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RADII OF STARLIKENESS AND CONVEXITY OF BESSEL FUNCTION DERIVATIVES*

РАДІУСИ ЗІРЧАСТОСТІ ТА ОПУКЛОСТІ ПОХІДНИХ ФУНКЦІЇ БЕССЕЛЯ

In this paper, our aim is to find the radii of starlikeness and convexity of Bessel function derivatives for three different kind of normalization. The key tools in the proof of our main results are the Mittag-Leffler expansion for nth derivative of Bessel function and properties of real zeros of it. In addition, by using the Euler – Rayleigh inequalities we obtain some tight lower and upper bounds for the radii of starlikeness and convexity of order zero for the normalized nth derivative of Bessel function. The main results of the paper are natural extensions of some known results on classical Bessel functions of the first kind.

Знайдено радіуси зірчастості та опуклості похідних функції Бесселя для трьох різних видів нормалізації. Ключовими інструментами доведення основних результатів є розклад Міттаг-Леффлера для n-ї похідної функції Бесселя та властивості його дійсних нулів. Крім того, за допомогою нерівностей Ейлера – Релея отримано деякі точні нижні й верхні межі для радіусів зірчастості та опуклості нульового порядку для нормованої n-ї похідної функції Бесселя. Основними результатами роботи є природні розширення деяких відомих результатів щодо класичних функцій Бесселя першого роду.

1. Introduction. Denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ $(r > 0)$ the disk of radius r and let $\mathbb{D} = \mathbb{D}_1$. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. We say that the function $f \in \mathcal{A}$ is starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(\dfrac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is starlike of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(\dfrac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{D}_r$. We define by the real number

$$r^*_\beta(f) = \sup\left\{ r \in (0, r_f) : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \ \text{ for all } \ z \in \mathbb{D}_r \right\}$$

the radius of starlikeness of order β of the function f. Note that $r^*(f) = r^*_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^*_\beta(f)})$ is a starlike domain with respect to the origin.

The function $f \in \mathcal{A}$ is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + \dfrac{zf''(z)}{f'(z)}\right) > 0$, $z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is convex of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + \dfrac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \mathbb{D}_r$. The radius of convexity of order β of the function f is defined by

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the real number

$$r_\beta^c(f) = \sup\left\{r \in (0, r_f) : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \ \text{ for all } \ z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r_\beta^c(f)})$ is a convex domain.

The first kind of Bessel function of order ν is defined by [18, p. 217]

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m+\nu}, \quad z \in \mathbb{C}.$$

Now, we consider the nth derivative of Bessel function of the first kind by

$$J_\nu^{(n)}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(2m+\nu+1)}{m! 2^n \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m-n+\nu}, \quad z \in \mathbb{C}.$$

Here, it is important mentioning that for $n = 0$ the $J_\nu^{(n)}$ reduce to classical Bessel function J_ν. Since the function $J_\nu^{(n)}$ is not belongs to $\mathcal{A}$, firstly, we form and focus on the following normalized forms:

$$\begin{aligned} f_{\nu,n}(z) &= \left[2^\nu \Gamma(\nu-n+1) J_\nu^{(n)}(z)\right]^{\frac{1}{\nu-n}}, \\ g_{\nu,n}(z) &= 2^\nu \Gamma(\nu-n+1) z^{1+n-\nu} J_\nu^{(n)}(z), \\ h_{\nu,n}(z) &= 2^\nu \Gamma(\nu-n+1) z^{1+\frac{n-\nu}{2}} J_\nu^{(n)}(\sqrt{z}), \end{aligned} \tag{1.1}$$

where $\nu > n - 1$.

The first studies on geometric properties of Bessel functions of first kind was conducted in 1960 by Brown, Kreyszig and Todd [10, 16]. They determined the radius of starlikeness of the functions $f_{\nu,0}(z)$ and $g_{\nu,0}(z)$ for the case $\nu > 0$. Recently, in 2014, Baricz et al. [3] and Baricz and Szász [4] obtained, respectively, the radius of starlikeness of order β and the radius of convexity of order β for the functions $f_{\nu,0}(z)$, $g_{\nu,0}(z)$ and $h_{\nu,0}(z)$ in the case when $\nu > -1$. On the other hand, we know that if $\nu \in (-2, -1)$, then the Bessel function has exactly two purely imaginary conjugate complex zeros, and all the other zeros are real [21, p. 483]. In 2015, Szász [20] investigated the radius of starlikeness of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu \in (-2, -1)$ by using some inequalities. In the same year, Baricz and Szász [5] obtained the radius of convexity of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu \in (-2, -1)$. Later, in 2016, Baricz et al. [7] determined the radius of α-convexity of the same three functions for $\nu > -1$. After a year, Çağlar et al. [11] extended it for the case when $\nu \in (-2, -1)$. In 2017, Deniz and Szász [12] determined the radius of uniform convexity of $f_{\nu,0}(z)$, $g_{\nu,0}(z)$ and $h_{\nu,0}(z)$ for $\nu > -1$. They also determined necessary and sufficient conditions on the parameters of these three normalized functions such that they are uniformly convex in the unit disk. Moreover, in [1, 2] authors determined tight lower and upper bounds for the radii of starlikeness and convexity of the functions $g_{\nu,0}(z)$ and $h_{\nu,0}(z)$. The key tools in their proofs were some new Mittag-Leffler expansions for quotients of Bessel functions of the first kind, special properties of the zeros of Bessel functions of the first kind and their derivatives, Euler – Rayleigh inequalities and the fact that the smallest positive zeros of some Dini functions are less than the first positive zero of the Bessel function of first kind.

Another study on Bessel functions investigate the properties of derivatives and the zeros of these derivatives. In the last three decades the zeros of the nth derivative of Bessel functions of the first kind for $n \in \{1, 2, 3\}$ have been also studied by researchers like Elbert, Ifantis, Ismail, Kokologiannaki, Laforgia, Landau, Lorch, Mercer, Muldoon, Petropoulou, Siafarikas and Szegö; for more details see the papers [13, 15] and the references therein. Very recently in 2018, Baricz et al. [8] obtained some results for the zeros of the nth derivative of Bessel functions of the first kind for all $n \in \mathbb{N}$ by using the Laguerre – Pólya class of entire functions and the so-called Laguerre inequalities.

Motivated by the above results in this paper, we deal with the radii of starlikeness and convexity of order β for the functions $f_{\nu,n}(z),\ g_{\nu,n}(z)$ and $h_{\nu,n}(z)$ in the case when $\nu > n-1$ for $n \in \mathbb{N}$. Also, we determined tight lower and upper bounds for the radii of starlikeness and convexity of these functions.

2. Preliminaries. In order to prove the main results we need the following preliminary results.

Lemma 2.1 [8]. *The following assertions are valid*:

(a) *If* $\nu > n-1$, *then* $z \mapsto J_\nu^{(n)}(z)$ *has infinitely many zeros, which are all real and simple, expect the origin.*

(b) *If* $\nu > n$, *then the positive zeros of the* n*th and* $(n+1)$*th derivative of* J_ν *are interlacing.*

(c) *If* $\nu > n-1$, *then all zeros of* $z \mapsto (n-\nu)J_\nu^{(n)}(z) + zJ_\nu^{(n+1)}(z)$ *are real and interlace with the zeros of* $z \mapsto J_\nu^{(n)}(z)$.

The lemma below (see [9, 19]) is also required for our work.

Lemma 2.2. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n,\ a_n \in \mathbb{R}$, *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n,\ b_n > 0$, *for all* $n \geq 0$, *converge on an interval* $(-r, r)$ *for some* $r > 0$. *If the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is decreasing* (*increasing*), *then the function* $x \to f(x)/g(x)$ *is decreasing* (*increasing*) *too on* $(0, r)$. *So the same result holds for the following*:

$$f(x) = \sum_{n=0}^{\infty} a_n x^{2n} \qquad \textit{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^{2n}.$$

2.1. Zeros of hyperbolic polynomials and the Laguerre – Pólya class of entire functions. In this subsection, some necessary knowledge about polynomials and entire functions with real zeros are given. An algebraic polynomial is named hyperbolic if its all zeros are real. We will be using the following lemma given in [6] and obtain new results.

