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INITIAL SEVEN COEFFICIENT ESTIMATES FOR A SUBCLASS OF BI-STARLIKE FUNCTIONS

ПОЧАТКОВІ ОЦІНКИ СЕМИ КОЕФІЦІЄНТІВ ДЛЯ ПІДКЛАСУ БІЗІРКОВИХ ФУНКЦІЙ

In the present article, a subclass of bi-starlike functions is studied and initial seven Taylor – Maclaurin coefficient estimates $|a_2|, |a_3|, \ldots, |a_7|$ for functions in the subclass of the function class Σ are obtained for the first time in the literature. Few new or known consequences of the results are also pointed out.

Досліджується підклас бізіркових функцій та вперше отримано оцінки сімох початкових коефіцієнтів Тейлора – Маклорена $|a_2|, |a_3|, \ldots, |a_7|$ для функцій у підкласі функціонального класу Σ. Також вказано деякі наслідки результатів, як нові, так і відомі раніше.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in Δ. The important and well analyzed subclasses of the univalent function class $\mathcal{S}$ includes, the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)$ of convex functions of order α, $0 \le \alpha < 1$, in Δ. It is well-known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad z \in \Delta,$$

and

$$f(f^{-1}(w)) = w, \qquad |w| < r_0(f), \quad r_0(f) \ge 1/4,$$

where

$$\begin{gathered} f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + (14a_2^4 - \\ - 21a_2^2a_3 + 6a_4a_2 + 3a_3^2 - a_5)w^5 + (-42a_2^5 + 84a_2^3a_3 - 28a_4a_2^2 - \\ - 28a_2a_3^2 + 7a_5a_2 + 7a_4a_3 - a_6)w^6 + (132a_2^6 - 330a_2^4a_3 + 120a_2^3a_4 + \\ + 180a_2^2a_3^2 - 36a_5a_2^2 - 72a_2a_3a_4 + 8a_6a_2 - 12a_3^3 + 8a_5a_3 + 4a_4^2 - a_7)w^7 + \ldots . \end{gathered} \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1). Earlier, Brannan and Taha [4] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order

α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex function of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively.

A function $f(z) \in \mathcal{A}$ is in the class of strongly bi-starlike functions $\mathcal{S}^*_\Sigma[\alpha]$ [4, 15] of order α, $0 < \alpha \leq 1$, if each of the following conditions is satisfied:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}$$

and strongly bi-convex functions $\mathcal{K}^*_\Sigma[\alpha]$ [4, 15] of order α, $0 < \alpha \leq 1$,

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad \left|\arg\left(1 + \frac{wg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2},$$

where g is given by (1.2). For each of the function classes $\mathcal{S}^*_\Sigma[\alpha]$ and $\mathcal{K}_\Sigma[\alpha]$, non-sharp estimates on the first two Taylor – Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [4, 15]. Though intensive research is happening to settle the coefficient problem of obtaining each of the following Taylor – Maclaurin coefficients:

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}, \qquad \mathbb{N} := \{1, 2, 3, \ldots\},$$

it is still an open problem (see [3, 4, 10, 12, 15]). Many researchers (see [14, 16, 17]) have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first few Taylor – Maclaurin coefficients $|a_2|, |a_3|$ and $|a_4|$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [19] unified various subclasses of starlike and convex functions for which either of the quantity $\dfrac{z\, f'(z)}{f(z)}$ or $1+\dfrac{z\, f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this analysis, an analytic function ϕ with positive real part in the unit disk Δ was considered, with $\phi(0) = 1, \phi'(0) > 0$, and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma – Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\dfrac{z\, f'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma – Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \dfrac{z\, f''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma – Minda type or bi-convex of Ma – Minda type if both f and f^{-1} are, respectively, Ma – Minda starlike or convex. These classes are denoted, respectively, by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.

Motivated by the earlier work of coefficient estimate analysis study of various subclasses of bi-univalent function class [1 – 3, 5 – 11, 13, 14, 16, 17], in the present article, a bi-starlike function subclass of Σ is considered and initial seven Taylor – Maclaurin coefficient estimates $|a_2|$, $|a_3|, \ldots,$ $|a_7|$ for functions in the subclass of the function class Σ are obtained. It can be easily observed in the literature that only $|a_2|$, $|a_3|$, $|a_4|$ coefficient estimates are obtained for many subclasses of function class Σ so far, where as $|a_5|$, $|a_6|$, $|a_7|$ are estimated for the first time in the literature without any assumptions.

