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EXISTENCE RESULTS FOR A PERTURBED DIRICHLET PROBLEM WITHOUT SIGN CONDITION IN ORLICZ SPACES

РЕЗУЛЬТАТИ ІСНУВАННЯ ДЛЯ ЗБУРЕНОЇ ЗАДАЧІ ДІРІХЛЕ ЗА ВІДСУТНОСТІ УМОВ ЩОДО ЗНАКА В ПРОСТОРАХ ОРЛІЧА

We deal with the existence result for nonlinear elliptic equations related to the form

$$Au + g(x, u, \nabla u) = f,$$

where the term $-\mathrm{div}\Big(a(x, u, \nabla u)\Big)$ is a Leray – Lions operator from a subset of $W_0^1 L_M(\Omega)$ into its dual. The growth and coercivity conditions on the monotone vector field a are prescribed by an N-function M which does not have to satisfy a Δ_2-condition. Therefore we use Orlicz – Sobolev spaces which are not necessarily reflexive and assume that the nonlinearity $g(x, u, \nabla u)$ is a Carathéodory function satisfying only a growth condition with no sign condition. The right-hand side f belongs to $W^{-1}E_{\overline{M}}(\Omega)$.

Розглядається задача існування для нелінійних еліптичних рівнянь у формі

$$Au + g(x, u, \nabla u) = f,$$

де $-\mathrm{div}\Big(a(x, u, \nabla u)\Big)$ — оператор Лере – Ліонса з підмножини $W_0^1 L_M(\Omega)$ у її дуальну множину. Умови зростання та коерцитивності в монотонному векторному полі a визначаються N-функцією M, яка не повинна задовольняти Δ_2-умови. Тому ми використовуємо простори Орліча – Соболєва, які не обов'язково є рефлексивними, і припускаємо, що нелінійність $g(x, u, \nabla u)$ є функцією Каратеодорі, що задовольняє лише умову зростання без умови знака. Права частина f належить $W^{-1}E_{\overline{M}}(\Omega)$.

1. Introduction. In the last decade, there has been an increasing interest in the study of various mathematical problems in modular spaces. These problems have many consideration in applications [13, 33, 34] and have resulted in a renewal interest in Modular spaces, the origins of which can be traced back to the work of Orlicz in the 1930s. In the 1950s, this study was carried on by Nakano [29]. Later on, Polish and Czechoslovak mathematicians investigated the modular function spaces (see, for instance, [25, 28]).

One of our motivations to study nonlinear problems in modular spaces comes from applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to highly change in their mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Růžička [32, 33], we refer for instance to [4, 9, 21 – 24, 27] for different non-standard growth conditions. Another important application is related to image processing [30] where this kind of diffusion operator is used to underline the borders of the distorted image and to eliminate the noise.

In this paper, we are interesting to prove an existence result for a nonlinear elliptic problem with nonlinearity. The studies will be undertaken for the case of rather general growth conditions for the highest order term. This formulation requires a general framework for the function space setting. The problems will be considered in Orlicz spaces. The level of generality of our considerations will have a crucial significance on the applied methods. This is a natural generalization of the numerous recent studies appearing on Lebesgue and Sobolev spaces, which may be considered as a particular case of our approach.

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, we consider the following nonlinear elliptic problem:

$$\begin{aligned} &Au + g(x,u,\nabla u) = f \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $Au = -\operatorname{div}(a(x,u,\nabla u))$ is a Leray–Lions operator defined on $W^{1,p}(\Omega)$, $1 < p < \infty$. Bensoussan, Boccardo, and Murat [8] proved the existence of solutions for the Dirichlet problem associated to the problem (1.1), where g is a nonlinearity satisfying the following (natural) growth condition:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^p),$$

and the sign condition $g(x,s,\xi)s \geq 0$. In the case, where the right-hand side f is assumed to belong to $W^{-1,p'}(\Omega)$ and g depends only on x and u, see the result of Brézis and Browder in [12]. In [31] Porretta has proved the existence result for the problem (1.1) but the result is restricted to $b(.) \equiv 1$ in Sobolev spaces, and in the case with $b(|s|) \leq \beta |s|^{r-1}$where $0 \leq r < p$ the same problem has been studied by Benkirane et al. in [7]. A different approach (without sign condition) was used in [10], under the assumption $g(x,s,\xi) = \lambda s - |\xi|^2$, with $\lambda > 0$.

We recall also that the authors used in [10] the methods of lower and uper solutions. In the literature of the same problems, the sign condition play a crucial role in the proof of the main result.

In [19], Gossez and Mustonen solved (1.1) in the case where g satisfies the classical sign condition $g(x,s).s \geq 0$ and data f in $W^{-1}E_{\overline{M}}(\Omega)$.

We find also some existence results in the same context for strongly nonlinear problem associated to (1.1) proved in [3, 5, 6, 16] when data f belongs either to $W^{-1}E_{\overline{M}}(\Omega)$ or $L^1(\Omega)$ with M satisfies Δ_2-condition. In the case where the Δ_2-condition is not fulfilled the above problem was studied in [2, 14, 15].

