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MONOGENIC FUNCTIONS TAKING VALUES IN GENERALIZED CLIFFORD ALGEBRAS

МОНОГЕННІ ФУНКЦІЇ ЗІ ЗНАЧЕННЯМИ В УЗАГАЛЬНЕНИХ АЛГЕБРАХ КЛІФФОРДА

Generalized Clifford algebras are constructed by various methods and have some applications in mathematics and physics. In this paper, we introduce a new type of generalized Clifford algebra such that all components of a monogenic function are solutions of an elliptic partial differential equation. One of our aims is to cover more partial differential equations in framework of Clifford analysis. We shall prove some Cauchy integral representation formulae for monogenic functions in those cases.

Узагальнені алгебри Кліффорда будуються різними методами і мають певні застосування в математиці та фізиці. У цій роботі введено новий тип узагальненої алгебри Кліффорда, такий, що всі компоненти моногенної функції є розв'язками еліптичного диференціального рівняння з частинними похідними. Однією з цілей є охоплення більш широкого класу диференціальних рівнянь з частинними похідними в рамках аналізу Кліффорда. У відповідних випадках доведено деякі формули інтегрального зображення Коші для моногенних функцій.

1. Introduction. In the classical Clifford analysis [1], Clifford algebra $\mathcal{A}_n$ is generated by the imaginary elements $e_1, e_2, \ldots, e_n$ with the following multiplication rules:

$$e_i^2 = -1, \quad i = 1, 2, \ldots, n,$$

$$e_i e_j + e_j e_i = 0, \quad i \neq j.$$

A basis of $\mathcal{A}_n$ is $\mathcal{B} = \big\{e_0 = 1;\ e_{i_1 i_2 \ldots i_k} = e_{i_1} e_{i_2} \ldots e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\big\}$. An $\mathcal{A}_n$-valued function f has 2^n real-valued components $f(x) = \sum_A f_A(x) e_A,\ f_A(x) \in \mathbb{R}$. The Cauchy–Riemann operator D and its adjoin $\overline{D}$ are

$$D = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}, \qquad \overline{D} = \frac{\partial}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

The operator D applies from the left or from the right to a function f as follows:

$$Df = \frac{\partial f}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}, \qquad fD = \frac{\partial f}{\partial x_0} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i.$$

A function f is called a left monogenic function if $Df = 0$ and a right monogenic function if $fD = 0$. We also call a left monogenic function shortly a monogenic function. Since $D\overline{D} = \overline{D}D = \Delta$, all components of a left (right) monogenic function satisfy the Laplace equation.

Generalized Clifford algebras are constructed by various methods and have some important applications [2–8]. We introduce a new type of generalized Clifford algebra $\mathcal{A}_n(2k_i, \alpha_{ij})$. It is generated by the imaginary elements $e_1, e_2, \ldots, e_n$ with the following multiplication rules:

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$$e_i^{2k_i} = \sum_{j=0}^{2k_i-1} \alpha_{ij} e_i^j, \quad i = 1, 2, \ldots, n,$$

$$e_i e_j + e_j e_i = 0, \quad i \neq j,$$

where $k_1, k_2, \ldots, k_n \in \mathbb{N}^*$, $\alpha_{ij} \in \mathbb{R}$.

A basis of $\mathcal{A}_n(2k_i, \alpha_{ij})$ is $\mathcal{B} = \left\{ e_1^{\lambda_1} e_2^{\lambda_2} \ldots e_n^{\lambda_n} \mid 0 \leq \lambda_i \leq 2k_i - 1, \ 1 \leq i \leq n \right\}$. With $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, denote $e_\lambda = e_1^{\lambda_1} e_2^{\lambda_2} \ldots e_n^{\lambda_n}$. An element $w \in \mathcal{A}_n(2k_i, \alpha_{ij})$ is in the form $w = \sum_{e_\lambda \in \mathcal{B}} w_\lambda e_\lambda$, $w_\lambda \in \mathbb{R}$.

