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ON SOME HERMITE – HADAMARD INEQUALITIES FOR FRACTIONAL INTEGRALS AND THEIR APPLICATIONS

ПРО ДЕЯКІ НЕРІВНОСТІ ЕРМІТА – АДАМАРА ДЛЯ ДРОБОВИХ ІНТЕГРАЛІВ ТА ЇХ ЗАСТОСУВАННЯ

We establish some new extensions of Hermite – Hadamard inequality for fractional integrals and present several applications for the Beta function.

Встановлено деякі нові розширення нерівності Ерміта – Адамара для дробових інтегралів та запропоновано кілька застосувань для бета-функції.

1. Introduction. Throughout in this paper, let $a \leq c < d \leq b$ in $\mathbb{R}$ with $a+b=c+d$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f\colon [a,b] \to \mathbb{R}$, is known in the literature as Hermite – Hadamard inequality [7].

For some results which generalize, improve, and extend inequality (1.1), see [1 – 6, 8 – 18].

In [4], Dragomir and Agarwal established the following results connected with the second inequality in inequality (1.1).

Theorem A. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\big(|f'(a)|+|f'(b)|\big)$$

which is the trapezoid inequality provided $|f'|$ is convex on $[a,b]$.

In [12], Kirmaci and Özdemir established the following results connected with the first inequality in (1.1).

Theorem B. *Under the assumptions of Theorem A, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\big(|f'(a)|+|f'(b)|\big)$$

which is the midpoint inequality provided $|f'|$ is convex on $[a,b]$.

In what follows we recall the following definition [14].

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Definition 1. *Let* $f \in L_1[a,b]$. *The Riemann–Liouville integrals* $J^{\alpha}_{a+}f$ *and* $J^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

In [14], Sarikaya et al. established the following Hermite–Hadamard-type inequalities for fractional integrals.

Theorem C. *Let* $f\colon [a,b] \to \mathbb{R}$ *be positive with* $0 \leq a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

for $\alpha > 0$.

Theorem D. *Under the assumptions of Theorem A, we have the following Hermite–Hadamard-type inequality for fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right]\right| \leq \frac{2^{\alpha}-1}{2^{\alpha+1}(\alpha+1)}(b-a)\big(\left|f'(a)\right| + \left|f'(b)\right|\big)$$

for $\alpha > 0$.

In [8], Hwang et al. established the following fractional integral inequality.

Theorem E. *Under the assumptions of Theorem A, we have the following Hermite–Hadamard-type inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \leq$$

$$\leq \frac{b-a}{4(\alpha+1)} \left(\alpha - 1 + \frac{1}{2^{\alpha-1}}\right) \big(|f'(a)| + \left|f'(b)\right|\big)$$

for $\alpha > 0$.

In [11], Hwang et al. established the following Hermite–Hadamard-type inequalities which are refinements and similar extensions of Theorems C–E.

Theorem F. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a convex function with* $a < b$. *Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \leq$$

$$\leq \frac{3^{\alpha}-1}{4^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leq$$

$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right] \leq$$

$$\leq \frac{3^{\alpha}-1}{2\cdot 4^{\alpha}}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\left[f(a)+f(b)\right] \leq$$

$$\leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

for $\alpha > 0$.

Theorem G. *Under the assumptions of Theorem A, we have the following inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\right.$$

$$\left.-\left\{\frac{3^{\alpha}-1}{2\cdot 4^{\alpha}}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\left[f(a)+f(b)\right]\right\}\right| \leq$$

$$\leq K(\alpha)(b-a)\big(|f'(a)|+|f'(b)|\big),$$

where

$$K(\alpha) := \frac{2^{\alpha}-1}{2^{\alpha+1}(\alpha+1)}-\frac{3^{\alpha}-1}{2\cdot 4^{\alpha+1}}$$

with $\alpha > 0$.

Theorem H. *Under the assumptions of Theorem A, we have the following inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\right.$$

$$\left.-\left\{\frac{3^{\alpha}-1}{4^{\alpha}} f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right\}\right| \leq$$

$$\leq \left(\frac{1}{8}-K(\alpha)\right)(b-a)\big(|f'(a)|+|f'(b)|\big),$$

where $K(\alpha)$ *is defined as in Theorem F and* $\alpha > 0$.

Remark **1.** 1. The assumptions $f\colon [a,b] \to \mathbb{R}$ is positive with $0 \leq a < b$ in Theorem C can be weakened as $f\colon [a,b] \to \mathbb{R}$ with $a < b$.

2. In Theorem D, let $\alpha = 1$. Then Theorem D reduces to Theorem A.

3. In Theorem E, let $\alpha = 1$. Then Theorem refte reduces to Theorem B.

In this paper, we establish some new extensions of Theorems D – H and present several applications for the Beta function.