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad B_1 > 0.$$

The following bi-starlike class definition is considered from the literature for the study.

Definition 1.1. *A function* $f(z) \in \Sigma$ *given by* (1.1) *is said to be in the class* $\mathcal{S}^*_\Sigma(\phi)$ *if the following conditions are satisfied*:

$$\frac{z\, f'(z)}{f(z)} \prec \phi(z) \tag{1.3}$$

and

$$\frac{w\, g'(w)}{g(w)} \prec \phi(w) \tag{1.4}$$

where $z,\ w \in \Delta$ *and the function* g *is given by* (1.2).

2. Coefficient estimates for the function class $\mathcal{S}^*_\Sigma(\phi)$. The following lemma is used to derive our main result.

Lemma 2.1 [18]. *If* $h \in \mathcal{P}$, *then* $|c_k| \le 2$ *for each* k, *where* $\mathcal{P}$ *is the family of all functions* h *analytic in* Δ *for which real part of* $h(z) > 0$ *and*

$$h(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad \textit{for} \quad z \in \Delta.$$

Theorem 2.1. *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}^*_\Sigma(\phi)$. *Then the initial seven Taylor–Maclaurin coefficient estimates are*

$$|a_2| \le B_1,$$

$$|a_3| \le B_1^2 + B_1/2,$$

$$|a_4| \le 2B_1^3/3 + 5B_1^2/4 + 4B_1/3 + 4|B_2|/3 + |B_3|/3,$$

$$|a_5| \le 7B_1/4 + |B_3|(B_1 + 3/4) + 35B_1^2/8 + 5B_1^3/4 + |B_2|(4B_1 + 9/4),$$

$$|a_6| \le 16B_1/5 + 8|B_4|/5 + |B_5|/5 + |B_3|(B_1^2 + 77B_1/24 + 24/5) + 63B_1^2/8 +$$
$$+ 19B_1^3/4 + 7B_1^4/12 + B_1^5/5 + |B_2|(4B_1^2 + 77B_1/8 + 32/5),$$

$$|a_7| \le 31B_1/6 + |B_3|(10B_1^3/9 + 115B_1^2/24 + 2023B_1/90 + 16|B_2|/9 + 35/3) +$$
$$+ 1787B_1^2/90 + 231B_1^3/16 + 32|B_2|^2/9 + 5909B_1^4/36 + 2|B_3|^2/9 +$$
$$+ 2899B_1^5/12 + 49B_1^6/45 + |B_5|(4B_1/5 + 5/6) + |B_4|(32B_1/5 + 5) +$$
$$+ |B_2|(1124B_1^3/9 + 385B_1^2/24 + 3349B_1/90 + 25/2).$$

Proof. It follows from (1.3) and (1.4) that

$$\frac{z\, f'(z)}{f(z)} = \phi(u(z)) \tag{2.1}$$

and

$$\frac{w\,g'(w)}{g(w)} = \phi(v(w)). \tag{2.2}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + p_5 z^5 + p_6 z^6 + p_7 z^7 + \ldots$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + q_4 z^4 + q_5 z^5 + q_6 z^6 + q_7 z^7 + \ldots,$$