The Cauchy – Riemann operator D is defined by $D = \dfrac{\partial}{\partial x_0} + \sum_{i=1}^n e_i \dfrac{\partial}{\partial x_i}$.

In this paper, we introduce some generalized Clifford algebras such that all components of a monogenic function are solutions of an elliptic partial differential equation. One of our aims is to cover more partial differential equations in framework of Clifford analysis. This ideal is in [9], there various partial differential equations are treated. There is another approach to this problem. Solutions of some partial differential equations by means of monogenic functions given in commutative algebras are investigated in [10 – 13]. In [12, 13], a theory of monogenic functions is developed in finite-dimensional commutative algebras. In particular, integral theorems for these functions are proved and their relations with partial differential equations are studied.

In Section 2, we introduce $\mathcal{A}_1(2m)$ and get the equation $\dfrac{\partial^{2m} f}{\partial x_0^{2m}} + \dfrac{\partial^{2m} f}{\partial x_1^{2m}} = 0$ in $\mathbb{R}^2$. In Section 3, we introduce $\mathcal{A}'_n$ and get the equation $\left(\sum_{i=0}^{n-1} \dfrac{\partial^2}{\partial x_i^2} \right)^2 f + \dfrac{\partial^4 f}{\partial x_n^4} = 0$. In Section 4, we introduce $\mathcal{A}''_n$ and get the biharmonic equation in $\mathbb{R}^{n+1}$. We shall prove some Cauchy integral representation formulae for monogenic functions in those cases.

2. Hypercomplex algebra $\boldsymbol{\mathcal{A}_1(2m)}$. Instead of the equality $i^2 = -1$ in complex numbers, we give the rule $e^{2m} = -1$ for a new imaginary element e. It generates a hypercomplex algebra $\mathcal{A}_1(2m)$, $m \in \mathbb{N}^*$. A basis of $\mathcal{A}_1(2m)$ is

$$\mathcal{B} = \left\{ 1, e, e^2, \ldots, e^{2m-1} \right\}.$$

An element $w \in \mathcal{A}_1(2m)$ has the form $w = \sum_{k=0}^{2m-1} w_k e^k$, $w_k \in \mathbb{R}$. Define a norm $\|w\| = \sqrt{w_0^2 + w_1^2 + \ldots + w_{2m-1}^2}$.

In [14] the algebra with the basis $\mathcal{B}_1 = \{1, e, e^2\}$ for which $e^3 = -1$ is considered, and an algebra isomorphic to this algebra is considered in [15].

The Cauchy – Riemann operator is $D = \dfrac{\partial}{\partial x_0} + e \dfrac{\partial}{\partial x_1}$. The adjoin Cauchy – Riemann is defined by

$$\overline{D} = \left(\frac{\partial}{\partial x_0} - e \frac{\partial}{\partial x_1} \right) \left(\frac{\partial^{2m-2}}{\partial x_0^{2m-2}} + e^2 \frac{\partial^{2m-2}}{\partial x_0^{2m-4} \partial x_1^2} + \ldots + e^{2m-2} \frac{\partial^{2m-2}}{\partial x_1^{2m-2}} \right).$$

We have $\overline{D} D = D \overline{D} = \dfrac{\partial^{2m}}{\partial x_0^{2m}} + \dfrac{\partial^{2m}}{\partial x_1^{2m}}$.

We consider a special function $z = e x_0 - x_1$. It satisfies the equation $Dz = 0$. Then

$$\frac{1}{ex_0 - x_1} = -\frac{x_1 + ex_0}{x_1^2 - e^2x_0^2} = -\frac{(x_1 + ex_0)\big(x_1^{2m-2} + e^2x_1^{2m-4}x_0^2 + \ldots + e^{2m-2}x_0^{2m-2}\big)}{x_0^{2m} + x_1^{2m}}$$

with $x_0^2 + x_1^2 > 0$.

The function $\dfrac{1}{ex_0 - x_1}$ is considered as a Cauchy kernel of the operator D.