In the present paper, we deal with the existence result for the following problem:

$$(P) \begin{cases} u \in W_0^1 L_M(\Omega), g(x,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega (a(x,u,\nabla u))\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u) T_k(u-v)\,dx \leq \\ \displaystyle\leq \int_\Omega f T_k(u-v)\,dx \\ \text{for every } v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) \text{ and for all } k > 0, \end{cases}$$

where $f \in W^{-1}E_{\overline{M}}(\Omega)$.

Note that the sign condition on a nonlinearity plays a crucial role to obtain a priori estimates and existence of solution, to overcome the difficulty of the elimination of the sign condition we use the following growth condition:

$$|g(x, s, \xi)| \leq b(|s|) + h(|s|)M(|\xi|) \tag{1.2}$$

with $h \in L^1(\mathbb{R}^+)$, $b(|s|) \leq \overline{P}^{-1} P(|s|)$ where M and P are two N-functions such that $P \ll M$.

In [2], the authors assume the same growth condition (1.2) on nonlinearities but the function b depends only on x not on u and belongs to $L^1(\Omega)$.

The main novelty of the paper is that the nonlinearity g does not have to satisfy any sign condition, beside this we have the function b depends on solution u of our problem.

Our principal goal in this paper is to prove the existence result for the problem (P) but without assuming any sign condition on nonlinearities and any restriction on the N-function M of Orlicz spaces.

This paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we state and prove our general results.

2. Preliminaries. Let $M\colon \mathbb{R} \to \mathbb{R}$ be an N-function, i.e., M is even, continuous and convex function, with $M(t) > 0$ for $t > 0$, $\dfrac{M(t)}{t} \to 0$ as $t \to 0$ and $\dfrac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation $M(t) = \displaystyle\int_0^{|t|} m(s)\, ds$, where $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \displaystyle\int_0^{|t|} \overline{m}(s)\, ds$, where $\overline{m}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = = \sup\{s : m(s) \leq t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some $k > 0$,

$$M(2t) \leq k\, M(t) \quad \text{for all } t \geq 0.$$

When this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2-condition near infinity.

Let P and M be two N-functions. $P \ll M$ means that P grows essentially less rapidly than M, i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{M(\varepsilon t)} \to 0 \quad \text{as} \quad t \to \infty.$$

This is the case if and only if

$$\frac{M^{-1}(t)}{P^{-1}(t)} \to 0 \quad \text{as} \quad t \to \infty.$$

We will extend these N-functions into even functions on all $\mathbb{R}$. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \qquad \left(\text{resp., } \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \ \text{ for some } \ \lambda > 0\right).$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 \colon \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx \leq 1\right\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not.

We define the Orlicz norm $||u||_{(M)}$ by

$$||u||_{(M)} = \sup \int_{Q_T} u(x)v(x)\,dx,$$

where the supremum is taken over all $v \in E_{\overline{M}(\Omega)}$ such that $||v||_{\overline{M}} \leq 1$, for which

$$||u||_M \leq ||u||_{(M)} \leq 2||u||_M$$

holds for all $u \in L_{M(\Omega)}$.

We now turn to the Orlicz–Sobolev space. $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|\nabla^\alpha u\|_{M,\Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$, $\int_\Omega M((D^\alpha u_n - D^\alpha u)/\lambda)dx \to 0$ for all $|\alpha| \leq 1$ with $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$.

If M satisfies the Δ_2 condition on $\mathbb{R}^+$(near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [18]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

For $k > 0$, we define the truncation at height k, $T_K : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \min(K, \max(s, -k)) = \begin{cases} s, & \text{if } |s| \leq k, \\ \dfrac{ks}{|s|}, & \text{if } |s| > k. \end{cases}$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1 (cf. [18]). *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian with $F(0)=0$. Let M be an N-function and let $u\in W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$).*

Then $F(u)\in W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(u)\dfrac{\partial u}{\partial x_i}, & \text{a.e. in } \{x\in\Omega : u(x)\notin D\},\\ 0, & \text{a.e. in } \{x\in\Omega : u(x)\in D\}.\end{cases}$$

Lemma 2.2 (cf. [18]). *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F: W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$.*

Lemma 2.3 (cf. [20]). *Let $u_k,\ u\in L_M(\Omega)$. If $u_k\to u$ with respect to the modular convergence, then $u_k\to u$ for $\sigma(L_M,L_{\overline{M}})$.*

Below, we will use the following technical lemmas.

Lemma 2.4 (cf. [5]). *Let $(f_n), f\in L^1(\Omega)$, such that*

(i) $f_n\geq 0$ *a.e. in* Ω,

(ii) $f_n\to f$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega f_n(x)\,dx\to\int_\Omega f(x)\,dx.$

Then $f_n\to f$ strongly in $L^1(\Omega)$.

Lemma 2.5 (Young's inequality). *Let M be an N-function and $\bar{M}$ its conjugate. Then we have*

$$st\leq M(s)+\bar{M}\quad \text{for all}\quad s,t\geq 0.$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [5]).