2. New refinements of Hermite – Hadamard-type inequality for fractional integrals. In this section, we establish some inequalities which refine the inequality (1.2) and generalize the inequality (1.3).

Theorem 1. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function and let $a \le c < d \le b$ in $\mathbb{R}$ with $a+b = c+d$. Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{(b-c)^\alpha - (c-a)^\alpha}{(b-a)^\alpha} f\left(\frac{a+b}{2}\right) +$$

$$+\frac{(b-a)^\alpha + (c-a)^\alpha - (b-c)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big] \le$$

$$\le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\big] \le \frac{(b-c)^\alpha - (c-a)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]+$$

$$+\frac{(b-a)^\alpha + (c-a)^\alpha - (b-c)^\alpha}{2(b-a)^\alpha}\big[f(a)+f(b)\big] \le \frac{f(a)+f(b)}{2} \tag{2.1}$$

for $\alpha > 0$.

Proof. It is easily observed from the convexity of f that the first and last inequalities of (2.1) hold.

By using simple computation, we have the following identities:

$$\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\big] =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^b \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big] f(x)\,dx =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\big[f(x) + f(a+b-x)\big]\,dx+$$

$$+\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}} \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\big[f(x) + f(a+b-x)\big]dx, \tag{2.2}$$

$$\frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\,dx = \frac{(b-a)^\alpha + (c-a)^\alpha - (b-c)^\alpha}{2(b-a)^\alpha}, \tag{2.3}$$

$$\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}} \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\,dx = \frac{(b-c)^\alpha - (c-a)^\alpha}{2(b-a)^\alpha}, \tag{2.4}$$

$$c = \frac{a+b-x-c}{a+b-2x}x + \frac{c-x}{a+b-2x}(a+b-x) = \frac{d-x}{c+d-2x}x + \frac{c-x}{c+d-2x}(a+b-x) \tag{2.5}$$

and

$$d = \frac{a+b-x-d}{a+b-2x}x + \frac{d-x}{a+b-2x}(a+b-x) =$$

$$= \frac{c-x}{c+d-2x}x + \frac{d-x}{c+d-2x}(a+b-x), \tag{2.6}$$

where $x \in [a,c]$ with $0 \le \dfrac{c-x}{c+d-2x}, \dfrac{d-x}{c+d-2x} \le 1$,

$$\frac{a+b}{2} = \frac{1}{2}\big[x + (a+b-x)\big], \tag{2.7}$$

where $x \in \left[c, \dfrac{a+b}{2}\right]$,

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \tag{2.8}$$

and

$$a+b-x = \frac{x-a}{b-a}a + \frac{b-x}{b-a}b, \tag{2.9}$$

where $x \in [a,c]$,

$$x = \frac{d-x}{d-c}c + \frac{x-c}{d-c}d \tag{2.10}$$

and

$$a+b-x = \frac{d-a-b+x}{d-c}c + \frac{a+b-x-c}{d-c}d\frac{x-c}{d-c}c + \frac{d-x}{d-c}d, \tag{2.11}$$

where $x \in \left[c, \dfrac{a+b}{2}\right]$ with $0 \le \dfrac{x-c}{d-c}, \dfrac{d-x}{d-c} \le 1$.

Now, by using the above identities and the convexity of f, we have the following inequalities:

$$\frac{(b-a)^\alpha + (c-a)^\alpha - (b-c)^\alpha}{2(b-a)^\alpha}\big[f(c) + f(d)\big] =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\big[f(c)+f(d)\big]dx \le$$

$$\le \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\times$$

$$\times\left[\frac{d-x}{c+d-2x}f(x) + \frac{c-x}{c+d-2x}f(a+b-x) + \right.$$

$$\left. + \frac{c-x}{c+d-2x}f(x) + \frac{d-x}{c+d-2x}f(a+b-x)\right]dx =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c \big[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\big]\big[f(x)+f(a+b-x)\big]dx \tag{2.12}$$

by identities and (2.5), (2.6),

$$\frac{(b-c)^\alpha-(c-a)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)=$$

$$=\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]2f\left(\frac{a+b}{2}\right)dx\leq$$

$$\leq\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\left[f(x)+f(a+b-x)\right]dx \tag{2.13}$$

by identities (2.4) and (2.7),

$$\frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\left[f(x)+f(a+b-x)\right]dx\leq$$

$$\leq\frac{\alpha}{2(b-a)^\alpha}\int\limits_a^c\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\times$$

$$\times\left[\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)+\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)\right]dx=$$