Lemma 2.1. *Let Ω be a domain in $\mathbb{R}^2$ and Ω_1 be a bounded domain with $\mathcal{C}^1$-boundary, $\overline{\Omega_1} \subset \Omega$. Suppose that $f \in \mathcal{C}^1(\Omega, \mathcal{A}_1(2m))$, then*

$$\oint\limits_{\partial\Omega_1} f(x_0, x_1)dz = -\iint\limits_{\Omega_1} \left(\frac{\partial f}{\partial x_0} + e\frac{\partial f}{\partial x_1}\right) dx_0dx_1 = -\iint\limits_{\Omega_1} Df dx_0dx_1.$$

Lemma 2.2.

$$I = \oint\limits_{x_0^2+x_1^2=\epsilon^2} \frac{edx_0 - dx_1}{ex_0 - x_1} = \frac{2\pi}{m}\sum_{k=0}^{m-1}\frac{1}{\sin\frac{(2k+1)\pi}{2m}}e^{2k+1} \quad \forall\epsilon > 0,$$

$$I^{-1} = \frac{-1}{2m\pi}\sum_{k=0}^{m-1}\frac{1}{\sin\dfrac{(2k+1)\pi}{2m}}e^{2k+1}.$$

Proof. We have

$$I = \oint\limits_{x_0^2+x_1^2=\epsilon^2} \frac{edx_0 - dx_1}{ex_0 - x_1} =$$

$$= \oint\limits_{x_0^2+x_1^2=\epsilon^2} \frac{(dx_1 - edx_0)(x_1 + ex_0)\big(x_1^{2m-2} + e^2x_1^{2m-4}x_0^2 + \ldots + e^{2m-2}x_0^{2m-2}\big)}{x_0^{2m} + x_1^{2m}},$$

$$x = \epsilon\cos t, \quad y = \epsilon\sin t,$$

$$I = \int\limits_{-\pi}^{\pi} \frac{(\cos t + e\sin t)(\sin t + e\cos t)\sum_{k=0}^{m-1} e^{2k}\sin^{2m-2k-2}t\cos^{2k}t}{\cos^{2m}t + \sin^{2m}t}dt =$$

$$= e\int\limits_{-\pi}^{\pi} \frac{\sum_{k=0}^{m-1} e^{2k}\sin^{2m-2k-2}t\cos^{2k}t}{\cos^{2m}t + \sin^{2m}t}dt =$$

$$= 4e\sum_{k=0}^{m-1} e^{2k}\int\limits_0^{\frac{\pi}{2}} \frac{\sin^{2m-2k-2}t\cos^{2k}t}{\cos^{2m}t + \sin^{2m}t}dt =$$

$$= 4e\sum_{k=0}^{m-1} e^{2k}\int\limits_0^{+\infty} \frac{u^{2m-2k-2}du}{u^{2m} + 1} =$$

$$= \frac{2e}{m} \sum_{k=0}^{m-1} e^{2k} \int_{0}^{+\infty} \frac{v^{\frac{2m-2k-1}{2m}-1} dv}{v+1} =$$

$$= \frac{2e}{m} \sum_{k=0}^{m-1} B\left(\frac{2m-2k-1}{2m}, \frac{2k+1}{2m}\right) e^{2k} =$$

$$= \frac{2\pi}{m} \sum_{k=0}^{m-1} \frac{1}{\sin \frac{(2k+1)\pi}{2m}} e^{2k+1},$$

where $B(\ , \ )$ is the Beta function.