Lemma 2.3. *Assume*

$$p(x) = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots + (-1)^n a_n x^n = (1 - x/x_1) \ldots (1 - x/x_n)$$

is a hyperbolic polynomial with positive zeros $0 < x_1 \leq x_2 \leq \ldots \leq x_n$, *and it is normalized by* $p(0) = 1$. *Then the polynomial* $q(x) = Cp(x) - xp'(x)$ *is hyperbolic for any constant* C. *Also, the smallest zero* η_1 *is in* $(0, x_1)$ *if and only if* $C < 0$.

Clearly, a real entire function ψ is in the Laguerre – Pólya class $\mathcal{LP}$ if it is in the form

$$\psi(x) = cx^m e^{-ax^2+\beta x} \prod_{k\geq 1} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R},\ a \geq 0,\ m \in \mathbb{N} \cup \{0\}$ and $\sum x_k^{-2} < \infty$. Similarly, we say ϕ is of type $\mathcal{I}$ in the Laguerre – Pólya class, denoted by $\phi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ is written as

$$\phi(x) = cx^m e^{\sigma x} \prod_{k \geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R},\ \sigma \geq 0,\ m \in \mathbb{N} \cup \{0\},\ x_k > 0$ and $\sum x_k^{-1} < \infty$. The complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on the compact sets of the complex plane is the class $\mathcal{LP}$ if the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign is $\mathcal{LPI}$. For any entire function φ in the form

$$\varphi(x) = \sum_{k \geq 0} \mu_k \frac{x^k}{k!},$$

its Jensen polynomials are given by

$$P_m(\varphi; x) = P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k.$$

The following lemma is a well-known characterization of functions in the class $\mathcal{LP}$ (see [14]).

Lemma 2.4. *φ is in the class $\mathcal{LP}$ ($\mathcal{LPI}$, respectively) if and only if all the polynomials $P_m(\varphi; x),\ m = 1, 2, \ldots,$ are hyperbolic such that they are hyperbolic with zeros of equal sign. Also, the sequence $P_m(\varphi; z/n)$ is locally uniformly convergent to $\varphi(z)$.*

The following lemma is necessary for the proof of main results.

Lemma 2.5. *Let $\nu > n - 1$ and $a < 0$. Then the functions $z \longmapsto (2a - n + \nu) J_\nu^{(n)}(z) - zJ_\nu^{(n+1)}(z)$ are written in the form*

$$2^{n-1}\Gamma(\nu + 1 - n)\Big((2a - n + \nu) J_\nu^{(n)}(z) - zJ_\nu^{(n+1)}(z)\Big) = \left(\frac{z}{2}\right)^{\nu - n} W_{\nu,n}(z),$$

where $W_{\nu,n}$ is entire functions belonging to the Laguerre – Pólya class $\mathcal{LP}$. Moreover, the smallest positive zero of $W_{\nu,n}$ cannot exceed the first positive zero $j_{\nu,1}^{(n)}$, where $j_{\nu,m}^{(n)}$ is the mth positive zero of $J_\nu^{(n)}(z),\ m \in \mathbb{N},\ n \in \mathbb{N}_0$.

Proof. It is obvious from the infinite product representation of $z \longmapsto \mathcal{J}_\nu^{(n)}(z) = 2^\nu \Gamma(\nu + 1 - n)(z)^{n-\nu} J_\nu^{(n)}(z)$ that this function is in the class $\mathcal{LP}$. This shows that the function $z \longmapsto \mathbb{J}_\nu^{(n)}(z) = \mathcal{J}_\nu^{(n)}(2\sqrt{z}\,)$ is in the class $\mathcal{LPI}$. Then, due to Lemma 2.4, its Jensen polynomials

$$P_m(\mathbb{J}_\nu^{(n)}; \varsigma) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k$$

are all hyperbolic. However, it can be seen that the Jensen polynomials of $\widetilde{W}_{\nu,n}(z) = W_{\nu,n}(2\sqrt{z}\,)$ are clearly

$$P_m\big(\widetilde{W}_{\nu,n}; \varsigma\big) = aP_m\big(\mathbb{J}_\nu^{(n)}; \varsigma\big) - \varsigma P_m'\big(\mathbb{J}_\nu^{(n)}; \varsigma\big).$$

Moreover, Lemma 2.3 tells us that all zeros of $P_m\big(\widetilde{W}_{\nu,n}; \varsigma\big)$ are real and positive and that the smallest one precedes the first zero of $P_m\big(\mathbb{J}_\nu^{(n)}; \varsigma\big)$. From Lemma 2.4, the latter result immediately implies that $\widetilde{W}_{\nu,n} \in \mathcal{LPI}$ and that its first zero precedes $j_{\nu,1}^{(n)}$. Finally, the first part of the statement of the lemma follows after we go back from $\widetilde{W}_{\nu,n}$ to $W_{\nu,n}$ by setting $\varsigma = \dfrac{z^2}{4}$.

Lemma 2.5 is proved.

2.2. Euler–Rayleigh sums for positive zeros of $J_\nu^{(n)}(z)$. Baricz et al. [8] proved the Weierstrassian decomposition of $J_\nu^{(n)}(z)$ as follows:

$$J_\nu^{(n)}(z) = \frac{z^{\nu-n}}{2^\nu\Gamma(\nu+1-n)} \prod_{m\geq 1}\left(1 - \frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right), \tag{2.1}$$

where $j_{\nu,m}^{(n)}$ is the mth positive zero of $J_\nu^{(n)}(z)$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. Therefore we can write

$$g_{\nu,n}(z) = 2^\nu\Gamma(\nu-n+1)z^{1+n-\nu}J_\nu^{(n)}(z) = z\prod_{m\geq 1}\left(1 - \frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right). \tag{2.2}$$

On the other hand, the series representation of $g_{\nu,n}(z)$

$$g_{\nu,n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^{2m+1}. \tag{2.3}$$

Now, we would like to mention that by using the equations (2.2) and (2.3) we can obtain the following Euler–Rayleigh sums for the positive zeros of the function $g_{\nu,n}$. From the equality (2.3) we have

$$g_{\nu,n}(z) = z - \frac{\nu+2}{4(\nu-n+2)(\nu-n+1)}z^3 +$$

$$+ \frac{(\nu+4)(\nu+3)}{32(\nu-n+4)(\nu-n+3)(\nu-n+2)(\nu-n+1)}\, z^5 - \ldots . \tag{2.4}$$

Now, if we consider (2.2), then some calculations yield that

$$g_{\nu,n}(z) = z - \sum_{m\geq 1}\frac{1}{\left(j_{\nu,m}^{(n)}\right)^2}z^3 + \frac{1}{2}\left(\left(\sum_{m\geq 1}\frac{1}{\left(j_{\nu,m}^{(n)}\right)^2}\right)^2 - \sum_{m\geq 1}\frac{1}{\left(j_{\nu,m}^{(n)}\right)^4}\right)z^5 - \ldots . \tag{2.5}$$

By equating the first few coefficients with the same degrees in equations (2.4) and (2.5), we get

$$\sum_{m\geq 1}\frac{1}{\left(j_{\nu,m}^{(n)}\right)^2} = \frac{\nu+2}{4(\nu-n+2)(\nu-n+1)} \tag{2.6}$$

and

$$\sum_{m\geq 1}\frac{1}{\left(j_{\nu,m}^{(n)}\right)^4} = \frac{1}{16(\nu-n+2)(\nu-n+1)}\times$$

$$\times\left(\frac{(\nu+2)^2}{(\nu-n+2)(\nu-n+1)} - \frac{(\nu+4)(\nu+3)}{(\nu-n+4)(\nu-n+3)}\right). \tag{2.7}$$

Here, it is important mentioning that for $n = 0$ the equations (2.6) and (2.7) reduce to

$$\sum_{m\geq 1}\frac{1}{(j_{\nu,m})^2}=\frac{1}{4(\nu+1)}\qquad\text{and}\qquad\sum_{m\geq 1}\frac{1}{(j_{\nu,m})^4}=\frac{1}{16(\nu+2)(\nu+1)^2},$$

respectively, where $j_{\nu,m}$ denotes the mth zero of classical Bessel function J_ν.