or, equivalently,

$$u(z) = \frac{p(z)-1}{p(z)+1} = (p_1 z)/2 + (p_2/2 - p_1^2/4)z^2 + (p_3/2 - p_1(p_2/4 - p_1^2/8) - (p_1p_2)/4)z^3 +$$
$$+ (p_4/2 - p_2(p_2/4 - p_1^2/8) - (p_1p_3)/4 + p_1((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 +$$
$$+ (p_1p_2)/8))z^4 + (p_5/2 - p_3(p_2/4 - p_1^2/8) - (p_1p_4)/4 + p_2((p_1(p_2/4 -$$
$$- p_1^2/8))/2 - p_3/4 + (p_1p_2)/8) - p_1(p_4/4 - (p_2(p_2/4 - p_1^2/8))/2 -$$
$$- (p_1p_3)/8 + (p_1((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 + (p_1p_2)/8))/2))z^5 + \ldots$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = (q_1 z)/2 + (q_2/2 - q_1^2/4)z^2 + (q_3/2 - q_1(q_2/4 - q_1^2/8) - (q_1q_2)/4)z^3 +$$
$$+ (q_4/2 - q_2(q_2/4 - q_1^2/8) - (q_1q_3)/4 + q_1((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 +$$
$$+ (q_1q_2)/8))z^4 + (q_5/2 - q_3(q_2/4 - q_1^2/8) - (q_1q_4)/4 + q_2((q_1(q_2/4 -$$
$$- q_1^2/8))/2 - q_3/4 + (q_1q_2)/8) - q_1(q_4/4 - (q_2(q_2/4 - q_1^2/8))/2 -$$
$$- (q_1q_3)/8 + (q_1((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 + (q_1q_2)/8))/2))z^5 + \ldots .$$

Then $p(z)$ and $q(z)$ are analytic in Δ with $p(0) = 1 = q(0)$. Since $u,\ v\colon \Delta \to \Delta$, the functions $p(z)$ and $q(z)$ have a positive real part in Δ, and, for each i,

$$|p_i| \le 2 \quad \text{and} \quad |q_i| \le 2.$$

Since $p(z)$ and $q(w)$ in $\mathcal{P}$ we have the following forms:

$$\phi(u(z)) = \phi\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \ldots\right]\right) =$$
$$= ((B_1p_1)/2)z + ((B_2p_1^2)/4 + B_1(p_2/2 - p_1^2/4))z^2 + (-B_1(p_1(p_2/4 - p_1^2/8) -$$
$$- p_3/2 + (p_1p_2)/4) + (B_3p_1^3)/8 + B_2p_1(p_2/2 - p_1^2/4))z^3 + ((B_4p_1^4)/16 -$$
$$- B_2(p_1(p_1(p_2/4 - p_1^2/8) - p_3/2 + (p_1p_2)/4) - (p_2/2 - p_1^2/4)^2) + B_1(p_4/2 -$$

$$- p_2(p_2/4 - p_1^2/8) - (p_1p_3)/4 + p_1((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 + (p_1p_2)/8)) +$$

$$+ (3B_3p_1^2(p_2/2 - p_1^2/4))/4)z^4 + (-B_3(-p_1(p_2/2 - p_1^2/4)^2 + (p_1(p_1(p_1(p_2/4 -$$

$$- p_1^2/8) - p_3/2 + (p_1p_2)/4) - (p_2/2 - p_1^2/4)^2))/2 + (p_1^2(p_1(p_2/4 - p_1^2/8) - p_3/2 +$$

$$+ (p_1p_2)/4))/4) - B_2(2(p_2/2 - p_1^2/4)(p_1(p_2/4 - p_1^2/8) - p_3/2 + (p_1p_2)/4) -$$

$$- p_1(p_4/2 - p_2(p_2/4 - p_1^2/8) - (p_1p_3)/4 + p_1((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 +$$

$$+ (p_1p_2)/8))) + (B_5p_1^5)/32 - B_1(-p_5/2 + p_3(p_2/4 - p_1^2/8) + (p_1p_4)/4 -$$

$$- p_2((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 + (p_1p_2)/8) + p_1(p_4/4 - (p_2(p_2/4 -$$

$$- p_1^2/8))/2 - (p_1p_3)/8 + (p_1((p_1(p_2/4 - p_1^2/8))/2 - p_3/4 + (p_1p_2)/8))/2)) +$$

$$+ (B_4p_1^3(p_2/2 - p_1^2/4))/2)z^5 + \ldots$$

and

$$\phi(v(w)) = \phi\left(\frac{1}{2}\left[q_1w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \ldots\right]\right) =$$

$$= ((B_1q_1)/2)w + ((B_2q_1^2)/4 + B_1(q_2/2 - q_1^2/4))w^2 + (-B_1(q_1(q_2/4 - q_1^2/8) -$$

$$- q_3/2 + (q_1q_2)/4) + (B_3q_1^3)/8 + B_2q_1(q_2/2 - q_1^2/4))w^3 + ((B_4q_1^4)/16 -$$