Lemma 2.6. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P, Q be N-functions such that $Q\ll P$ and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that, for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$,*

$$|f(x,s)|\leq c(x)+k_1P^{-1}M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x)\in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x)=f(x,u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega),\frac{1}{k_2}\right)=\left\{u\in L_M(\Omega): d(u,E_M(\Omega))<\frac{1}{k_2}\right\}$$

into $E_Q(\Omega)$.

Lemma 2.7 [17]. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and M is an N-function, so there exist two positive constants δ and λ such that*

$$\int_\Omega M(\delta|v|)dx\leq\int_\Omega \lambda M(|\nabla v|)dx\quad \text{for all}\quad v\in W_0^1L_M(\Omega).$$

3. Main results. Throughout this paper, we assume that the following assumptions hold true:

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property, M be an N-function and P be an N-function such that $P \ll M$. We consider the Leray–Lions operator

$$Au = -\mathrm{div}(a(x, u, \nabla u)), \tag{3.1}$$

defined on $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$ where $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that, for a.e. $x \in \Omega$, for all $\zeta, \xi \in \mathbb{R}^N (\zeta \neq \xi)$ and for all $s \in \mathbb{R}$,

$$|a(x, s, \zeta)| \leq c(x) + k_1 \overline{M}^{-1} P(k_2 |s|) + k_3 \overline{M}^{-1} M(k_4 |\zeta|), \tag{3.2}$$

$$(a(x, s, \zeta) - a(x, s, \xi), \zeta - \xi) > 0, \tag{3.3}$$

$$a(x, s, \zeta)\zeta \geq \alpha M(|\zeta|) \tag{3.4}$$

with $\alpha > 0$ $k_1,\ k_2,\ k_3,\ k_4 \geq 0,\ c \in E_{\overline{M}}(\Omega)$.

Furthermore, let $g(x, s, \xi)$ be a Carathéodory function satisfying the following assumptions:

$$|g(x, s, \xi)| \leq b(|s|) + h(s) M(|\xi|) \tag{3.5}$$

with $h \in L^1(\mathbb{R}^+)$ and $b(|s|) \leq \overline{P}^{-1} P(|s|)$, where $b\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $P \ll M$.

Let us give and prove the following lemmas which will be needed later.

Lemma 3.1 (cf. [1]). *Assume that the assumptions* (3.2)–(3.4) *hold and let* (z_n) *be a sequence in* $W_0^1 L_M(\Omega)$ *such that*

$$z_n \rightharpoonup z \ \text{ in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)), \tag{3.6}$$

$$(a(x, z_n, \nabla z_n))_n \ \text{ is bounded in } \ (L_{\overline{M}}(\Omega))^N, \tag{3.7}$$

$$\int_\Omega \Big[a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z \chi_s)\Big]\Big[\nabla z_n - \nabla z \chi_s\Big]\, dx \longrightarrow 0 \tag{3.8}$$

as n *and* s *tend to* $+\infty$, *and where* χ_s *is the characteristic function of*

$$\Omega_s = \Big\{x \in \Omega;\ |\nabla z| \leq s\Big\}.$$

Then

$$\nabla z_n \to \nabla z \ \text{ a.e. in } \ \Omega,$$

$$\lim_{n\to\infty} \int_\Omega a(x, z_n, \nabla z_n) \nabla z_n \, dx = \int_\Omega a(x, z, \nabla z) \nabla z \, dx\,,$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \ \text{ in } \ L^1(\Omega).$$

***Remark* 3.1.** It should be interesting to note that the condition (3.7) is not necessary in the case where the N-function M satisfies the Δ_2-condition.

Lemma 3.2. *Let define the function* φ *as follows*:

$$\varphi(t) = c' t \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right)$$

with $\alpha, c' > 0,\ s, t \in \mathbb{R}$ *and* $h \in L^1(\mathbb{R}^+)$, *then*

$$\varphi'(t) - \frac{h(|t|)}{\alpha}|\varphi(t)| \geq c'.$$

Proof. If $t \geq 0$, then

$$\varphi'(t) = c' \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) + c' t \frac{h(|t|)}{\alpha} \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) =$$

$$= c' \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) + \frac{h(|t|)}{\alpha}|\varphi(t)|,$$

and if $t \leq 0$, then

$$\varphi'(t) = c' \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) + c' t \frac{h(|t|)}{\alpha} \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) =$$

$$= c' \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) + \frac{h(|t|)}{\alpha}|\varphi(t)|$$

which implies that

$$\varphi'(t) - \frac{h(|t|)}{\alpha}|\varphi(t)| = c' \exp\left(\int\limits_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right) \geq c'.$$

Lemma 3.2 is proved.

***Remark* 3.2.** If we have $u \in W^1 L_M(\Omega)$ by using Lemma 2.2, we can conclude that $\varphi(u)$ belongs to $W^1 L_M(\Omega)$.