Another special case for $n=1,2$ the equations (2.6) and (2.7) reduce to

$$\sum_{m\geq 1}\frac{1}{\left(j'_{\nu,m}\right)^2}=\frac{\nu+2}{4\nu(\nu+1)},\qquad\sum_{m\geq 1}\frac{1}{\left(j'_{\nu,m}\right)^4}=\frac{\nu^2+8\nu+8}{16\nu^2(\nu+1)^2(\nu+2)}$$

and

$$\sum_{m\geq 1}\frac{1}{\left(j''_{\nu,m}\right)^2}=\frac{\nu+2}{4(\nu-1)\nu},\qquad\sum_{m\geq 1}\frac{1}{\left(j''_{\nu,m}\right)^4}=\frac{13\nu^3+19\nu^2+26\nu+8}{16(\nu-1)^2\nu^2(\nu+1)(\nu+2)},$$

where $j'_{\nu,m}$ and $j''_{\nu,m}$ denotes the mth zeros of function J'_ν and J''_ν, respectively.

3. Main results. ***3.1. Radii of starlikeness and convexity of the functions $f_{\nu,n}$, $g_{\nu,n}$ and $h_{\nu,n}$.*** The first principal result we established concerns the radii of starlikeness and reads as follows. Here and in the sequel I_ν denotes the modified Bessel function of the first kind and order ν. Note that $I_\nu(z)=i^{-\nu}J_\nu(iz)$.

Theorem 3.1. *The followings hold*:

(a) *If $\nu>n$ and $\beta\in[0,1)$, then $r^*_\beta(f_{\nu,n})=x^{(n)}_{\nu,1}$, where $x^{(n)}_{\nu,1}$ is the smallest positive root of the equation*

$$\frac{rJ^{(n+1)}_\nu(r)}{(\nu-n)J^{(n)}_\nu(r)}-\beta=0.$$

Besides, if $n-1<\nu<n$ and $\beta\in[0,1)$, then we have $r^*_\beta(f_{\nu,n})=x^{(n)}_{\nu,2}$, where $x^{(n)}_{\nu,2}$ is the smallest positive root of the equation

$$\frac{rI^{(n+1)}_\nu(r)}{(\nu-n)I^{(n)}_\nu(r)}-\beta=0.$$

(b) *If $\nu>n-1$ and $\beta\in[0,1)$, then $r^*_\beta(g_{\nu,n})=y^{(n)}_{\nu,1}$, where $y^{(n)}_{\nu,1}$ is the smallest positive root of the equation*

$$\frac{rJ^{(n+1)}_\nu(r)}{J^{(n)}_\nu(r)}+n+1-\nu-\beta=0.$$

(c) *If $\nu>n-1$ and $\beta\in[0,1)$, then $r^*_\beta(h_{\nu,n})=z^{(n)}_{\nu,1}$, where $z^{(n)}_{\nu,1}$ is the smallest positive root of the equation*

$$\frac{\sqrt{r}J^{(n+1)}_\nu(\sqrt{r})}{J^{(n)}_\nu(\sqrt{r})}+n+2-\nu-2\beta=0.$$

Proof. Firstly, we prove part (a) for $\nu>n$ and parts (b) and (c) for $\nu>n-1$. We need to show that the following inequalities:

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right)>\beta,\qquad\operatorname{Re}\left(\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)}\right)>\beta\qquad\text{and}\qquad\operatorname{Re}\left(\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)}\right)>\beta\qquad(3.1)$$

are valid for $z\in\mathbb{D}_{r^*_\beta(f_{\nu,n})}$, $z\in\mathbb{D}_{r^*_\beta(g_{\nu,n})}$ and $z\in\mathbb{D}_{r^*_\beta(h_{\nu,n})}$, respectively, and each inequality above cannot holds in larger disks.

When we write the equation (2.1) in definition of the functions $f_{\nu,n}(z)$, $g_{\nu,n}(z)$ and $h_{\nu,n}(z)$, we get by using logarithmic derivation

$$\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)} = \frac{1}{\nu-n}\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)} = 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}, \quad \nu > n,$$

$$\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)} = n+1-\nu+\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)} = 1 - \sum_{m\geq 1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}, \quad \nu > n-1,$$

$$\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)} = 1+\frac{n-\nu}{2}+\frac{1}{2}\frac{\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})}{J_\nu^{(n)}(\sqrt{z})} = 1 - \sum_{m\geq 1}\frac{z}{\left(j_{\nu,m}^{(n)}\right)^2 - z}, \quad \nu > n-1.$$

It is known [4] that if $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $\lambda > |z|$, then

$$\frac{|z|}{\lambda - |z|} \geq \operatorname{Re}\left(\frac{z}{\lambda - z}\right). \tag{3.2}$$

Then the inequality

$$\frac{|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} \geq \operatorname{Re}\left(\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right)$$

holds for every $\nu > n-1$. Therefore,

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right) = 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right) \geq$$

$$\geq 1 - \frac{1}{\nu-n}\sum_{m\geq 1}\frac{2|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} = \frac{|z|f'_{\nu,n}(|z|)}{f_{\nu,n}(|z|)},$$

$$\operatorname{Re}\left(\frac{zg'_{\nu,n}(z)}{g_{\nu,n}(z)}\right) = 1 - \sum\nolimits_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right) \geq$$

$$\geq 1 - \sum_{m\geq 1}\frac{2|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|^2} = \frac{|z|g'_{\nu,n}(|z|)}{g_{\nu,n}(|z|)},$$

$$\operatorname{Re}\left(\frac{zh'_{\nu,n}(z)}{h_{\nu,n}(z)}\right) = 1 - \sum_{m\geq 1}\operatorname{Re}\left(\frac{z}{\left(j_{\nu,m}^{(n)}\right)^2 - z}\right) \geq$$

$$\geq 1 - \sum_{m\geq 1}\frac{|z|}{\left(j_{\nu,m}^{(n)}\right)^2 - |z|} = \frac{|z|h'_{\nu,n}(|z|)}{h_{\nu,n}(|z|)},$$

where equalities are obtained only if $z = |z| = r$. From the latest inequalities and the minimum principle for harmonic functions, we conclude that the corresponding inequalities in (3.1) hold if and only if $|z| < x_{\nu,1}^{(n)}$, $|z| < y_{\nu,1}^{(n)}$ and $|z| < z_{\nu,1}^{(n)}$, respectively, where $x_{\nu,1}^{(n)}$, $y_{\nu,1}^{(n)}$ and $z_{\nu,1}^{(n)}$ is the smallest positive roots of the equations

$$\frac{rf'_{\nu,n}(r)}{f_{\nu,n}(r)} = \beta, \qquad \frac{rg'_{\nu,n}(r)}{g_{\nu,n}(r)} = \beta \qquad \text{and} \qquad \frac{rh'_{\nu,n}(r)}{h_{\nu,n}(r)} = \beta,$$

which are equivalent to

$$\frac{rJ_\nu^{(n+1)}(r)}{(\nu - n)J_\nu^{(n)}(r)} - \beta = 0, \qquad \frac{rJ_\nu^{(n+1)}(r)}{J_\nu^{(n)}(r)} + n + 1 - \nu - \beta = 0$$

and

$$\frac{\sqrt{r}J_\nu^{(n+1)}(\sqrt{r})}{J_\nu^{(n)}(\sqrt{r})} + n + 2 - \nu - 2\beta = 0.$$

The result follows from Lemma 2.5 by taking instead of a the values $\frac{(\beta-1)(\nu-n)}{2}$, $\frac{\beta-1}{2}$ and $\beta - 1$, respectively. In other words, Lemma 2.5 show that all the zeros of the above three functions are real and their first positive zeros do not exceed the first positive zeros $j_{v,1}^{(n)}$ and $\sqrt{j_{v,1}^{(n)}}$. This guarantees that the above inequalities hold. This completes the proof of part (a) if $\nu > n$ and parts (b) and (c) if $\nu > n - 1$.