$$- B_2(q_1(q_1(q_2/4 - q_1^2/8) - q_3/2 + (q_1q_2)/4) - (q_2/2 - q_1^2/4)^2) + B_1(q_4/2 -$$

$$- q_2(q_2/4 - q_1^2/8) - (q_1q_3)/4 + q_1((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 + (q_1q_2)/8)) +$$

$$+ (3B_3q_1^2(q_2/2 - q_1^2/4))/4)w^4 + (-B_3(-q_1(q_2/2 - q_1^2/4)^2 + (q_1(q_1(q_1(q_2/4 -$$

$$- q_1^2/8) - q_3/2 + (q_1q_2)/4) - (q_2/2 - q_1^2/4)^2))/2 + (q_1^2(q_1(q_2/4 - q_1^2/8) - q_3/2 +$$

$$+ (q_1q_2)/4))/4) - B_2(2(q_2/2 - q_1^2/4)(q_1(q_2/4 - q_1^2/8) - q_3/2 + (q_1q_2)/4) -$$

$$- q_1(q_4/2 - q_2(q_2/4 - q_1^2/8) - (q_1q_3)/4 + q_1((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 +$$

$$+ (q_1q_2)/8))) + (B_5q_1^5)/32 - B_1(-q_5/2 + q_3(q_2/4 - q_1^2/8) + (q_1q_4)/4 -$$

$$- q_2((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 + (q_1q_2)/8) + q_1(q_4/4 - (q_2(q_2/4 - q_1^2/8))/2 -$$

$$- (q_1q_3)/8 + (q_1((q_1(q_2/4 - q_1^2/8))/2 - q_3/4 + (q_1q_2)/8))/2)) +$$

$$+ (B_4q_1^3(q_2/2 - q_1^2/4))/2)w^5 + \ldots .$$

The analytic conditions given in the equation (2.1) and (2.2) are estimated using MatLab functions and by comparing the coefficients on both sides of the conditions given, we can easily get the following:

$$p_1 = -q_1,$$

$$a_2 = (p_1/2)B_1,$$

$$a_3 = (B_1^2 p_1^2)/4 + (p_2/8 - q_2/8)B_1,$$

$$a_4 = (-q_1^3/24)B_3 + (p_1^3/12)B_1^3 + (p_3/12 - q_3/12 + (p_2q_1)/12 + (q_1q_2)/12 -$$

$$- q_1^3/24)B_1 + ((5p_1p_2)/32 - (5p_1q_2)/32)B_1^2 + (q_1^3/12 - (q_1q_2)/12 - (p_2q_1)/12)B_2,$$

$$a_5 = ((3p_2^2)/128 - (3p_2q_2)/64 + (p_1q_1p_2)/8 + (3q_2^2)/128 + (p_1q_1q_2)/8 +$$

$$+ (p_1p_3)/8 - (p_1q_3)/8 - (p_1q_1^3)/16)B_1^2 + ((5p_1^2p_2)/64 - (5p_1^2q_2)/64)B_1^3 +$$

$$+ B_1(p_4/16 - q_4/16 + (p_3q_1)/16 + (q_1q_3)/16 + (3p_2q_1^2)/64 - (3q_1^2q_2)/64 -$$

$$- p_2^2/32 + q_2^2/32) + ((3p_2q_1^2)/64 - (3q_1^2q_2)/64 - (B_1p_1q_1^3)/16)B_3 +$$

$$+ ((3q_1^2q_2)/32 - (p_3q_1)/16 - (q_1q_3)/16 - (3p_2q_1^2)/32 - B_1((p_1p_2q_1)/8 -$$

$$- (p_1q_1^3)/8 + (p_1q_1q_2)/8) + p_2^2/32 - q_2^2/32)B_2,$$

$$a_6 = (-q_1^5/160)B_5 + B_1^2((7p_1p_4)/64 + (7p_2p_3)/192 - (7p_1q_4)/64 - (7p_2q_3)/192 -$$

$$- (7p_3q_2)/192 + (7q_2q_3)/192 - (7p_1p_2^2)/128 + (7p_1q_2^2)/128 + (7p_2^2q_1)/192 -$$