Theorem 3.1. *Assume that the assumptions* (3.2)–(3.5) *hold and let* f *belongs to* $W^{-1} E_{\overline{M}}(\Omega)$. *Then there exists a measurable function* u *solution of the following problem*:

$$(P)\begin{cases} u \in W_0^1 L_M(\Omega), g(x, u, \nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega (a(x, u, \nabla u)) \nabla T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u) T_k(u - v)\, dx \leq \\ \displaystyle\leq \int_\Omega f T_k(u - v)\, dx \\ \textit{for every } v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) \textit{ and for every } k > 0. \end{cases}$$

Proof. The proof is divided into several steps, first we introduce a sequence of approximate problems and derive a priori estimates for the approximate problem and we show two intermediate results, namely, the the almost everywhere convergente of ∇u_n and the strong convergence in $L^1(\Omega)$ of the nonlinearity $g_n(x, u_n, \nabla u_n)$.

Let us consider the sequence of approximate problem

$$(P_n)\begin{cases} u_n \in W_0^1 L_M(\Omega), \\ \int_\Omega (a(x, u_n, \nabla u_n))\nabla T_k(u_n - v)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n - v)\, dx \leq \\ \leq \int_\Omega f T_k(u_n - v)\, dx \\ \text{for every } v \in W_0^1 L_M(\Omega), \end{cases}$$

where

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n}|g(x, s, \xi)|}.$$

Note that $g_n(x, s, \xi)$ satisfies the following conditions:

$$|g_n(x, s, \xi)| \leq |g(x, s, \xi)|, \qquad |g_n(x, s, \xi)| \leq n.$$

By the classical result of [19], the approximate problem (P_n) has at least one solution.

Lemma 3.3. *Let u_n be a solution of the problem (P_n), then we have*

$$\int_\Omega M(|\nabla u_n|)\, dx \leq C,$$

where C is a positive constant not depending on n.

Proof. Let $v = u_n - \varphi(u_n)$, where $\varphi(t) = t \exp\left(\int_0^{|t|} \frac{h(|s|)}{\alpha}\, ds\right)$ (the function h appears in (3.5)). Since $v \in W_0^1 L_M(\Omega)$, v is admissible test function in (P_n), then

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla(\varphi(u_n))\, dx + \int_\Omega g_n(x, u_n, \nabla u_n)\varphi(u_n)\, dx \leq \int_\Omega f\varphi(u_n)\, dx,$$

by (3.5) we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \varphi'(u_n)\, dx \leq \int_\Omega b(|u_n|)|\varphi(u_n)|\, dx + \int_\Omega h(|u_n|)|\varphi(u_n)| M(|\nabla u_n|)\, dx + \\ + \int_\Omega |f||\varphi(u_n)|\, dx.$$

Since $\varphi' \geq 0$ and by (3.4), we obtain

$$\alpha \int_\Omega M(|\nabla u_n|)\Big(\varphi'(u_n) - \frac{h(|u_n|)}{\alpha}|\varphi(u_n)|\Big)\, dx \leq c_0 \int_\Omega \overline{P}^{-1} P(|u_n|)|u_n|\, dx + c_0 \int_\Omega |f||u_n|\, dx$$

with $c_0 = \exp\left(\int_0^{+\infty} \frac{h(|s|)}{\alpha}\, ds\right)$.

By Lemma 3.2, we deduce

$$\alpha \int_\Omega M(|\nabla u_n|)\, dx \le c_0 \int_\Omega \overline{P}^{-1} P(|u_n|)|u_n|\, dx + c_0 \int_\Omega |f||u_n|\, dx. \tag{3.9}$$

Since $P \ll M$, for all $\varepsilon > 0$, there exists a constant that K_ε depending on ε such that

$$P(t) \le M(\varepsilon t) + K_\varepsilon \quad \forall t \ge 0 \tag{3.10}$$

without loss of generality, we can assume that $\varepsilon = \dfrac{\alpha\delta}{4c_0\lambda + \alpha(\delta+1)} < 1$, where δ and λ are two positive constants in Lemma 2.7 and $c_0 = \exp\left(\int_0^{+\infty} \frac{h(|s|)}{\alpha}\, ds\right)$, so, by convexity, we have

$$P(t) \le \varepsilon M(t) + K_\varepsilon \quad \forall t \ge 0. \tag{3.11}$$

By Young inequality and in view of (3.10), we deduce

$$c_0 \int_\Omega \overline{P}^{-1} P(|u_n|)|u_n|\, dx \le 2c_0 \int_\Omega M(\varepsilon |u_n|)\, dx + C^1_\varepsilon,$$

by Lemma 2.7 and the fact that $\dfrac{\varepsilon}{\delta} < 1$, we get

$$c_0 \int_\Omega \overline{P}^{-1} P(|u_n|)|u_n|\, dx \le \frac{2c_0\varepsilon\lambda}{\delta} \int_\Omega M(|\nabla u_n|)\, dx + C^1_\varepsilon. \tag{3.12}$$

On the other hand, f can be written as $f = f_0 - \operatorname{div} F$, where $f_0 \in E_{\overline{M}}(\Omega)$, $F \in (E_{\overline{M}}(\Omega))^N$, using Lemma 5.7 in [18] and Young's inequality we obtain

$$\begin{aligned} \int_\Omega f_0 u_n\, dx &\le C_1 + \frac{\alpha}{4} \int_\Omega M(|\nabla u_n|)\, dx, \\ \int_\Omega F \nabla u_n\, dx &\le C_2 + \frac{\alpha}{4} \int_\Omega M(|\nabla u_n|)\, dx. \end{aligned} \tag{3.13}$$

