n}{2}}\psi^{\binom{n+1}{2}}\prod_{1\le i<j\le n}(y_j-y_i)\prod_{k=1}^{n}W_k(0).$$

By following the same proof as that for Theorem 4 and invoking Lemma 3, we can evaluate another similar determinant involving formal power series in the matrix entries.

Theorem 5. *Let* $F_k(x)$, $G_k(x)$ *and* $W_i(x)$ *be the formal power series with the initial terms of* F_k *and* G_k *being given by*

$$[x]F_k(x)=1, \qquad [x^2]F_k(x)=\phi_k, \qquad \text{and} \qquad [x]G_k(x)=\psi_k.$$

For the constants $\{\beta_k\}_{k=1}^n$ *independent of* x, *define the determinant* $\mathbf{D}$ *by coefficients of composite series*

$$\mathbf{D} := \det_{1\le i,j\le m}\left[\mathbf{d}_{i,j}\right] \quad \text{with} \quad d_{i,j} = [x^{i-M_{k-1}}]\beta_k^{y+j}\left\{W_i(x)F_k^{\langle y+j\rangle}\left[G_k(x)\right]\right\},$$

where $j\in[1,m]$ *and* $i\in(M_{k-1},M_k]$ *for* $k\in[1,n]$. *Then we have the determinantal evaluation*

$$\mathbf{D} = \prod_{i=1}^{m}W_i(0)\prod_{1\le\imath<\jmath\le n}\{\beta_\jmath-\beta_\imath\}^{m_\imath m_\jmath}\prod_{k=1}^{n}\big(\beta_k^{1+y}\psi_k\big)^{m_k}\big(\phi_k\psi_k\beta_k\big)^{\binom{m_k}{2}}\prod_{j=1}^{m_k}(j-1)!.$$

If $G_k(x)\equiv x$ and $W_i(x)\equiv 1$, this theorem reduces to the following one which may be considered as a counterpart of Theorem 4.

***Corollary* 2.** *Let $F_k(x)$ be the unitary formal power series subject to condition* (3)*. Define the determinant T by*

$$T := \det_{1\le i,j\le m}\Big[t_{i,j}\Big] \quad \textit{with} \quad t_{i,j} = [x^{i-M_{k-1}}]\left\{\beta_k^{y+j}F_k^{\langle y+j\rangle}(x)\right\},$$

where $j \in [1,m]$ and $i \in (M_{k-1}, M_k]$ for $k \in [1,n]$. Then the following determinant identity holds:

$$T = \prod_{1\le\imath<\jmath\le n}\{\beta_\jmath - \beta_\imath\}^{m_\imath m_\jmath}\prod_{k=1}^{n}\beta_k^{(1+y)m_k}\big(\phi_k\beta_k\big)^{\binom{m_k}{2}}\prod_{j=1}^{m_k}(j-1)!.$$

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