$$- (7p_2q_1^3)/384 - (7q_1q_2^2)/192 + (7q_1^3q_2)/384 + (21p_1p_2q_1^2)/256 -$$

$$- (21p_1q_1^2q_2)/256 + (7p_1p_3q_1)/64 + (7p_1q_1q_3)/64) + B_1(p_5/20 - q_5/20 -$$

$$- (p_2p_3)/20 + (p_4q_1)/20 + (q_1q_4)/20 + (q_2q_3)/20 - (3p_2^2q_1)/80 + (p_2q_1^3)/40 +$$

$$+ (3p_3q_1^2)/80 - (3q_1q_2^2)/80 - (3q_1^2q_3)/80 + (q_1^3q_2)/40 - q_1^5/160) + ((7p_1^3q_2)/384 -$$

$$-(7p_1^3p_2)/384)B_1^4 + ((3p_1p_2^2)/128 + (p_1^2p_3)/16 + (3p_1q_2^2)/128 - (p_1^2q_3)/16 -$$

$$- (p_1^2q_1^3)/32 + (p_1^2p_2q_1)/16 + (p_1^2q_1q_2)/16 - (3p_1p_2q_2)/64)B_1^3 + (-p_1^5/160)B_1^5 +$$

$$+ (q_1^5/40 - (q_1^3q_2)/40 - (p_2q_1^3)/40)B_4 + ((p_2p_3)/20 - (p_4q_1)/20 - (q_1q_4)/20 -$$

$$- (q_2q_3)/20 - B_1((7p_1q_2^2)/128 - (7p_1p_2^2)/128 + (7p_2^2q_1)/192 - (7p_2q_1^3)/192 -$$

$$- (7q_1q_2^2)/192 + (7q_1^3q_2)/192 + (21p_1p_2q_1^2)/128 - (21p_1q_1^2q_2)/128 + (7p_1p_3q_1)/64 +$$

$$+ (7p_1q_1q_3)/64) - B_1^2((p_1^2p_2q_1)/16 - (p_1^2q_1^3)/16 + (p_1^2q_1q_2)/16) + (3p_2^2q_1)/40 -$$

$$- (3p_2q_1^3)/40 - (3p_3q_1^2)/40 + (3q_1q_2^2)/40 + (3q_1^2q_3)/40 - (3q_1^3q_2)/40 + q_1^5/40)B_2 +$$

$$+ ((3p_2q_1^3)/40 - (3p_2^2q_1)/80 - B_1((7p_2q_1^3)/384 - (7q_1^3q_2)/384 - (21p_1p_2q_1^2)/256 +$$

$$+ (21p_1q_1^2q_2)/256) + (3p_3q_1^2)/80 - (3q_1q_2^2)/80 - (3q_1^2q_3)/80 + (3q_1^3q_2)/40 -$$

$$- (3q_1^5)/80 - (B_1^2p_1^2q_1^3)/32)B_3,$$

$$a_7 = ((3q_1^2q_2^2)/32 - B_1^2((q_1^3(15p_1p_2 - 15p_1q_2))/576 - (p_1p_2q_1^3)/128 + (p_1q_1^3q_2)/128 -$$

$$- (21p_1^2p_2q_1^2)/512 + (21p_1^2q_1^2q_2)/512) + (5p_2q_1^4)/64 + (p_3q_1^3)/16 + (p_4q_1^2)/32 -$$

$$- (q_1^2 q_4)/32 + (q_1^3 q_3)/16 - (5q_1^4 q_2)/64 + p_2^3/96 - q_2^3/96 - B_1((3p_1 q_1^5)/40 -$$

$$- (3q_1^2 q_2^2)/128 + (q_1^3(8p_3 - 8q_3 + 8p_2 q_1 + 8q_1 q_2 - 4q_1^3))/576 - (3p_2^2 q_1^2)/128 +$$

$$+ (3p_1 p_2^2 q_1)/40 - (3p_1 p_2 q_1^3)/20 - (3p_1 p_3 q_1^2)/40 + (3p_1 q_1 q_2^2)/40 + (3p_1 q_1^2 q_3)/40 -$$

$$- (3p_1 q_1^3 q_2)/20 + (3p_2 q_1^2 q_2)/64) - (3p_2^2 q_1^2)/32 + (B_2 q_1^3(8p_2 q_1 + 8q_1 q_2 - 8q_1^3))/576 +$$