$$=\frac{\alpha\left[f(a)+f(b)\right]}{2(b-a)^\alpha}\int\limits_a^c\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]dx=$$

$$=\frac{(b-c)^\alpha-(c-a)^\alpha}{2(b-a)^\alpha}\left[f(a)+f(b)\right] \tag{2.14}$$

by identities (2.3) and (2.8), (2.9),

$$\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\left[f(x)+f(a+b-x)\right]dx\leq$$

$$\leq\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\times$$

$$\times\left[\frac{d-x}{d-c}f(c)+\frac{x-c}{d-c}f(d)+\frac{x-c}{d-c}f(c)+\frac{d-x}{d-c}f(d)\right]dx=$$

$$=\frac{\alpha}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]\left[f(c)+f(d)\right]dx=$$

$$=\frac{(b-c)^\alpha-(c-a)^\alpha}{2(b-a)^\alpha}\left[f(c)+f(d)\right] \tag{2.15}$$

by identities (2.4) and (2.10), (2.11).

The second and third inequalities of (2.1) follow from identity (2.2) and inequalities (2.12) – (2.15).

Theorem 1 is proved.

***Remark* 2.** In Theorem 1, inequality (2.1) refines Hermite – Hadamard-type inequality (1.2).

***Corollary* 1.** *In Theorem* 1, *let* $c = (1-\beta)a + \beta b$ *and* $d = \beta a + (1-\beta)b$ *with* $0 \le \beta < \dfrac{1}{2}$. *Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \le [(1-\beta)^\alpha - \beta^\alpha]\, f\left(\frac{a+b}{2}\right) +$$

$$+\big[1-(1-\beta)^\alpha + \beta^\alpha\big]\frac{f((1-\beta)a+\beta b) + f(\beta a + (1-\beta)\, b)}{2} \le$$

$$\le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\big] \le$$

$$\le \big[(1-\beta)^\alpha - \beta^\alpha\big]\frac{f((1-\beta)a+\beta b) + f(\beta a + (1-\beta) b)}{2} +$$

$$+\big[1-(1-\beta)^\alpha + \beta^\alpha\big]\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}.$$

***Remark* 3.** In Corollary 1, let $\beta = \dfrac{1}{4}$. Then Corollary 1 reduces to Theorem F.

***Remark* 4.** In Theorem 1, let $\alpha = 1$. Then we have the inequality

$$f\left(\frac{a+b}{2}\right) \le$$

$$\le (1-2\beta)\, f\left(\frac{a+b}{2}\right) + \beta\big[f((1-\beta)a+\beta b) + f(\beta a + (1-\beta)\, b)\big] \le$$

$$\le \frac{1}{b-a}\int_a^b f(x)\, dx \le$$

$$\le \frac{1-2\beta}{2}\big[f((1-\beta)a+\beta b) + f\big(\beta a + (1-\beta) b\big)\big] + \beta\big[f(a)+f(b)\big] \le$$

$$\le \frac{f(a)+f(b)}{2}$$

which refines Hermite – Hadamard inequality (1.1).

3. Some extended inequalities for fractional integrals. In this section, we establish two theorems which are similar extensions of Theorems A – B and D – E.

Theorem 2. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a convex function and let* $a \le c < d \le b$ *in* $\mathbb{R}$ *with* $a+b = c+d$. *Then we have the following inequality for fractional integrals*:

$$\Bigg|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\big] -$$

$$-\left(\frac{(b-c)^\alpha - (c-a)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big] + \frac{(b-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha}{2(b-a)^\alpha}[f(a)+f(b)]\right)\Bigg| \le$$

$$\leq H_\alpha(c)(b-a)\big(|f'(a)|+|f'(b)|\big), \tag{3.1}$$

where

$$H_\alpha(c) := \frac{2^\alpha-1}{2^{\alpha+1}(\alpha+1)} - \frac{c-a}{2(b-a)}\left[\left(\frac{b-c}{b-a}\right)^\alpha - \left(\frac{c-a}{b-a}\right)^\alpha\right]$$

with $\alpha > 0$.