Using (3.17) and (3.13) in (3.9), we get

$$\alpha \int_\Omega M(|\nabla u_n|)\, dx \le \frac{2c_0\varepsilon\lambda}{\delta} \int_\Omega M(|\nabla u_n|)\, dx + \frac{\alpha}{2} \int_\Omega M(|\nabla u_n|)\, dx + C_\varepsilon,$$

which implies

$$\left(\frac{\alpha}{2} - \frac{2c_0\varepsilon\lambda}{\delta}\right) \int_\Omega M(|\nabla u_n|)\, dx \le C_\varepsilon,$$

we can easily verify that $\left(\dfrac{\alpha}{2}-\dfrac{2c_0\varepsilon\lambda}{\delta}\right)>0$, where $\varepsilon=\dfrac{\alpha\delta}{4c_0\lambda+\alpha(\delta+1)}$.

Thus,

$$\int\limits_{\Omega} M(|\nabla u_n|)\,dx \le C, \tag{3.14}$$

where C is a positive constant independent on n.

It follows that the sequence $\{u_n\}$ is bounded in $W_0^1L_M(\Omega)$. Consequently, there exist a subsequence on $\{u_n\}$, still denoted by $\{u_n\}$ and a function $u\in W_0^1L_M(\Omega)$ such that

$$u_n \rightharpoonup u \ \text{ in } \ W_0^1L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{3.15}$$

$$u_n \to u \ \text{ in } \ E_M(\Omega) \ \text{ strongly and a.e. in } \ \Omega. \tag{3.16}$$

Lemma 3.4. *Let u_n be a solution of the approximate problem (P_n). Then*

$$\Big(a(x,u_n,\nabla u_n)\Big)_n \ \textit{ is bounded in } \ (L_{\overline{M}}(\Omega))^N.$$

Proof. Let $\varphi\in(E_M(\Omega))^N$ be arbitrary. In view of the monotonicity of a, one easily has

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\varphi\,dx \le \int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n\,dx +$$

$$+\int\limits_{\Omega} a(x,u_n,\varphi)(\nabla u_n-\varphi)\,dx.$$

First, let take $v=u_n-u_ne^{G(u_n)}$ with $G(r)=\displaystyle\int_0^{|r|}\frac{h(|s|)}{\alpha}\,ds$ as a test function in (P_n), then

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla(u_ne^{G(u_n)})\,dx+\int\limits_{\Omega} g_n(x,u_n,\nabla u_n)u_ne^{G(u_n)}\,dx \le \int\limits_{\Omega} fu_ne^{G(u_n)}\,dx,$$

by (3.5), we get

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla u_ne^{G(u_n)}\,dx+\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n\frac{h(|u_n|)}{\alpha}e^{G(u_n)}u_n\,dx \le$$

$$\le \int\limits_{\Omega} b(|u_n|)u_ne^{G(u_n)}\,dx+\int\limits_{\Omega} h(|u_n|)u_ne^{G(u_n)}M(|\nabla u_n|)\,dx+\int\limits_{\Omega} |f||u_ne^{G(u_n)}|\,dx.$$

By (3.4) and the fact that $1\le e^{G(u_n)}\le c_0$, we obtain

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n\,dx+\int\limits_{\Omega} M(|\nabla u_n|)h(|u_n|)e^{G(u_n)}u_n\,dx \le$$

$$\le c_0\int\limits_{\Omega} b(|u_n|)|u_n|\,dx+\int\limits_{\Omega} M(|\nabla u_n|)h(|u_n|)u_ne^{G(u_n)}\,dx+\int\limits_{\Omega} |f||u_n|\,dx,$$

with $c_0 = \exp\left(\int_0^{+\infty} \frac{h(|s|)}{\alpha}\, ds\right)$, which gives

$$\int\limits_\Omega a(x, u_n, \nabla u_n)\nabla u_n\, dx \le c_0 \int\limits_\Omega b(|u_n|)|u_n|\, dx + c_0 \int\limits_\Omega |f||u_n|\, dx,$$

as in (3.9) and by (3.14), we have

$$\int\limits_\Omega a(x, u_n, \nabla u_n)\nabla u_n\, dx \le C_a.$$

Otherwise, for λ large enough, we get, by using (3.2) and convexity of $\overline{M}$,

$$\int\limits_\Omega \overline{M}\left(\frac{a(x, u_n, \varphi)}{\lambda}\right) dx \le \int\limits_\Omega \overline{M}\left(\frac{\beta\left[c(x) + k_1 \overline{M}^{-1} P(k_2|u_n|) + \overline{M}^{-1} M(k_3|\varphi|)\right]}{\lambda}\right) dx \le$$

$$\le \frac{\beta}{\lambda}\int\limits_\Omega \overline{M}(c(x)) + \frac{\beta k_1}{\lambda}\int\limits_\Omega \overline{M}\overline{M}^{-1} P(k_2|u_n|)\, dx + \frac{\beta}{\lambda}\int\limits_\Omega \overline{M}\overline{M}^{-1} M(k_3|\varphi|)\, dx \le$$

$$\le \frac{\beta}{\lambda}\int\limits_\Omega \overline{M}(c(x)) + \frac{\beta k_1}{\lambda}\int\limits_\Omega P(k_2|u_n|)\, dx + \frac{\beta}{\lambda}\int\limits_\Omega M(k_3|\varphi|)\, dx.$$