Now, to prove the statement for part (a) when $\nu \in (n-1, n)$, we use the counterpart of (3.2), that is,

$$\operatorname{Re}\left(\frac{z}{\lambda - z}\right) \geq \frac{-|z|}{\lambda + |z|}, \tag{3.3}$$

which holds for all $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are such that $\lambda > |z|$ (see [3]). If in the inequality (3.3), we replace z by z^2 and λ by $\left(j_{\nu,m}^{(n)}\right)^2$, it follows that

$$\operatorname{Re}\left(\frac{z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right) \geq \frac{-|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 + |z|^2},$$

provided that $|z| < j_{\nu,1}^{(n)}$. Thus, for $n - 1 < \nu < n$, we obtain

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right) = 1 - \frac{1}{\nu - n}\sum_{m\geq 1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2 - z^2}\right) \geq$$

$$\geq 1 + \frac{1}{\nu - n}\sum_{m\geq 1}\frac{2|z|^2}{\left(j_{\nu,m}^{(n)}\right)^2 + |z|^2} = \frac{i|z|f'_{\nu,n}(i|z|)}{f_{\nu,n}(i|z|)}.$$

In this case equality is obtained when $z = i|z| = ir$. Also, the latter inequality tells us that

$$\operatorname{Re}\left(\frac{zf'_{\nu,n}(z)}{f_{\nu,n}(z)}\right) > \beta$$

if and only if $|z| < x_{v,2}^{(n)}$, where $x_{v,2}^{(n)}$ denotes the smallest positive root of the equations

$$\frac{irf'_{\nu,n}(ir)}{f_{\nu,n}(ir)} = \beta,$$

which is equivalent to

$$\frac{irJ_\nu^{(n+1)}(ir)}{(\nu-n)J_\nu^{(n)}(ir)} - \beta = 0 \qquad \text{or} \qquad \frac{rI_\nu^{(n+1)}(r)}{(\nu-n)I_\nu^{(n)}(r)} - \beta = 0$$

for $n-1 < \nu < n$. It follows from Lemma 2.5 that the first positive zero of $z \mapsto irJ_\nu^{(n+1)}(ir) - - \beta(\nu-n)J_\nu^{(n)}(ir)$ cannot exceed $j_{\nu,1}^{(n)}$ so the above inequalities are verified. So we would only need to prove that the above function has actually only one zero in $(0,\infty)$. Note that, due to Lemma 2.2, the function

$$r \mapsto \frac{irJ_\nu^{(n+1)}(ir)}{J_\nu^{(n)}(ir)} = \frac{Q_1}{Q_2},$$

where

$$Q_1 = \sum_{m=0}^{\infty} \frac{(2m-n+\nu)\Gamma(2m+\nu+1)}{m!2^{2m+\nu}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} r^{2m},$$

$$Q_2 = \sum_{m=0}^{\infty} \frac{\Gamma(2m+\nu+1)}{m!2^{2m+\nu}\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} r^{2m},$$

is increasing on $(0,\infty)$ as a quotient of two power series whose positive coefficients form the increasing “quotient sequence” $\{2m-n+\nu\}_{m\geq 0}$. On the other hand, the above function tends to $\nu - n$ when $r \to 0$, so that its graph can intersect the horizontal line $y = \beta(\nu-n) > \nu - n$ only once. Thus, proof for part (a) of the theorem is completed if $\nu \in (n-1, n)$.

Theorem 3.1 is proved.

With regards to Theorem 3.1, we tabulate the radius of starlikeness for $f_{\nu,n}$, $g_{\nu,n}$ and $h_{\nu,n}$ for a fixed $\nu = 2.5$, $n = 0,1,2,3$ and, respectively, $\beta = 0$ and $\beta = 0.5$. These are given in Table 3.1. Also, in Table 3.1, we see that radius of starlikeness is decreasing according to the order of derivative and the order of starlikeness. On the other words, from all these results we concluded that $r_\beta^*(f_{\nu,0}) > r_\beta^*(f_{\nu,1}) > r_\beta^*(f_{\nu,2}) > \ldots > r_\beta^*(f_{\nu,n}) > \ldots$ for $\beta \in [0,1)$ and $\nu > n-1$, $n \in \mathbb{N}_0$. In addition to, we can write $r_{\beta_1}^*(f_{\nu,n}) < r_{\beta_0}^*(f_{\nu,n})$ for $0 \leq \beta_0 < \beta_1 < 1$ and $\nu > n-1$, $n \in \mathbb{N}_0$. Same inequalities is also true for $r_\beta^*(g_{\nu,n})$ and $r_\beta^*(h_{\nu,n})$.

For $n = 0$ in the Theorem 3.1 we obtain the results of Baricz et al. [3]. Our results is a common generalization of these results.

Table 3.1

n	$r_\beta^*(f_{2.5,n})$		$r_\beta^*(g_{2.5,n})$		$r_\beta^*(h_{2.5,n})$	
	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$
0	3.6328	2.7569	2.5011	1.8192	11.1696	6.2556
1	2.1056	1.5926	1.7975	1.3307	5.4265	3.2312
2	0.8512	0.6229	1.1285	0.8512	2.0284	1.2735
3	0.4586	0.3051	0.4819	0.3703	0.3543	0.2323

The second principal result we established concerns the radii of convexity and reads as follows.

Theorem 3.2. *The following statements hold*:

(a) *If $\nu > n$ and $\beta \in [0,1)$, then the radius $r_\beta^c(f_{\nu,n})$ is the smallest positive root of the equation*

$$1-\beta+\frac{rJ_\nu^{(n+2)}(r)}{J_\nu^{(n+1)}(r)}+\left(\frac{1}{\nu-n}-1\right)\frac{rJ_\nu^{(n+1)}(r)}{J_\nu^{(n)}(r)}=0.$$

Moreover, $r_\beta^c(f_{\nu,n}) < j_{\nu,1}^{(n+1)} < j_{\nu,1}^{(n)}$.

(b) *If $\nu > n-1$ and $\beta \in [0,1)$, then the radius $r_\beta^c(g_{\nu,n})$ is the smallest positive root of the equation*

$$n+1-\nu-\beta+\frac{(n-\nu+2)rJ_\nu^{(n+1)}(r)+r^2J_\nu^{(n+2)}(r)}{(n-\nu+1)J_\nu^{(n)}(r)+rJ_\nu^{(n+1)}(r)}=0.$$

(c) *If $\nu > n-1$ and $\beta \in [0,1)$, then the radius $r_\beta^c(h_{\nu,n})$ is the smallest positive root of the equation*

$$\frac{n+2-\nu-2\beta}{2}+\frac{\sqrt{r}}{2}\frac{(n-\nu+3)J_\nu^{(n+1)}(\sqrt{r})+\sqrt{r}J_\nu^{(n+2)}(\sqrt{r})}{(n-\nu+2)J_\nu^{(n)}(\sqrt{r})+\sqrt{r}J_\nu^{(n+1)}(\sqrt{r})}=0.$$

Proof. (a) Since

$$1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}=1+\frac{zJ_\nu^{(n+2)}(z)}{J_\nu^{(n+1)}(z)}+\left(\frac{1}{\nu-n}-1\right)\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)}$$

and by means of (2.1) we have

$$\frac{zJ_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)}=\nu-n-\sum_{m\ge1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2},$$

it follows that

$$1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}=1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\ge1}\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2}-\sum_{m\ge1}\frac{2z^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-z^2}.$$