Denote a polynomial $P(x) = \sum_{k=0}^{m-1} \frac{1}{\sin \frac{(2k+1)\pi}{2m}} x^{2k+1}$. We obtain

$$P\left[\cos\left(\frac{\pi}{2m} + k\frac{2\pi}{2m}\right) + i \sin\left(\frac{\pi}{2m} + k\frac{2\pi}{2m}\right)\right] = mi \quad (\forall k = 0, 1, \ldots, 2m-1) \Rightarrow$$

$$\Rightarrow [P(x)]^2 + m^2 = (x^{2m}+1)Q(x) \quad \text{(for some polynomial } Q(x) \text{ of order } 2m-2) \Rightarrow$$

$$\Rightarrow [P(e)]^2 = -m^2,$$

$$I = \frac{2\pi}{m} P(e) \Rightarrow I^{-1} = -\frac{P(e)}{2m\pi} = \frac{-1}{2m\pi} \sum_{k=0}^{m-1} \frac{1}{\sin \frac{(2k+1)\pi}{2m}} e^{2k+1}.$$

***Example* 2.1.** We get

$$m = 1 \Rightarrow I^{-1} = \frac{1}{2\pi i},$$

$$m = 2 \Rightarrow I^{-1} = \frac{-\sqrt{2}}{4\pi} \left(e + e^3\right),$$

$$m = 3 \Rightarrow I^{-1} = \frac{-1}{6\pi} \left(2e + e^3 + 2e^5\right),$$

$$m = 4 \Rightarrow I^{-1} = \frac{-1}{8\pi} \left[\sqrt{4+2\sqrt{2}}(e+e^7) + \sqrt{4-2\sqrt{2}}(e^3+e^5)\right],$$

$$m = 5 \Rightarrow I^{-1} = \frac{-1}{10\pi} \left[(\sqrt{5}+1)(e+e^9) + e^5 + (\sqrt{5}-1)(e^3+e^7)\right].$$

Apply Lemmas 2.1 and 2.2 we get the following Cauchy integral representation formula for monogenic functions taking value in $\mathcal{A}_1(2m)$.

Theorem 2.1. *Let Ω be an open domain in $\mathbb{R}^2$ and Ω_1 be a bounded domain with $\mathcal{C}^1$-boundary, $\overline{\Omega_1} \subset \Omega$. Suppose that $f \in \mathcal{C}^1(\Omega, \mathcal{A}_1(2m))$ and $Df = 0$ in Ω, then*

$$f(y) = \frac{-1}{2m\pi} \sum_{k=0}^{m-1} \frac{e^{2k+1}}{\sin \frac{(2k+1)\pi}{2m}} \oint_{\partial\Omega_1} \frac{f(x)(edx_0 - dx_1)}{e(x_0-y_0) - (x_1-y_1)} \quad \forall y \in \Omega_1.$$

3. Generalized Clifford algebra $\mathcal{A}_n'$. We consider the generalized Clifford algebra $\mathcal{A}_n'$ which is generated by n imaginary elements $e_1, e_2, \ldots, e_n$ with the following multiplication rules:

$$e_j^2 = -1, \quad j = 1, 2, \ldots, n-1,$$

$$e_n^4 = -1,$$

$$e_i e_j + e_j e_i = 0, \quad i \neq j.$$

A basis of $\mathcal{A}_n'$ is $\mathcal{B} = \{e_1^{i_1} e_2^{i_1} \ldots e_n^{i_n} \mid i_k \in \{0,1\},\ 1 \leq k \leq n-1,\ i_n \in \{0,1,2,3\}\}$. The dimension of $\mathcal{A}_n$ is 2^{n+1}. With $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ denote $e_\lambda = e_1^{\lambda_1} e_2^{\lambda_2} \ldots e_n^{\lambda_n}$, an element $w \in \mathcal{A}_n'$ has the form $w = \sum_{e_\lambda \in \mathcal{B}} w_\lambda e_\lambda,\ w_\lambda \in \mathbb{R}$.

The generalized Cauchy–Riemann operator and its adjoin are given by

$$D = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i},$$

$$\overline{D} = \left(\frac{\partial}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}\right)\left(\sum_{i=0}^{n-1} \frac{\partial^2}{\partial x_i^2} + e_n^2 \frac{\partial^2}{\partial x_n^2}\right),$$

$$\overline{D} D = D \overline{D} = \left(\sum_{i=0}^{n-1} \frac{\partial^2}{\partial x_i^2}\right)^2 + \frac{\partial^4}{\partial x_n^4}.$$

***Example* 3.1.** Matrix representation of $\mathcal{A}_2'$:

$$E_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

We have $E_1^2 = -I,\ E_2^4 = -I,\ E_1E_2 + E_2E_1 = 0,\ \mathcal{A}'_2$ can be considered as a subalgebra of the space of matrices $M_{8\times 8},\ \mathcal{A}'_2$ is generated by $E_1,\ E_2$.