We have $\varphi \in (E_M(\Omega))^N$ and $c \in E_{\overline{M}}(\Omega)$, then

$$\int\limits_\Omega \overline{M}\left(\frac{a(x, T_K(u_n), w)}{\lambda}\right) dx \le C_1 + C_2 \int\limits_\Omega P(k_2|u_n|)\, dx.$$

In view of (3.10), we can take ε small enough, the way that $\dfrac{\varepsilon k_2}{\delta} \le 1$, thus by Lemma 2.7 and the convexity of M, we get

$$C_2 \int\limits_\Omega P(k_2|u_n|)\, dx \le \frac{C_2 k_2 \varepsilon \lambda}{\delta} \int\limits_\Omega M(|\nabla u_n|)\, dx + C_\varepsilon(\Omega). \tag{3.17}$$

By (3.9) and (3.17), we obtain

$$\int\limits_\Omega \overline{M}\left(\frac{a(x, T_K(u_n), w)}{\lambda}\right) dx \le C_b.$$

Hence, since u_n is bounded in $W_0^1 L_M(\Omega)$, one easily deduce that $a(x, u_n, \nabla u_n)$ is a bounded sequence in $(L_{\overline{M}}(\Omega))^N$. Thus, up to a subsequence

$$a(x, u_n, \nabla u_n) \rightharpoonup \varphi_k \;\text{ in }\; (L_{\overline{M}}(\Omega))^N \;\text{ for }\; \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

for some $\varphi_k \in (L_{\overline{M}}(\Omega))^N$.

Lemma 3.4 is proved.

Lemma 3.5. *Let u_n be a solution of the problem (P_n), then we have*

$$\lim_{j\to\infty}\lim_{n\to\infty}\int\limits_{\{j\le|u_n|\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx=0.$$

Proof. Consider the function $v=u_n-e^{G(u_n)}T_1(u_n-T_j(u_n))^+$ for $j>1$, where $G(u_n)=$ $=\int_0^{|u_n|}\frac{h(|s|)}{\alpha}\,ds$. Then we obtain

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{h(|u_n|)}{\alpha}e^{G(u_n)}T_1(u_n-T_j(u_n))^+ +$$

$$+\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^+e^{G(u_n)}+$$

$$+\int\limits_\Omega g_n(x,u_n,\nabla u_n)e^{G(u_n)}T_1(u_n-T_j(u_n))^+\le\int\limits_\Omega fe^{G(u_n)}T_1(u_n-T_j(u_n))^+\,dx.$$

From the growth condition (3.5), we have

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{h(|u_n|)}{\alpha}e^{G(u_n)}T_1(u_n-T_j(u_n))^+ +$$

$$+\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^+e^{G(u_n)}\le$$

$$\le\int\limits_\Omega \overline{P}^{-1}P(|u_n|)e^{G(u_n)}T_1(u_n-T_j(u_n))^+ +$$

$$+\int\limits_\Omega h(|u_n|)M(|\nabla u_n|)e^{G(u_n)}T_1(u_n-T_j(u_n))^+ +\int\limits_\Omega fe^{G(u_n)}T_1(u_n-T_j(u_n))^+\,dx.$$

Thanks to (3.4), we get

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^+e^{G(u_n)}\le$$

$$\le\int\limits_{\{u_n>j\}} e^{G(u_n)}\overline{P}^{-1}P(|u_n|)T_1(u_n-T_j(u_n))^+ +\int\limits_\Omega fe^{G(u_n)}T_1(u_n-T_j(u_n))^+\,dx.$$

By Young's inequality and the fact that

$$1\le e^{G(u_n)}=\exp\left(\int\limits_0^{|u_n|}\frac{h(s)}{\alpha}\,ds\right)\le\exp\left(\int\limits_0^{+\infty}\frac{h(s)}{\alpha}\,ds\right)=c_0,$$

we obtain

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^+ \leq$$

$$\leq c_0 \int\limits_{\{u_n>j\}} (P(|u_n|)+P(T_1(u_n-T_j(u_n))^+)) + c_0 \int\limits_\Omega f T_1(u_n-T_j(u_n))^+\, dx.$$

By the inequality (3.11), we get

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^+ \leq$$

$$\leq c_0\varepsilon \int\limits_{\{u_n>j\}} \Big(M(|u_n|)+M(T_1(u_n-T_j(u_n))^+)+2K_\varepsilon\Big) + c_0 \int\limits_\Omega f T_1(u_n-T_j(u_n))^+\, dx. \quad (3.18)$$