Now, suppose that $\nu \in (n, n+1]$. If we use the inequality (3.2), for all $z \in \mathbb{D}_{j_{\nu,1}^{(n)}}$ we get the inequality

$$\operatorname{Re}\left(1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right)=1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\ge1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n)}\right)^2-z^2}\right)-$$

$$-\sum_{m\ge1}\operatorname{Re}\left(\frac{2z^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-z^2}\right)\ge$$

$$\ge1-\left(\frac{1}{\nu-n}-1\right)\sum_{m\ge1}\frac{2r^2}{\left(j_{\nu,m}^{(n)}\right)^2-r^2}-\sum_{m\ge1}\frac{2r^2}{\left(j_{\nu,m}^{(n+1)}\right)^2-r^2},$$

where $|z| = r$. Also, observe that if we use the inequality [4] (Lemma 2.1)

$$\mu \mathrm{Re}\left(\frac{z}{a-z}\right) - \mathrm{Re}\left(\frac{z}{b-z}\right) \geq \mu \frac{|z|}{a-|z|} - \frac{|z|}{b-|z|},$$

where $a > b > 0$, $\mu \in [0,1]$ and $z \in \mathbb{C}$ such that $|z| < b$, then we get that the above inequality is also valid when $\nu > n+1$. Here we used that the zeros of the nth and $(n+1)$th derivative of J_v are interlacing according to Lemma 2.1. The above inequality implies for $r \in (0, j_{\nu,1}^{(n)})$

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right)\right\} = 1 + \frac{rf''_{\nu,n}(r)}{f'_{\nu,n}(r)}.$$

On the other hand, we define the function $\varphi_{\nu,n}\colon (n, j_{\nu,1}^{(n)}) \to \mathbb{R}$,

$$\varphi_{\nu,n}(r) = 1 + \frac{rf''_{\nu,n}(r)}{f'_{\nu,n}(r)}.$$

Since the zeros of the nth and $(n+1)$th derivative of J_v are interlacing according to Lemma 2.1 and $r < j_{\nu,1}^{(n+1)} < j_{\nu,1}^{(n)}$ $\left(\text{or } r < \sqrt{j_{\nu,1}^{(n)} j_{\nu,1}^{(n+1)}}\right)$ for all $\nu > n$, we have

$$\left(j_{\nu,m}^{(n)}\right)\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right) - \left(j_{\nu,m}^{(n+1)}\right)\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right) < 0.$$

Thus, the inequality

$$\frac{d\varphi_{\nu,n}(r)}{dr} = -\left(\frac{1}{\nu-n} - 1\right)\sum_{m\geq1}\frac{4r\left(j_{\nu,m}^{(n)}\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2} - \sum_{m\geq1}\frac{4r\left(j_{\nu,m}^{(n+1)}\right)^2}{\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2} <$$

$$< \sum_{m\geq1}\frac{4r\left(j_{\nu,m}^{(n)}\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2} - \sum_{m\geq1}\frac{4r\left(j_{\nu,m}^{(n+1)}\right)^2}{\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2} =$$

$$= 4r\sum_{m\geq1}\frac{\left(j_{\nu,m}^{(n)}\right)^2\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2 - \left(j_{\nu,m}^{(n+1)}\right)^2\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - r^2\right)^2\left(\left(j_{\nu,m}^{(n+1)}\right)^2 - r^2\right)^2} < 0$$

is satisfied. Consequently, the function $\varphi_{\nu,n}$ is strictly decreasing. Observe also that $\lim_{r\searrow0}\varphi_{\nu,n}(r) = 1 > \beta$ and $\lim_{r\nearrow j_{\nu,1}^{(n)}}\varphi_{\nu,n}(r) = -\infty$, which means that for $z \in \mathbb{D}_{r_1}$ we have

$$\mathrm{Re}\left(1+\frac{zf''_{\nu,n}(z)}{f'_{\nu,n}(z)}\right) > \beta$$

if and only if r_1 is the unique root of

$$1+\frac{rf''_{\nu,n}(r)}{f'_{\nu,n}(r)}=\beta,$$

situated in $(0, j_{\nu,1}^{(n)})$.

(b) Observe that

$$1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}=(n-\nu+1)+\frac{(n-\nu+2)zJ_\nu^{(n+1)}(z)+z^2J_\nu^{(n+2)}(z)}{(n-\nu+1)J_\nu^{(n)}(z)+zJ_\nu^{(n+1)}(z)}.$$

By using (1.1) and (2.1), we have

$$g'_{\nu,n}(z)=2^\nu\Gamma(\nu-n+1)z^{n-\nu}\left[(n-\nu+1)J_\nu^{(n)}(z)+zJ_\nu^{(n+1)}(z)\right]=$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m} \tag{3.4}$$

and

$$\lim_{m\to\infty}\frac{m\log m}{\lambda(m,n,\nu)}=\frac{1}{2},$$

where

$$\lambda(m,n,\nu)=[2m\log 2+\log\Gamma(m+1)+\log\Gamma(2m-n+\nu+1)+\log\Gamma(m+\nu+1)-$$

$$-\log\Gamma(2m+\nu+1)-\log\Gamma(\nu-n+1)-\log(2m+1)].$$

Here, we used $m! = \Gamma(m+1)$ and $\lim_{m\to\infty}\dfrac{\log\Gamma(am+b)}{m\log m} = a$, where a and b are positive constants. So, by applying Hadamard's theorem [17, p. 26], we can write the infinite product representation of $g'_{\nu,n}(z)$ as follows:

$$g'_{\nu,n}(z)=\prod_{m\geq 1}\left(1-\frac{z^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2}\right), \tag{3.5}$$

where $\gamma_{\nu,m}^{(n)}$ denotes the mth positive zero of the function $g'_{\nu,n}$. From Lemma 2.5 for $\nu > n-1$ the function $g'_{\nu,n} \in \mathcal{LP}$, and the smallest positive zero of $g'_{\nu,n}$ does not exceed the first positive zero of $J_\nu^{(n)}$.

By means of (3.5) we have

$$1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}=1-\sum_{m\geq 1}\frac{2z^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2-z^2}.$$

If we use the inequality (3.2), for all $z \in \mathbb{D}_{\gamma_{\nu,m}^{(n)}}$, we get the inequality

$$\operatorname{Re}\left(1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right)\geq 1-\sum_{m\geq 1}\frac{2r^2}{\left(\gamma_{\nu,m}^{(n)}\right)^2-r^2},$$

where $|z| = r$. Thus, for $r \in (0, \gamma_{\nu,1}^{(n)})$, we have

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right)\right\} = 1+\frac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)}.$$

The function $G_{\nu,n} : (0, \gamma_{\nu,1}^{(n)}) \to \mathbb{R}$, defined by

$$G_{\nu,n}(r) = 1+\frac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)},$$

is strictly decreasing and $\lim_{r\searrow 0} G_{\nu,n}(r) = 1 > \beta$ and $\lim_{r\nearrow\gamma_{\nu,1}^{(n)}} G_{\nu,n}(r) = -\infty$. Herewith, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_2}$, we get that

$$\mathrm{Re}\left(1+\frac{zg''_{\nu,n}(z)}{g'_{\nu,n}(z)}\right) > \beta$$

if and only if r_2 is the unique root of

$$1+\frac{rg''_{\nu,n}(r)}{g'_{\nu,n}(r)} = \beta,$$

situated in $(0, \gamma_{\nu,1}^{(n)})$.

(c) Observe that

$$1+\frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)} = \frac{n-\nu+2}{2}+\frac{\sqrt{z}}{2}\,\frac{(n-\nu+3)J_\nu^{(n+1)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+2)}(\sqrt{z})}{(n-\nu+2)J_\nu^{(n)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})}.$$

By using (1.1) and (2.1), we have

$$h'_{\nu,n}(z) = 2^{\nu-1}\Gamma(\nu-n+1)z^{\frac{n-\nu}{2}}\left[(n-\nu+2)J_\nu^{(n)}(\sqrt{z})+\sqrt{z}J_\nu^{(n+1)}(\sqrt{z})\right] =$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m(m+1)\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}\left(\frac{z}{4}\right)^m \tag{3.6}$$

and

$$\lim_{m\to\infty}\frac{m\log m}{\tau(m,n,\nu)} = \frac{1}{2},$$

where

$$\tau(m,n,\nu) = [2m\log 2+\log\Gamma(m+1)+\log\Gamma(2m-n+\nu+1)+\log\Gamma(m+\nu+1)-$$

$$-\log\Gamma(2m+\nu+1)-\log\Gamma(\nu-n+1)-\log(m+1)].$$

So, by applying Hadamard's theorem [17, p. 26] we can write the infinite product representation of $h'_{\nu,n}(z)$ as follows:

$$h'_{\nu,n}(z) = \prod_{m\geq 1}\left(1-\frac{z}{\delta_{\nu,m}^{(n)}}\right), \tag{3.7}$$

where $\delta_{\nu,m}^{(n)}$ denotes the mth positive zero of the function $h'_{\nu,n}$. From Lemma 2.5 for $\nu > n-1$ the function $h'_{\nu,n} \in \mathcal{LP}$, and the smallest positive zero of $h'_{\nu,n}$ does not exceed the first positive zero of $J_\nu^{(n)}$.