Proof. Define

$$h_1(x) = \begin{cases} (b-x)^\alpha-(x-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha, & x\in[a,c),\\ (b-x)^\alpha-(x-a)^\alpha, & x\in[c,d),\\ (b-x)^\alpha-(x-a)^\alpha+(b-c)^\alpha-(c-a)^\alpha, & x\in[d,b]. \end{cases}$$

By using the integration by parts, we have the following identities:

$$\frac{1}{2(b-a)^\alpha}\int_a^b h_1(x)f'(x)\,dx =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^b \big[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\big]f(x)\,dx-$$

$$-\left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]+\right.$$

$$\left.+\frac{(b-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha}{2(b-a)^\alpha}\big[f(a)+f(b)\big]\right\} =$$

$$= \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big] - \left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]+\right.$$

$$\left.+\frac{(b-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha}{2(b-a)^\alpha}\,[f(a)+f(b)]\right\} =$$

$$= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big] - \left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]+\right.$$

$$\left.+\frac{(b-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha}{2(b-a)^\alpha}\big[f(a)+f(b)\big]\right\}, \tag{3.2}$$

$$\int_a^c \big[(b-x)^\alpha-(x-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha\big]\frac{b-x}{b-a}\big|f'(a)\big|dx+$$

$$+\int_d^b \big[(x-a)^\alpha-(b-x)^\alpha-(b-c)^\alpha+(c-a)^\alpha\big]\frac{b-x}{b-a}\big|f'(a)\big|dx =$$

$$= \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{b-x}{b-a} \big|f'(a)\big| dx +$$

$$+ \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{x-a}{b-a} \big|f'(a)\big| dx =$$

$$= \big|f'(a)\big| \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] dx = P_1, \tag{3.3}$$

where

$$P_1 := \big|f'(a)\big| \Bigg\{ \frac{1}{\alpha+1} \big[(b-a)^{\alpha+1} - (b-c)^{\alpha+1} - (c-a)^{\alpha+1}\big] - \\ - (c-a)\big[(b-c)^\alpha - (c-a)^\alpha\big] \Bigg\},$$

$$\tag{3.4}$$

$$\int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{x-a}{b-a} \big|f'(b)\big| dx +$$

$$+ \int_d^b \big[(x-a)^\alpha - (b-x)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{x-a}{b-a} \big|f'(b)\big| dx =$$

$$= \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{x-a}{b-a} \big|f'(b)\big| dx +$$

$$+ \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] \frac{b-x}{b-a} \big|f'(b)\big| dx =$$

$$= \big|f'(b)\big| \int_a^c \big[(b-x)^\alpha - (x-a)^\alpha - (b-c)^\alpha + (c-a)^\alpha\big] dx = P_2,$$

where

$$P_2 := \big|f'(b)\big| \Bigg\{ \frac{1}{\alpha+1} \big[(b-a)^{\alpha+1} - (b-c)^{\alpha+1} - (c-a)^{\alpha+1}\big] - \\ - (c-a)\big[(b-c)^\alpha - (c-a)^\alpha\big] \Bigg\},$$

$$\tag{3.5}$$

$$\int_c^{\frac{a+b}{2}} \big[(b-x)^\alpha - (x-a)^\alpha\big] \frac{b-x}{b-a} \left|f'(a)\right| dx +$$

$$+\int\limits_{\frac{a+b}{2}}^{d}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx=$$

$$=\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|dx+$$

$$+\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|dx=$$

$$=\left|f'(a)\right|\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx=P_3,$$

where

$$P_3:=\frac{\left|f'(a)\right|}{\alpha+1}\left[(b-c)^{\alpha+1}+(c-a)^{\alpha+1}-\frac{(b-a)^{\alpha+1}}{2^{\alpha}}\right],$$

$$\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx+ \tag{3.6}$$

$$+\int\limits_{\frac{a+b}{2}}^{d}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx=$$

$$=\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|dx+$$

$$+\int\limits_{c}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|dx=$$

$$=\left|f'(b)\right|\int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx=P_4,$$

where

$$P_4:=\frac{\left|f'(b)\right|}{\alpha+1}\left[(b-c)^{\alpha+1}+(c-a)^{\alpha+1}-\frac{(b-a)^{\alpha+1}}{2^{\alpha}}\right].$$

Now, by using simple computation and identities (2.8) and (3.3) – (3.6), we have the inequality

$$\left|\frac{1}{2(b-a)^\alpha}\int\limits_a^b h_1(x)f'(x)\,dx\right| \leq \frac{1}{2(b-a)^\alpha}\int\limits_a^b |h_1(x)|\left|f'(x)\right|dx =$$

$$= \frac{1}{2(b-a)^\alpha}\int\limits_a^c \left[(b-x)^\alpha-(x-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha\right]\left|f'(x)\right|dx+$$

$$+\frac{1}{2(b-a)^\alpha}\int\limits_c^{\frac{a+b}{2}}\left[(b-x)^\alpha-(x-a)^\alpha\right]\left|f'(x)\right|dx+$$

$$+\frac{1}{2(b-a)^\alpha}+\int\limits_{\frac{a+b}{2}}^d\left[(x-a)^\alpha-(b-x)^\alpha\right]\left|f'(x)\right|dx+$$