In view of (3.15) and (3.16), we have

$$T_1(u_n-T_j(u_n))^+ \rightharpoonup T_1(u-T_j(u))^+ \text{ in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

In addition, since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L_M(\Omega)$, we deduce by Lebesgue's theorem that the right-hand side of the last inequality goes to zero as n and j tend to infinity. Then (3.18) becomes

$$\lim_{j\to\infty}\lim_{n\to\infty} \int\limits_{\{j\leq u_n\leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\, dx = 0. \quad (3.19)$$

Furthermore, consider the test function $v = u_n + e^{-G(u_n)}T_1(u_n-T_j(u_n))^-$ in (P_n), and reasoning as in the proof of (3.19), we deduce that

$$\lim_{j\to\infty}\lim_{n\to\infty} \int\limits_{\{-j-1\leq u_n\leq -j\}} a(x,u_n,\nabla u_n)\nabla u_n\, dx = 0. \quad (3.20)$$

Finally, combining (3.19) and (3.20), we have

$$\lim_{j\to\infty}\lim_{n\to\infty} \int\limits_{\{j\leq |u_n|\leq j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\, dx = 0. \quad (3.21)$$

Lemma 3.5 is proved.

Proposition 3.1. *Let u_n be a solution of the approximate problem (P_n). Then we have (for a subsequence noted again u_n)*

$$\nabla u_n \to \nabla u \;\; \text{a.e. in } \;\Omega$$

as n tends to $+\infty$.

Proof. We will use the following function of one real variable, which is defined as follows:

$$S_j(s) = \begin{cases} 1, & \text{if } \ |s|\leq j, \\ 0, & \text{if } \ |s|\geq j+1, \\ j+1-s, & \text{if } \ j\leq s\leq j+1, \\ s+j+1, & \text{if } \ -j-1\leq s\leq -j, \end{cases}$$

with j a nonnegative real parameter.

Let $\Omega_s = \{x \in \Omega, |\nabla T_k(u(x))| \leq s\}$ and denote by χ_s the characteristic function of Ω_s, clearly, $\Omega_s \subset \Omega_{s+1}$ and meas$(\Omega \setminus \Omega_s) \to 0$ as $s \to 0$. Let $v_i \in \mathcal{D}(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$. Using $v = u_n - \eta e^{-G(u_n)}(T_k(u_n) - T_k(v_i))^+ S_j(u_n)$ as a test function in (P_n), we obtain, by using (3.4) and (3.5),

$$\int\limits_{\{T_k(u_n)-T_k(v_i)\geq 0\}} e^{G(u_n)} a(x, u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(v_i))S_j(u_n)\, dx -$$

$$- \int\limits_{\{j\leq u_n\leq j+1\}} e^{G(u_n)} a(x, u_n, \nabla u_n)\nabla u_n (T_k(u_n) - T_k(v_i))^+\, dx \leq$$

$$\leq \int\limits_{\Omega} \overline{P}^{-1} P(|u_n|)(T_k(u_n) - T_k(v_i))^+ S_j(u_n) e^{G(u_n)}\, dx +$$

$$+ \int\limits_{\Omega} f(T_k(u_n) - T_k(v_i))^+ S_j(u_n) e^{G(u_n)}\, dx.$$

Thanks to (3.21) the second integral tend to zero as n and j tend to infinity, and by Lebesgue theorem, we deduce that the right-hand side converge to zero as n and j goes to infinity.

Using the same argument as in [2], we get

$$\int\limits_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\, dx \to 0.$$

By the Lemma 3.1, we obtain

$$M(|\nabla u_n|) \longrightarrow M(|\nabla u|) \quad \text{in} \quad L^1(\Omega). \tag{3.22}$$

Thanks to (3.22), we obtain for a subsequence

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in} \quad \Omega.$$

Proposition 3.1 is proved.

***Proof of Theorem* 3.1. *Step* 1.** Equi-integrability of the nonlinearities.
We show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \ \text{ strongly in } \ L^1(\Omega). \tag{3.23}$$

Let $v = u_n + e^{(-G(u_n))} \int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\, ds$. Since $v \in W_0^1 L_M(\Omega)$, v is an admissible test function in (P_n). Then

$$\int\limits_{\Omega} a(x, u_n, \nabla u_n)\nabla \left(-e^{(-G(u_n))} \int\limits_{u_n}^{0} h(s)\chi_{\{s<-l\}}\, ds \right) dx +$$

$$+\int_{\Omega} g_n(x,u_n,\nabla u_n)\left(-e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\right)\,dx \le$$

$$\le \int_{\Omega} f\left(-e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\right)\,dx,$$

which implies that, by using (3.5), we have

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \frac{h(u_n)}{\alpha} e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\,dx+$$

$$+\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n e^{(-G(u_n))}h(u_n)\chi_{\{u_n<-l\}}\,dx \le$$

$$\le \int_{\Omega} \overline{P}^{-1}P(|u_n|)e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\,dx+$$

$$+\int_{\Omega} h(u_n)M(|\nabla u_n|)e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\,dx - \int_{\Omega} f e^{(-G(u_n))}\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds\,dx.$$