4. Generalized Clifford algebra $\mathcal{A}''_n$. We change the multiplication rules of the imaginary elements in $\mathcal{A}''_n$ as the following:

$$e_j^2 = -1, \quad j = 1, 2, \ldots, n-1,$$

$$e_n^4 = -2e_n^2 - 1,$$

$$e_ie_j + e_je_i = 0, \quad i \neq j.$$

The generalized Cauchy – Riemann operator and its adjoin are given by

$$D = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i},$$

$$\overline{D} = \left(\frac{\partial}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}\right)\left(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2} + (e_n^2+1)\frac{\partial^2}{\partial x_n^2}\right),$$

$$\overline{D}D = D\overline{D} = \left(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2} - (e_n^2+1)\frac{\partial^2}{\partial x_n^2}\right)\left(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2} + (e_n^2+1)\frac{\partial^2}{\partial x_n^2}\right) =$$

$$= \left(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}\right)^2 - (e_n^2+1)^2\frac{\partial^4}{\partial x_n^4} = \left(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}\right)^2 = \Delta^2.$$

***Example* 4.1.** Matrix representation of $\mathcal{A}''_2$:

$$E_1 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

We have $E_1^2=-I,\ E_2^4=-2E_2^2-I,\ E_1E_2+E_2E_1=0$. $\mathcal{A}_2''$ can be considered as a subalgebra of the space of matrices $M_{8\times 8}$, $\mathcal{A}_2''$ is generated by $E_1,\ E_2$.

Definition 4.1. *Define a Cauchy kernel of the operator D by*

$$E(x,y)=\overline{D}H(x,y),$$

where $H(x,y)$ is the fundamental solution of the biharmonic equation in $\mathbb{R}^{n+1}$.

In the following lemma we consider the case $n\geq 4$:

$$H(x,y)=\frac{1}{2(n-1)(n-3)\omega_n|x-y|^{n-3}},$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$.

Lemma 4.1.

$$I=\int\limits_{|x-y|=\epsilon}E(x,y)\mathbf{n}(x)dS(x)=1\quad\forall\epsilon>0,$$

where $\mathbf{n}(x)=\nu_0+e_1\nu_1+\ldots+e_n\nu_n,\ \overrightarrow{n}=(\nu_0,\nu_1,\ldots,\nu_n)$ is the outer unit normal vector of the boundary.

Proof. We have

$$\frac{\partial H}{\partial x_i}=\frac{-(x_i-y_i)}{2(n-1)\omega_n|x-y|^{n-1}},\qquad\frac{\partial^2H}{\partial x_i^2}=\frac{(x_i-y_i)^2}{2\omega_n|x-y|^{n+1}}-\frac{1}{2(n-1)\omega_n|x-y|^{n-1}},$$

$$\Phi(x,y)=\Delta H+(e_n^2+1)\frac{\partial^2H}{\partial x_n^2}=\frac{-(e_n^2+3)}{2(n-1)\omega_n|x-y|^{n-1}}+(e_n^2+1)\frac{(x_n-y_n)^2}{2\omega_n|x-y|^{n+1}},$$

$$\int\limits_{|x-y|=\epsilon}E(x,y)\mathbf{n}(x)dS(x)=\int\limits_{|x-y|=\epsilon}\left(\frac{\partial}{\partial x_0}-\sum_{i=1}^{n}e_i\frac{\partial}{\partial x_i}\right)\Phi(x,y)\mathbf{n}(x)dS(x)=$$