By means of (3.5) we have

$$1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)} = 1 - \sum_{m\geq 1} \frac{z}{\delta_{\nu,m}^{(n)} - z}.$$

By using the inequality (3.2), for all $z \in \mathbb{D}_{\delta_{\nu,m}^{(n)}}$, we get the inequality

$$\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right) \geq 1 - \sum_{m\geq 1} \frac{r}{\delta_{\nu,m}^{(n)} - r},$$

where $|z| = r$. Thus, for $r \in \left(0, \delta_{\nu,1}^{(n)}\right)$, we have

$$\inf_{z\in\mathbb{D}_r} \left\{\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right)\right\} = 1 + \frac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)}.$$

The function $H_{\nu,n} : \left(0, \delta_{\nu,1}^{(n)}\right) \to \mathbb{R}$, defined by

$$H_{\nu,n}(r) = 1 + \frac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)},$$

is strictly decreasing and $\lim_{r\searrow 0} H_{\nu,n}(r) = 1 > \beta$ and $\lim_{r\nearrow\delta_{\nu,1}^{(n)}} H_{\nu,n}(r) = -\infty$. As a result, in view of the minimum principle for harmonic functions for $z \in \mathbb{D}_{r_3}$ we obtain that

$$\operatorname{Re}\left(1 + \frac{zh''_{\nu,n}(z)}{h'_{\nu,n}(z)}\right) > \beta$$

if and only if r_3 is the unique root of

$$1 + \frac{rh''_{\nu,n}(r)}{h'_{\nu,n}(r)} = \beta,$$

situated in $\left(0, \delta_{\nu,1}^{(n)}\right)$.

Theorem 3.2 is proved.

With regards to Theorem 3.2, we tabulate the radius of convexity for $f_{\nu,n},\ g_{\nu,n}$ and $h_{\nu,n}$ for a fixed $\nu = 3.5,\ n = 0, 1, 2, 3$ and, respectively, $\beta = 0$ and $\beta = 0.5$. These are given in Table 3.2. Also, in Table 3.2, we see that radius of convexity is decreasing according to the order of derivative and the order of convexity. On the other words, from all these results we concluded that $r_\beta^c(f_{\nu,0}) > > r_\beta^c(f_{\nu,1}) > r_\beta^c(f_{\nu,2}) > \ldots > r_\beta^c(f_{\nu,n}) > \ldots$ for $\beta \in [0, 1)$ and $\nu > n-1,\ n \in \mathbb{N}_0$. In addition to, we can write $r_{\beta_1}^c(f_{\nu,n}) < r_{\beta_0}^c(f_{\nu,n})$ for $0 \leq \beta_0 < \beta_1 < 1$ and $\nu > n-1,\ n \in \mathbb{N}_0$. Same inequalities is also true for $r_\beta^c(g_{\nu,n})$ and $r_\beta^c(h_{\nu,n})$.

Table 3.2

n	$r_\beta^c(f_{3.5,n})$		$r_\beta^c(g_{3.5,n})$		$r_\beta^c(h_{3.5,n})$	
	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$	$\beta=0$	$\beta=0.5$
0	2.7183	2.0865	0.5234	1.1461	6.2189	3.7194
1	1.8179	1.3998	1.2017	0.9084	3.7394	2.2873
2	1.0592	0.8123	0.8833	0.6715	1.9450	1.2190
3	0.4141	0.3131	0.5683	0.4350	0.7726	0.4968

For $n=0$ in the Theorem 3.2 we obtain the results of Baricz and Szász [4]. Our results is a common generalization of these results.

3.2. Bounds for radii of starlikeness and convexity of the functions $g_{\nu,n}$ ***and*** $h_{\nu,n}$**.** In this subsection, we consider two different functions $g_{\nu,n}$ and $h_{\nu,n}$ which are normalized forms of the Bessel function derivatives of the first kind given by (1.1). Here, firstly, our aim is to show that the radii of univalence of these functions correspond to the radii of starlikeness.

Theorem 3.3. *The following inequalities hold*:

(a) *If* $\nu>n-1$, *then* $r^*(g_{\nu,n})$ *satisfies the inequalities*

$$r^*(g_{\nu,n})<\sqrt{2}\sqrt{a_{\nu,n}^{-1}},$$

$$\frac{2\sqrt{3}}{3}\sqrt{a_{\nu,n}^{-1}}<r^*(g_{\nu,n})<2\sqrt{3}\sqrt{\frac{1}{9a_{\nu,n}-5b_{\nu,n}}}.$$

(b) *If* $\nu>n-1$, *then* $r^*(h_{\nu,n})$ *satisfies the inequalities*

$$r^*(h_{\nu,n})<2a_{\nu,n}^{-1},$$

$$2a_{\nu,n}^{-1}<r^*(h_{\nu,n})<\frac{8}{4a_{\nu,n}-3b_{\nu,n}},$$

where $a_{\nu,n}=\dfrac{\nu+2}{(\nu-n+2)(\nu-n+1)}$ *and* $b_{\nu,n}=\dfrac{(\nu+4)(\nu+3)}{(\nu-n+4)(\nu-n+3)(\nu+2)}$.

Proof. (a) By using the first Rayleigh sum (2.6) and the implict relation for $r^*(g_{\nu,n})$, obtained by Kreyszing and Todd [16], we get, for all $\nu>n-1$,

$$\frac{1}{\left(r^*(g_{\nu,n})\right)^2}=\sum_{m\geq 1}\frac{2}{\left(j_{\nu,m}^{(n)}\right)^2-\left(r^*(g_{\nu,n})\right)^2}>\sum_{m\geq 1}\frac{2}{\left(j_{\nu,m}^{(n)}\right)^2}=\frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}.$$

Now, by using the Euler – Rayleigh inequalities it is possible to have more tight bounds for the radius of univalence (and starlikeness) $r^*(g_{\nu,n})$. We define the function $\Psi_{\nu,n}(z)=g'_{\nu,n}(z)$, where $g'_{\nu,n}$ defined by (3.5). Now, taking logarithmic derivative of both sides of (3.5) for $|z|<\gamma_{\nu,1}^{(n)}$, we have

$$\frac{\Psi'_{\nu,n}(z)}{\Psi_{\nu,n}(z)}=-\sum_{m\geq 1}\frac{2z}{\left(\gamma_{\nu,m}^{(n)}\right)^2-z^2}=$$

$$= -2\sum_{m\geq 1}\sum_{k\geq 0}\frac{1}{\left(\gamma_{\nu,m}^{(n)}\right)^{2(k+1)}}z^{2k+1} = -2\sum_{k\geq 0}\sigma_{k+1}z^{2k+1}, \tag{3.8}$$

where $\sigma_k = \sum_{m\geq 1}\left(\gamma_{\nu,m}^{(n)}\right)^{-k}$ is Euler–Rayleigh sum for the zeros of $\Psi_{\nu,n}$. Also, using (3.4) from the infinite sum representation of $\Psi_{\nu,n}$, we obtain

$$\frac{\Psi'_{\nu,n}(z)}{\Psi_{\nu,n}(z)} = \frac{\sum_{m\geq 0}U_m z^{2m+1}}{\sum_{m\geq 0}V_m z^{2m}}, \tag{3.9}$$

where

$$U_m = \frac{2(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(2m+3)}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$V_m = \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(2m+1)}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients with the same degrees of (3.8) and (3.9), we obtain the Euler–Rayleigh sums

$$\sigma_1 = \frac{3(\nu+2)}{4(\nu-n+2)(\nu-n+1)}$$

and

$$\sigma_2 = \frac{3(\nu+2)}{16(\nu-n+2)(\nu-n+1)}\times$$

$$\times\left(\frac{3(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \frac{5(\nu+4)(\nu+3)}{3(\nu-n+4)(\nu-n+3)(\nu+2)}\right).$$