$$+\frac{1}{2(b-a)^\alpha}\int\limits_d^b\left[(x-a)^\alpha-(b-x)^\alpha-(b-c)^\alpha+(c-a)^\alpha\right]\left|f'(x)\right|dx \leq$$

$$\leq \frac{P_1+P_2+P_3+P_4}{2(b-a)^\alpha} = \frac{(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)}{2}\times$$

$$\times\left\{\frac{1}{\alpha+1}\left[1-\left(\frac{b-c}{b-a}\right)^{\alpha+1}-\left(\frac{c-a}{b-a}\right)^{\alpha+1}\right]-\frac{c-a}{b-a}\left[\left(\frac{b-c}{b-a}\right)^{\alpha}-\left(\frac{c-a}{b-a}\right)^{\alpha}\right]\right\}+$$

$$+\frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{2(\alpha+1)}\left(\left(\frac{b-c}{b-a}\right)^{\alpha+1}+\left(\frac{c-a}{b-a}\right)^{\alpha+1}-\frac{1}{2^\alpha}\right)=$$

$$= H_\alpha(c)(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{3.7}$$

Inequality (3.1) follows from identity (3.2) and inequality (3.7).

Theorem 2 is proved.

***Remark* 5.** In Theorem 2, let $c=a$. Then Theorem 2 reduces to Theorem D.

***Corollary* 2.** *In Theorem* 2, *let* $c=(1-\beta)a+\beta b$ *and* $d=\beta a+(1-\beta)b$ *with* $0\leq\beta<\dfrac{1}{2}$. *Then we have the inequality*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right]-\left\{\frac{(1-\beta)^\alpha-\beta^\alpha}{2}\left[f\big((1-\beta)a+\beta b\big)+\right.\right.$$

$$\left.\left.+f(\beta a+(1-\beta)b)\right]+\frac{1-(1-\beta)^\alpha+\beta^\alpha}{2}\left[f(a)+f(b)\right]\right\}\right| \leq$$

$$\leq M_\alpha(\beta)(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right),$$

where

$$M_\alpha(\beta) := \frac{2^\alpha-1}{2^{\alpha+1}(\alpha+1)}-\frac{\beta}{2}\left[(1-\beta)^\alpha-\beta^\alpha\right]$$

with $\alpha>0$.

***Remark* 6.** In Corollary 2, let $\beta = \dfrac{1}{4}$. Then Corollary 2 reduces to Theorem G.

***Remark* 7.** In Corollary 2, let $\alpha = 1$. Then we have Hermite – Hadamard-type inequality

$$\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - \left\{\frac{1-2\beta}{2}\big[f((1-\beta)a+\beta b)+f(\beta a+(1-\beta)b)\big]+\beta\big[f(a)+f(b)\big]\right\}\right| \le$$

$$\le \left[\frac{1}{8}-\beta\left(\frac{1}{2}-\beta\right)\right](b-a)\left(|f'(a)|+|f'(b)|\right).$$

***Remark* 8.** In Remark 6, let $\beta = 0$. Then Remark 6 reduces to Theorem A.

Theorem 3. *Let $f\colon [a,b]\to\mathbb{R}$ be a convex function and $a \le c < d \le b$ in $\mathbb{R}$ with $a+b=c+d$. Then we have the following inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\big]-\left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)+\right.\right.$$

$$\left.\left.+\frac{(b-a)^\alpha+(c-a)^\alpha-(b-c)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]\right\}\right| \le I_\alpha(c)(b-a)\left(|f'(a)|+|f'(b)|\right), \tag{3.8}$$

where

$$I_\alpha(c) := \frac{c-a}{2(b-a)}+\left[\frac{1}{4}-\frac{c-a}{2(b-a)}\right]\left[\left(\frac{b-c}{b-a}\right)^\alpha-\left(\frac{c-a}{b-a}\right)^\alpha\right]-\frac{2^\alpha-1}{2^{\alpha+1}(\alpha+1)}$$

with $\alpha > 0$.