By using (3.4) and since $\int_{u_n}^{0} h(s)\chi_{\{s<-l\}}\,ds \le \int_{-\infty}^{-l} h(s)\,ds$, we get

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n e^{(-G(u_n))}h(u_n)\chi_{\{u_n<-l\}}\,dx \le$$

$$\le e^{\left(\frac{\|h\|_{L^1(\mathbb{R})}}{\alpha}\right)}\int_{-\infty}^{-l} h(s)\,ds\left(\int_{\Omega}\overline{P}^{-1}P(|u_n|)+\int_{\Omega} f_0\,dx\right).$$

Applying Young inequality, (3.10) and using Lemma 2.7, one has

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n e^{(-G(u_n))}h(u_n)\chi_{\{u_n<-l\}}\,dx \le$$

$$\le e^{(\frac{\|h\|_{L^1(\mathbb{R})}}{\alpha})}\int_{-\infty}^{-l} h(s)\,ds\left(\frac{\varepsilon\lambda}{\delta}\int_{\Omega} M(|\nabla u_n|)\,dx + K'_{\varepsilon} + \int_{\Omega} f_0\,dx\right).$$

By using again (3.4) and the fact that u_n is bounded in $W_0^1L_M(\Omega)$, we obtain

$$\int_{\{u_n<-l\}} h(u_n)M(|\nabla u_n|)\,dx \le c_5\int_{-\infty}^{-l} h(s)\,ds,$$

and, since $h \in L^1(\mathbb{R})$, we deduce

$$\lim_{l \to +\infty} \sup_{n \in \mathbb{N}} \int\limits_{\{u_n < -l\}} h(u_n) M(|\nabla u_n|)\, dx = 0. \tag{3.24}$$

Otherwise, considering $v = u_n - e^{(-G(u_n))} \int_0^{u_n} h(s) \chi_{\{s > l\}}\, ds$ as a test function in (P_n). Thus, similarly to (3.24), we deduce

$$\lim_{l \to +\infty} \sup_{n \in \mathbb{N}} \int\limits_{\{u_n > l\}} h(u_n) M(|\nabla u_n|)\, dx = 0. \tag{3.25}$$

Moreover, on the set $\{u_n > l\}$ we have $1 < \dfrac{u_n}{l}$, then

$$\int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx \le \frac{1}{l} \int\limits_{\Omega} \overline{P}^{-1} P(|u_n|) u_n\, dx.$$

Applying Young inequality, one has

$$\int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx \le \frac{1}{l} \int\limits_{\Omega} (P(|u_n|) + P(|u_n|))\, dx.$$

In view of (3.10), we get

$$\int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx \le \frac{2}{l} \int\limits_{\Omega} (M(\varepsilon |u_n|) + K_\varepsilon)\, dx.$$

By using Lemma 2.7, we have

$$\int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx \le \frac{2\varepsilon\delta}{l\lambda} \int\limits_{\Omega} \Big(M(|\nabla u_n|) + K_\varepsilon \Big)\, dx.$$

By (3.14), we obtain

$$\int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx \le \frac{2\varepsilon\delta (C + C(\Omega))}{l\lambda},$$

then we conclude that

$$\lim_{l \to +\infty} \sup_{n \in \mathbb{N}} \int\limits_{\{u_n > l\}} \overline{P}^{-1} P(|u_n|)\, dx = 0. \tag{3.26}$$

Combining (3.5), (3.22), (3.24), (3.25), (3.26) and Vitali's theorem, we get (3.23).

***Step* 2.** Passing to the limit.

Let $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, we take $u_n - T_k(u_n - v)$ as test function in (P_n), we can write

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx+\int\limits_{\Omega} g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\le$$
$$\le\int\limits_{\Omega} fT_k(u_n-v)\,dx$$

which implies that

$$\int\limits_{\{|u_n-v|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx+\int\limits_{\{|u_n-v|\le k\}} a(x,T_{k+\|v\|_\infty}u_n,\nabla T_{k+\|v\|_\infty}(u_n))\nabla v\,dx+$$
$$+\int\limits_{\Omega} g_n(x,u_n,\nabla u_n)T_k(u_n-v)\,dx\le\int\limits_{\Omega} fT_k(u_n-v)\,dx.$$

By Fatou's lemma and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}},\Pi E_M)$, one easily see that

$$\int\limits_{\{|u-v|\le k\}} a(x,u,\nabla u)\nabla u\,dx-\int\limits_{\{|u-v|\le k\}} a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla v\,dx+$$
$$+\int\limits_{\Omega} g(x,u,\nabla u)T_k(u-v)\,dx\le\int\limits_{\Omega} fT_k(u-v)\,dx.$$

Hence,

$$\int\limits_{\Omega} a(x,u,\nabla u)\nabla T_k(u-v)\,dx+\int\limits_{\Omega} g(x,u,\nabla u)T_k(u-v)\,dx\le$$
$$\le\int_{\Omega} fT_k(u-v)\,dx\quad\forall v\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)\quad\forall k>0.$$

Theorem 3.1 is proved.

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