By using the Euler–Rayleigh inequalities

$$\sigma_k^{-\frac{1}{k}} < \left(\gamma_{\nu,1}^{(n)}\right)^2 < \frac{\sigma_k}{\sigma_{k+1}}$$

for $\nu > n-1,\ k\in\mathbb{N}$ and $k=1$, we get the following inequality:

$$\frac{4(\nu-n+2)(\nu-n+1)}{3(\nu+2)} < \left(r^*(g_{\nu,n})\right)^2 <$$

$$< \frac{4}{\dfrac{3(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \dfrac{5(\nu+4)(\nu+3)}{3(\nu-n+4)(\nu-n+3)(\nu+2)}},$$

and it is possible to have more tighter bounds for other values of $k\in\mathbb{N}$.

(b) By using the first Rayleigh sum (2.6) and the implict relation for $r^*(h_{\nu,n})$, obtained by Kreyszing and Todd [16], we get, for all $\nu > n-1$,

$$\frac{1}{r^*(h_{\nu,n})} = \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2 - r^*(h_{\nu,n})} > \sum_{m\geq 1} \frac{1}{\left(j_{\nu,m}^{(n)}\right)^2} = \frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}.$$

Now, by using the Euler–Rayleigh inequalities it is possible to have more tight bounds for the radius of univalence (and starlikeness) $r^*(h_{\nu,n})$. We define the function $\Phi_{\nu,n}(z) = h'_{\nu,n}(z)$, where $h'_{\nu,n}$ defined by (3.6) or (3.7). Now, taking logarithmic derivative of both sides of (3.7), we have

$$\frac{\Phi'_{\nu,n}(z)}{\Phi_{\nu,n}(z)} = -\sum_{m\geq 1} \frac{1}{\delta_{\nu,m}^{(n)} - z} = -\sum_{m\geq 1}\sum_{k\geq 0} \frac{1}{\left(\delta_{\nu,m}^{(n)}\right)^{k+1}} z^k =$$

$$= -\sum_{k\geq 0} \rho_{k+1} z^k, \quad |z| < \delta_{\nu,1}^{(n)}, \tag{3.10}$$

where $\rho_k = \sum_{m\geq 1} \left(\delta_{\nu,m}^{(n)}\right)^{-k}$ is Euler–Rayleigh sum for the zeros of $\Phi_{\nu,n}$. Also, using (3.6) from the infinite sum representation of $\Phi_{\nu,n}$, we obtain

$$\frac{\Phi'_{\nu,n}(z)}{\Phi_{\nu,n}(z)} = \frac{\sum_{m\geq 0} K_m z^m}{\sum_{m\geq 0} L_m z^m}, \tag{3.11}$$

where

$$K_m = \frac{(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(m+2)}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$L_m = \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(m+1)}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients with the same degrees of (3.10) and (3.11), we get the Euler–Rayleigh sums

$$\rho_1 = \frac{\nu+2}{2(\nu-n+2)(\nu-n+1)}$$

and

$$\rho_2 = \frac{\nu+2}{4(\nu-n+2)(\nu-n+1)} \times$$

$$\times \left(\frac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \frac{3(\nu+4)(\nu+3)}{4(\nu-n+4)(\nu-n+3)(\nu+2)} \right).$$

If we use the Euler–Rayleigh inequalities

$$\rho_k^{-\frac{1}{k}} < \delta_{\nu,1}^{(n)} < \frac{\rho_k}{\rho_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$ and $k = 1$, then we obtain the following inequality:

$$\frac{2(\nu-n+2)(\nu-n+1)}{\nu+2} < r^*(h_{\nu,n}) <$$

$$< \frac{2}{\dfrac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \dfrac{3(\nu+4)(\nu+3)}{4(\nu-n+4)(\nu-n+3)(\nu+2)}}$$

and it is possible to have more tighter bounds for other values of $k \in \mathbb{N}$.

Theorem 3.3 is proved.

If we take $n = 0$ in the Theorem 3.3 we obtain the results of Aktaş et al. [1]. Our results is a common generalization of these results. For special cases of parameters ν and n, Theorem 3.3 reduces tight lower and upper bounds for the radii of starlikeness and convexity of many elemanter functions. For example, for $\nu = \dfrac{3}{2}$ and $n = 2$ in Theorem 3.3, we have

$$\sqrt{\frac{2}{7}} < r^*\Big(g_{\frac{3}{2},2}(z) = 4\sin z - 4z\cos z\Big) < \sqrt{\frac{3}{7}}$$

and

$$\frac{3}{7} < r^*\Big(h_{\frac{3}{2},2}(z) = 4\sqrt{z}\sin\sqrt{z} - 4z\cos\sqrt{z}\Big) < \frac{2940}{5969}.$$

The next result concerning bounds for radii of convexity of functions $g_{\nu,n}$ and $h_{\nu,n}$.

Theorem 3.4. *The following statements hold*:

(a) *If* $\nu > n-1$, *then* $r^c(g_{\nu,n})$ *satisfies the inequalities*

$$\frac{2}{3}\sqrt{a_{\nu,n}^{-1}} < r^c(g_{\nu,n}) < 6\sqrt{\frac{1}{81a_{\nu,n} - 25b_{\nu,n}}}.$$

(b) *If* $\nu > n-1$, *then* $r^c(h_{\nu,n})$ *satisfies the inequalities*

$$a_{\nu,n}^{-1} < r^c(h_{\nu,n}) < \frac{16}{16a_{\nu,n} - 9b_{\nu,n}},$$

where $a_{\nu,n}$ *and* $b_{\nu,n}$ *given by in Theorem* 3.3.

Proof. (a) By using the Alexander duality theorem for starlike and convex functions we can say that the function $g_{\nu,n}(z)$ is convex if and only if $zg'_{\nu,n}(z)$ is starlike. But, the smallest positive zero of $z \mapsto z(zg'_{\nu,n}(z))'$ is actually the radius of starlikeness of $z \mapsto (zg'_{\nu,n}(z))$, according to Theorems 3.1 and 3.2. Therefore, the radius of convexity $r^c(g_{\nu,n})$ is the smallest positive root of the equation $(zg'_{\nu,n}(z))' = 0$. Therefore, from (3.4), we have

$$\Delta_{\nu,n}(z) = (zg'_{\nu,n}(z))' = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)^2\Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^{2m}.$$

Since the function $g_{\nu,n}(z)$ belongs to the Laguerre – Pólya class of entire functions and $\mathcal{LP}$ is closed under differentiation, we can say that the function $\Delta_{\nu,n}(z) \in \mathcal{LP}$. Therefore, the zeros of the function $\Delta_{\nu,n}$ are all real. Suppose that $d_{\nu,m}^{(n)}$ are the zeros of the function $\Delta_{\nu,n}$. Then the function $\Delta_{\nu,n}$ has the infinite product representation as follows:

$$\Delta_{\nu,n}(z) = \prod_{m\geq 1}\left(1 - \frac{z^2}{\left(d_{\nu,m}^{(n)}\right)^2}\right). \tag{3.12}$$