Proof. Define

$$h_2(x) = \begin{cases} (b-x)^\alpha-(x-a)^\alpha-(b-a)^\alpha, & x\in[a,c),\\ (b-x)^\alpha-(x-a)^\alpha-(b-c)^\alpha+(c-a)^\alpha, & x\in\left[c,\dfrac{a+b}{2}\right),\\ (b-x)^\alpha-(x-a)^\alpha+(b-c)^\alpha-(c-a)^\alpha, & x\in\left[\dfrac{a+b}{2},d\right),\\ (b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha, & x\in[d,b]. \end{cases}$$

By using the integration by parts, we have the following identities:

$$\frac{1}{2(b-a)^\alpha}\int\limits_a^b h_2(x)f'(x)\,dx = \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,dx-$$

$$-\left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)+\right.$$

$$\left.+\frac{(b-a)^\alpha+(c-a)^\alpha-(b-c)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]\right\} =$$

$$=\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\big[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\big]-\left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)+\right.$$

$$+\frac{(b-a)^\alpha+(c-a)^\alpha-(b-c)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]\Bigg\}=$$

$$=\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\big[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\big]-\left\{\frac{(b-c)^\alpha-(c-a)^\alpha}{(b-a)^\alpha}f\left(\frac{a+b}{2}\right)+\right.$$

$$\left.+\frac{(b-a)^\alpha+(c-a)^\alpha-(b-c)^\alpha}{2(b-a)^\alpha}\big[f(c)+f(d)\big]\right\}, \tag{3.9}$$

$$\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{b-x}{b-a}\big|f'(a)\big|dx+$$

$$+\int\limits_d^b\big[(b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha\big]\frac{b-x}{b-a}\big|f'(a)\big|\,dx=$$

$$=\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{b-x}{b-a}\big|f'(a)\big|\,dx+$$

$$+\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{x-a}{b-a}\big|f'(a)\big|\,dx=$$

$$=\big|f'(a)\big|\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]dx=Q_1, \tag{3.10}$$

where

$$Q_1:=\big|f'(a)\big|\left\{\frac{1}{\alpha+1}\big[(b-c)^{\alpha+1}+(c-a)^{\alpha+1}-(b-a)^{\alpha+1}\big]+(c-a)(b-a)^\alpha\right\}, \tag{3.11}$$

$$\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{x-a}{b-a}\big|f'(b)\big|\,dx+$$

$$+\int\limits_d^b\big[(b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha\big]\frac{x-a}{b-a}|f'(b)|dx=$$

$$=\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{x-a}{b-a}|f'(b)|dx+$$

$$+\int\limits_a^c\big[(x-a)^\alpha-(b-x)^\alpha+(b-a)^\alpha\big]\frac{b-x}{b-a}|f'(b)|dx=$$

$$= \left|f'(b)\right| \int\limits_a^c \left[(x-a)^\alpha - (b-x)^\alpha + (b-a)^\alpha\right] dx = Q_2,$$

where

$$Q_2 := \left|f'(b)\right| \left\{ \frac{1}{\alpha+1} \left[(b-c)^{\alpha+1} + (c-a)^{\alpha+1} - (b-a)^{\alpha+1}\right] + (c-a)(b-a)^\alpha \right\},$$

(3.12)

$$\int\limits_c^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right| dx +$$

$$+ \int\limits_{\frac{a+b}{2}}^{d} \left[(b-x)^\alpha - (x-a)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right| dx =$$

$$= \int\limits_c^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right| dx +$$

$$+ \int\limits_c^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right| dx =$$

$$= \left|f'(a)\right| \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] dx = Q_3,$$

where

$$Q_3 := \left|f'(a)\right| \left\{ \frac{1}{\alpha+1} \left[\frac{(b-a)^{\alpha+1}}{2^\alpha} - (b-c)^{\alpha+1} - (c-a)^{\alpha+1}\right] + \right.$$

$$\left. + \frac{a+b-2c}{2} \left[(b-c)^\alpha - (c-a)^\alpha\right] \right\},$$

(3.13)

$$\int\limits_c^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right| dx +$$

$$+ \int\limits_{\frac{a+b}{2}}^{d} \left[(b-x)^\alpha - (x-a)^\alpha + (b-c)^\alpha - (c-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right| dx =$$

$$= \int_{c}^{\frac{a+b}{2}} \big[(x-a)^{\alpha}-(b-x)^{\alpha}+(b-c)^{\alpha}-(c-a)^{\alpha}\big]\frac{x-a}{b-a}\big|f'(b)\big|dx+$$

$$+\int_{c}^{\frac{a+b}{2}} \big[(x-a)^{\alpha}-(b-x)^{\alpha}+(b-c)^{\alpha}-(c-a)^{\alpha}\big]\frac{b-x}{b-a}\big|f'(b)\big|\,dx=$$

$$=\big|f'(b)\big|\int_{c}^{\frac{a+b}{2}} \big[(x-a)^{\alpha}-(b-x)^{\alpha}+(b-c)^{\alpha}-(c-a)^{\alpha}\big]dx=Q_4,$$

where

$$Q_4:=\big|f'(b)\big|\left\{\frac{1}{\alpha+1}\left[\frac{(b-a)^{\alpha+1}}{2^{\alpha}}-(b-c)^{\alpha+1}-(c-a)^{\alpha+1}\right]+\right.$$
$$\left.+\frac{a+b-2c}{2}\big[(b-c)^{\alpha}-(c-a)^{\alpha}\big]\right\}.$$