$$=\int\limits_{|x-y|=\epsilon}\left(\nu_0\frac{\partial\Phi}{\partial x_0}-\sum_{i=1}^{n}e_i^2\nu_i\frac{\partial\Phi}{\partial x_i}\right)dS(x)=$$

$$=\int\limits_{|x-y|=\epsilon}\left[\left(\nu_0\frac{\partial\Phi}{\partial x_0}+\sum_{i=1}^{n}\nu_i\frac{\partial\Phi}{\partial x_i}\right)-(e_n^2+1)\nu_n\frac{\partial\Phi}{\partial x_n}\right]dS(x),$$

$$\int\limits_{|x-y|=\epsilon}\left(\nu_0\frac{\partial}{\partial x_0}+\sum_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}\right)\frac{1}{|x-y|^{n-1}}dS(x)=-(n-1)\omega_n,$$

$$\int\limits_{|x-y|=\epsilon}\left(\nu_0\frac{\partial}{\partial x_0}+\sum_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}\right)\frac{(x_n-y_n)^2}{|x-y|^{n+1}}dS(x)=\frac{-(n-1)}{n+1}\omega_n,$$

$$(e_n^2+1)\int\limits_{|x-y|=\epsilon}\nu_n\frac{\partial\Phi}{\partial x_n}dS(x)=\frac{e_n^2+1}{n+1},$$

$$I = \frac{e_n^2 + 3}{2} - (e_n^2 + 1)\frac{(n-1)}{2(n+1)} - \frac{e_n^2 + 1}{n+1} = 1.$$

Remark **4.1.** The fundamental solutions of the biharmonic equation in the cases $n = 2$, $n = 3$ are given by

$$H(x,y) = \frac{-|x-y|}{8\pi}, \quad n = 2,$$

$$H(x,y) = \frac{-\ln|x-y|}{8\pi^2}, \quad n = 3.$$

In these cases, Lemma 4.1 can be proved similarly.

Lemma 4.2. *Let Ω be a domain in $\mathbb{R}^{n+1}$, $n \geq 4$, and Ω_1 be a bounded domain with $\mathcal{C}^1$-boundary, $\overline{\Omega_1} \subset \Omega$. Suppose that $f, g \in \mathcal{C}^1(\Omega, \mathcal{A}_n'')$, then*

$$\int\limits_{\partial\Omega_1} g(x)\mathbf{n}(x)f(x)dS(x) = \int\limits_{\Omega_1} \big[g(x)D.f(x) + g(x).Df(x)\big]dx.$$

Apply Lemmas 4.1 and 4.2 we get the following Cauchy integral representation formula for monogenic functions taking value in $\mathcal{A}_n''$.

Theorem 4.1. *Let Ω be a domain in $\mathbb{R}^{n+1}$ and Ω_1 be a bounded domain with $\mathcal{C}^1$-boundary, $\overline{\Omega_1} \subset \Omega$. Suppose that $f \in \mathcal{C}^1(\Omega, \mathcal{A}_n'')$ and $Df = 0$ in Ω, then*

$$f(y) = \int\limits_{\partial\Omega_1} E(x,y)\mathbf{n}(x)f(x)dS(x) \quad \forall y \in \Omega_1.$$

With the similar purpose, the biharmonic equation is investigated in Clifford analysis [16] and in the theory of monogenic functions talking values in commutative algebras [17 – 19].

An open problem. There is an open question: Consider a given partial differential equation. Could we find a suitable generalized Clifford algebra such that all components of a monogenic function are solutions of the given partial differential equation? For example, we can answer this question in a simple case:

Let

$$\frac{\partial^{2m} f}{\partial x_0^{2m}} - \sum_{k=0}^{2m-1} a_k \frac{\partial^{2m} f}{\partial x_0^k \partial x_1^{2m-k}} = 0$$

be an elliptic equation in $\mathbb{R}^2$, $a_k \in \mathbb{R}$. The imaginary element e must obey the multiplication rule $e^{2m} = \sum_{k=0}^{2m-1} (-1)^k a_k e^k$.

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