By taking the logarithmic derivative of (3.12), we get

$$\frac{\Delta'_{\nu,n}(z)}{\Delta_{\nu,n}(z)} = -2\sum_{m\geq 1}\frac{z}{\left(d_{\nu,m}^{(n)}\right)^2 - z^2} =$$

$$= -2\sum_{m\geq 1}\sum_{k\geq 0}\frac{1}{\left(d_{\nu,m}^{(n)}\right)^{2(k+1)}}z^{2k+1} = -2\sum_{k\geq 0}\kappa_{k+1}z^{2k+1}, \quad |z| < d_{\nu,1}^{(n)}, \tag{3.13}$$

where $\kappa_k = \sum_{m\geq 1}\left(d_{\nu,m}^{(n)}\right)^{-k}$ is Euler – Rayleigh sum for the zeros of $\Delta_{\nu,n}$. On the other hand, by considering infinite sum representation of $\Delta_{\nu,n}(z)$, we obtain

$$\frac{\Delta'_{\nu,n}(z)}{\Delta_{\nu,n}(z)} = \frac{\sum_{m\geq 0} X_m z^{2m+1}}{\sum_{m\geq 0} Y_m z^{2m}}, \tag{3.14}$$

where

$$X_m = \frac{2(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(2m+3)^2}{m!4^{m+1}\Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$Y_m = \frac{(-1)^m\Gamma(2m+\nu+1)\Gamma(\nu-n+1)(2m+1)^2}{m!4^m\Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients of (3.13) and (3.14), we have

$$\kappa_1 = \frac{9(\nu+2)}{4(\nu-n+2)(\nu-n+1)}$$

and

$$\kappa_2 = \frac{9(\nu+2)}{16(\nu-n+2)(\nu-n+1)} \times$$

$$\times\left(\frac{9(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \frac{25(\nu+4)(\nu+3)}{9(\nu-n+4)(\nu-n+3)(\nu+2)}\right).$$

By using the Euler – Rayleigh inequalities

$$\kappa_k^{-\frac{1}{k}} < \left(d_{\nu,1}^{(n)}\right)^2 < \frac{\kappa_k}{\kappa_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$ and $k = 1$, we get the inequality

$$\frac{4(\nu-n+2)(\nu-n+1)}{9(\nu+2)} < \left(r^c(g_{\nu,n})\right)^2 <$$

$$< \frac{4}{\dfrac{9(\nu+2)}{(\nu-n+2)(\nu-n+1)} - \dfrac{25(\nu+4)(\nu+3)}{9(\nu-n+4)(\nu-n+3)(\nu+2)}}$$

and it is possible to have more tighter bounds for other values of $k \in \mathbb{N}$.

(b) By using the same procedure as in the previous proof, we can say that the radius of convexity $r^c(h_{\nu,n})$ is the smallest positive root of the equation $\left(zh'_{\nu,n}(z)\right)' = 0$ according to Theorem 3.2. From (3.6), we get

$$\Theta_{\nu,n}(z) = \left(zh'_{\nu,n}(z)\right)' = \sum_{m=0}^{\infty} \frac{(-1)^m (m+1)^2 \Gamma(2m+\nu+1)\Gamma(\nu-n+1)}{m! 4^m \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)} z^m. \tag{3.15}$$

Moreover, we know $h_{\nu,n}(z)$ belongs to the Laguerre – Pólya class of entire functions and $\mathcal{LP}$, consequently, $\Theta_{\nu,n}(z) \in \mathcal{LP}$. On the other words, the zeros of the function $\Theta_{\nu,n}$ are all real. Assume that $l^{(n)}_{\nu,m}$ are the zeros of the function $\Theta_{\nu,n}$. In this case, the function $\Theta_{\nu,n}$ has the infinite product representation as follows:

$$\Theta_{\nu,n}(z) = \prod_{m\geq 1} \left(1 - \frac{z^2}{\left(l^{(n)}_{\nu,m}\right)^2} \right). \tag{3.16}$$

By taking the logarithmic derivative of both sides of (3.16) for $|z| < l^{(n)}_{\nu,1}$, we have

$$\frac{\Theta'_{\nu,n}(z)}{\Theta_{\nu,n}(z)} = -\sum_{m\geq 1} \frac{1}{l^{(n)}_{\nu,m} - z} = -\sum_{m\geq 1}\sum_{k\geq 0} \frac{1}{\left(l^{(n)}_{\nu,m}\right)^{k+1}} z^k = -\sum_{k\geq 0} \omega_{k+1} z^k, \tag{3.17}$$

where $\omega_k = \sum_{m\geq 1} \left(l^{(n)}_{\nu,m}\right)^{-k}$. In addition, by using the derivative of infinite sum representation considering infinite sum representation of (3.15), we obtain

$$\frac{\Theta'_{\nu,n}(z)}{\Theta_{\nu,n}(z)} = \sum_{m\geq 0} T_m z^m \Big/ \sum_{m\geq 0} S_m z^m, \tag{3.18}$$

where

$$T_m = \frac{(-1)^{m+1}\Gamma(2m+\nu+3)\Gamma(\nu-n+1)(m+2)^2}{m! 4^{m+1} \Gamma(2m-n+\nu+3)\Gamma(m+\nu+2)}$$

and

$$S_m = \frac{(-1)^m \Gamma(2m+\nu+1)\Gamma(\nu-n+1)(m+1)^2}{m! 4^m \Gamma(2m-n+\nu+1)\Gamma(m+\nu+1)}.$$

By comparing the coefficients of (3.17) and (3.18), we get

$$\omega_1 = \frac{\nu+2}{(\nu-n+2)(\nu-n+1)}$$

and

$$\omega_2 = \frac{\nu+2}{(\nu-n+2)(\nu-n+1)} \times$$

$$\times \left(\frac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \frac{9(\nu+4)(\nu+3)}{16(\nu-n+4)(\nu-n+3)(\nu+2)} \right).$$

By using the Euler – Rayleigh inequalities

$$\omega_k^{-\frac{1}{k}} < l_{\nu,1}^{(n)} < \frac{\omega_k}{\omega_{k+1}}$$

for $\nu > n-1,\ k \in \mathbb{N}$ and $k = 1$, we have the following inequality:

$$\frac{(\nu-n+2)(\nu-n+1)}{\nu+2} < r^c(h_{\nu,n}) <$$

$$< \frac{1}{\dfrac{\nu+2}{(\nu-n+2)(\nu-n+1)} - \dfrac{9(\nu+4)(\nu+3)}{16(\nu-n+4)(\nu-n+3)(\nu+2)}}$$

and it is possible to have more tighter bounds for other values of $k \in \mathbb{N}$.

Theorem 3.4 is proved.

If we take $n = 0$ in the Theorem 3.4 we obtain the results of Aktaş et al. [2]. For special cases $n = 1, 2, 3,$ we obtain following result.

***Corollary* 3.1.** *The following inequalities hold*:

$$\frac{2}{3}\sqrt{a_{\nu,1}^{-1}} < r^c(g_{\nu,1}) < 6\sqrt{\frac{1}{81a_{\nu,1} - 25b_{\nu,1}}}, \quad \nu > 0,$$

$$a_{\nu,1}^{-1} < r^c(h_{\nu,1}) < \frac{16}{16a_{\nu,1} - 9b_{\nu,1}}, \quad \nu > 0,$$

$$\frac{2}{3}\sqrt{a_{\nu,2}^{-1}} < r^c(g_{\nu,2}) < 6\sqrt{\frac{1}{81a_{\nu,2} - 25b_{\nu,2}}}, \quad \nu > 1,$$

$$a_{\nu,2}^{-1} < r^c(h_{\nu,2}) < \frac{16}{16a_{\nu,2} - 9b_{\nu,2}}, \quad \nu > 1,$$

$$\frac{2}{3}\sqrt{a_{\nu,3}^{-1}} < r^c(g_{\nu,3}) < 6\sqrt{\frac{1}{81a_{\nu,3} - 25b_{\nu,3}}}, \quad \nu > 2,$$

$$a_{\nu,3}^{-1} < r^c(h_{\nu,3}) < \frac{16}{16a_{\nu,3} - 9b_{\nu,3}}, \quad \nu > 2,$$

where $a_{\nu,n}$ *and* $b_{\nu,n}$ *for* $n = 1, 2, 3$ *given by in Theorem* 3.3.

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