Now, by using simple computation and identities (2.8) and (3.10) – (3.13), we have the inequality

$$\left|\frac{1}{2(b-a)^{\alpha}}\int_{a}^{b} h_2(x)f'(x)\,dx\right|\le\frac{1}{2(b-a)^{\alpha}}\int_{a}^{b}|h_2(x)|\,\big|f'(x)\big|\,dx=$$

$$=\frac{1}{2(b-a)^{\alpha}}\int_{a}^{c}[(x-a)^{\alpha}-(b-x)^{\alpha}+(b-a)^{\alpha}]\,\big|f'(x)\big|\,dx+$$

$$+\frac{1}{2(b-a)^{\alpha}}\int_{c}^{\frac{a+b}{2}}[(x-a)^{\alpha}-(b-x)^{\alpha}+(b-c)^{\alpha}-(c-a)^{\alpha}]\,\big|f'(x)\big|\,dx+$$

$$+\frac{1}{2(b-a)^{\alpha}}+\int_{\frac{a+b}{2}}^{d}[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-c)^{\alpha}-(c-a)^{\alpha}]\,\big|f'(x)\big|\,dx+$$

$$+\frac{1}{2(b-a)^{\alpha}}\int_{d}^{b}[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}]\,\big|f'(x)\big|\,dx\le\frac{Q_1+Q_2+Q_3+Q_4}{2(b-a)^{\alpha}}=$$

$$=\frac{(b-a)\left(\big|f'(a)\big|+\big|f'(b)\big|\right)}{2}\left\{\frac{1}{\alpha+1}\left[\left(\frac{b-c}{b-a}\right)^{\alpha+1}++\left(\frac{c-a}{b-a}\right)^{\alpha+1}-1\right]+\frac{c-a}{b-a}\right\}+$$

$$+\frac{(b-a)\left(\big|f'(a)\big|+\big|f'(b)\big|\right)}{2}\left\{\frac{1}{\alpha+1}\left[\frac{1}{2^{\alpha}}-\left(\frac{b-c}{b-a}\right)^{\alpha+1}-\right.\right.$$

$$
-\left(\frac{c-a}{b-a}\right)^{\alpha+1}\Bigg] + \left(\frac{1}{2}-\frac{b-c}{b-a}\right)\left[\left(\frac{b-c}{b-a}\right)^{\alpha}-\left(\frac{c-a}{b-a}\right)^{\alpha}\right]\Bigg\} =
$$

$$
= I_\alpha(c)(b-a)\Big(\left|f'(a)\right| + \left|f'(b)\right|\Big). \tag{3.14}
$$

Inequality (3.8) follows from identity (3.9) and inequality (3.14).

Theorem 3 is proved.

***Remark* 9.** In Theorem 3, let $c = a$. Then Theorem 3 reduces to Theorem E.

***Corollary* 3.** *In Theorem* 3, *let* $c = (1-\beta)a+\beta b$ *and* $d = \beta a+(1-\beta)b$ *with* $0 \le \beta < \dfrac{1}{2}$. *Then we have the inequality*

$$
\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] - \left\{\left[(1-\beta)^\alpha-\beta^\alpha\right]f\left(\frac{a+b}{2}\right) + \right.\right.
$$

$$
\left.\left. + \frac{1-(1-\beta)^\alpha+\beta^\alpha}{2}\left[f\big((1-\beta)a+\beta b\big)+f(\beta a+(1-\beta)b)\right]\right\}\right| \le
$$

$$
\le N_\alpha(\beta)(b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right),
$$

where

$$
N_\alpha(\beta) := \frac{\beta}{2} + \left(\frac{1}{4}-\frac{\beta}{2}\right)\left[(1-\beta)^\alpha-\beta^\alpha\right] - \frac{2^\alpha-1}{2^{\alpha+1}(\alpha+1)}
$$

with $\alpha > 0$.

***Remark* 10.** In Corollary 2, let $\beta = \dfrac{1}{4}$. Then Corollary 2 reduces to Theorem H.

***Remark* 11.** In Corollary 3, let $\alpha = 1$. Then we have the Hermite – Hadamard-type inequality

$$
\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - \left\{(1-2\beta)f\left(\frac{a+b}{2}\right) + \beta\big[f((1-\beta)a+\beta b)+f(\beta a+(1-\beta)b)\big]\right\}\right| \le
$$

$$
\le \left[\frac{1}{8}-\beta\left(\frac{1}{2}-\beta\right)\right](b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big).
$$

***Remark* 12.** In Remark 6, let $\beta = 0$. Then Remark 11 reduces to Theorem B.

***Remark* 13.** In Corollaries 2 and 3, let $\beta = \dfrac{1}{4}$. Then we obtain the following inequalities:

$$
\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - \frac{1}{4}\left[f(a)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f(b)\right]\right| \le
$$

$$
\le \frac{b-a}{16}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
$$

and

$$
\left|\frac{1}{b-a}\int\limits_a^b f(x)\,dx - \frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \le
$$

$$\leq \frac{b-a}{16}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

which are similar extensions of Theorems A and B.

4. Applications for the Beta functions. Throughout this section, let $\alpha > 0,\ a = 0,\ b = 1,$ $\Gamma(\alpha)$ be the Gamma function and $f(x) = x^{\rho-1}$ $(\rho > 1, x \in [0,1])$.

Let us recall the *Beta function*

$$B(p,q) = \int\limits_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad p,q > 0.$$

***Remark* 14.** In Sections 2 and 3, we obtain

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} J^\alpha_{a^+} f(b) = \frac{\alpha}{2}\int\limits_0^1 (1-x)^{\alpha-1} x^{\rho-1} dx = \frac{\alpha}{2} B(\rho,\alpha)$$

and

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} J^\alpha_{b^-} f(a) = \frac{\alpha}{2}\int\limits_0^1 x^{\alpha+\rho-2} dx = \frac{\alpha}{2(\alpha+\rho-1)}.$$

By using Corollaries 1 – 3 and Remark 14, we have the following propositions.

Proposition 1. *Let* $\rho \geq 2,\ 0 \leq \beta < \dfrac{1}{2},\ c = \beta$ *and* $d = 1-\beta$ *in Corollary* 1. *Then the following inequality holds*:

$$\frac{1}{2^{\rho-1}}\frac{(1-\beta)^\alpha - \beta^\alpha}{2^{\rho-1}} + \frac{1-(1-\beta)^\alpha+\beta^\alpha}{2}\left[(1-\beta)^{\rho-1} + \beta^{\rho-1}\right] \leq$$

$$\leq \frac{\alpha}{2}B(\rho,\alpha) + \frac{\alpha}{2(\alpha+\rho-1)} \leq$$

$$\leq \frac{(1-\beta)^\alpha - \beta^\alpha}{2}\left[(1-\beta)^{\rho-1} + \beta^{\rho-1}\right] + \frac{1-(1-\beta)^\alpha+\beta^\alpha}{2} \leq \frac{1}{2}.$$

Proposition 2. *Let* $\rho \geq 3,\ 0 \leq \beta < \dfrac{1}{2},\ c = \beta$ *and* $d = 1-\beta$ *in Corollary* 2. *Then*, *on the basis of Proposition* 1, *the following inequality holds:*

$$0 \leq \frac{(1-\beta)^\alpha - \beta^\alpha}{2}\left[(1-\beta)^{\rho-1} + \beta^{\rho-1}\right] +$$

$$+\frac{1-(1-\beta)^\alpha+\beta^\alpha}{2} - \left[\frac{\alpha}{2}B\left(\rho,\alpha\right) + \frac{\alpha}{2\left(\alpha+\rho-1\right)}\right] \leq$$

$$\leq (\rho-1)\left\{\frac{2^\alpha - 1}{2^{\alpha+1}\left(\alpha+1\right)} - \frac{\beta}{2}\left[(1-\beta)^\alpha - \beta^\alpha\right]\right\}.$$

Proposition 3. *Let* $\rho \geq 3,\ 0 \leq \beta < \dfrac{1}{2},\ c = \beta$ *and* $d = 1 - \beta$ *in Corollary* 3. *Then, on the basis of Proposition* 1, *the following inequality holds*:

$$0 \leq \frac{\alpha}{2} B(\rho, \alpha) + \frac{\alpha}{2(\alpha + \rho - 1)} -$$

$$- \frac{(1-\beta)^{\alpha} - \beta^{\alpha}}{2^{\rho - 1}} - \frac{1 - (1-\beta)^{\alpha} + \beta^{\alpha}}{2} \left[(1-\beta)^{\rho - 1} + \beta^{\rho - 1} \right] \leq$$

$$\leq (\rho - 1) \left\{ \frac{\beta}{2} + \left(\frac{1}{4} - \frac{\beta}{2} \right) \left[(1-\beta)^{\alpha} - \beta^{\alpha} \right] - \frac{2^{\alpha} - 1}{2^{\alpha+1}(\alpha + 1)} \right\}.$$

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