$$+ (5B_1^3 p_1^3 q_1^3)/288 - (p_2 p_3 q_1)/16 - (q_1 q_2 q_3)/16)B_3 + B_1^3((7p_1^2 p_4)/128 + (7p_2 q_2^2)/1024 -$$

$$- (7p_2^2 q_2)/1024 - (7p_1^2 q_4)/128 + (7p_2^3)/3072 - (7q_2^3)/3072 + ((15p_1 p_2 - 15p_1 q_2)$$

$$(8p_3 - 8q_3 + 8p_2 q_1 + 8q_1 q_2 - 4q_1^3))/2304 - (7p_1^2 p_2^2)/256 + (7p_1^2 q_2^2)/256 - (p_1 p_2^2 q_1)/64 +$$

$$+ (p_1 p_2 q_1^3)/128 + (7p_1^2 p_3 q_1)/128 + (p_1 q_1 q_2^2)/64 - (p_1 q_1^3 q_2)/128 + (7p_1^2 q_1 q_3)/128 +$$

$$+ (21p_1^2 p_2 q_1^2)/512 - (21p_1^2 q_1^2 q_2)/512 - (p_1 p_2 p_3)/64 + (p_1 p_2 q_3)/64 + (p_1 p_3 q_2)/64 -$$

$$- (p_1 q_2 q_3)/64) + ((49p_1^6)/2880)B_1^6 + (q_1^6/288)B_3^2 - B_2((3q_1^2 q_2^2)/32 + B_1^3((p_1^3(8p_2 q_1 +$$

$$+ 8q_1 q_2 - 8q_1^3))/288 + (85p_1^3((p_2 q_1)/12 + (q_1 q_2)/12 - q_1^3/12))/16 + (97p_1^3 q_1^3)/192 -$$

$$- (97p_1^3 p_2 q_1)/192 - (97p_1^3 q_1 q_2)/192) - (p_2 p_4)/24 + (p_5 q_1)/24 + (q_1 q_5)/24 + (q_2 q_4)/24 +$$

$$+ B_1((3q_1^2 q_2^2)/64 + ((8p_2 q_1 + 8q_1 q_2 - 8q_1^3)(8p_3 - 8q_3 + 8p_2 q_1 + 8q_1 q_2 - 4q_1^3))/2304 +$$

$$+ (p_2 q_2^2)/64 + (p_2^2 q_2)/64 - (p_1 q_1^5)/20 - p_2^3/64 - q_2^3/64 + (3p_2^2 q_1^2)/64 - (3p_1 p_2^2 q_1)/20 +$$

$$+ (3p_1 p_2 q_1^3)/20 + (3p_1 p_3 q_1^2)/20 - (3p_1 q_1 q_2^2)/20 - (3p_1 q_1^2 q_3)/20 + (3p_1 q_1^3 q_2)/20 -$$

$$- (3p_2 q_1^2 q_2)/32 - (p_1 p_2 p_3)/10 + (p_1 p_4 q_1)/10 + (p_2 p_3 q_1)/32 + (p_1 q_1 q_4)/10 + (p_1 q_2 q_3)/10 + \ldots$$

$$\ldots + (p_2 q_1 q_3)/32 - (p_3 q_1 q_2)/32 - (q_1 q_2 q_3)/32) + (5p_2 q_1^4)/96 + (p_3 q_1^3)/16 + (p_4 q_1^2)/16 -$$

$$- (q_1^2 q_4)/16 + (q_1^3 q_3)/16 - (5q_1^4 q_2)/96 + p_2^3/48 - p_3^2/48 - q_2^3/48 + q_3^2/48 +$$

$$+ B_1^2((((15p_1 p_2 - 15p_1 q_2)(8p_2 q_1 + 8q_1 q_2 - 8q_1^3))/2304 - (7p_1^2 p_2^2)/256 + (7p_1^2 q_2^2)/256 -$$

$$- (p_1 p_2^2 q_1)/64 + (p_1 p_2 q_1^3)/64 + (7p_1^2 p_3 q_1)/128 + (p_1 q_1 q_2^2)/64 - (p_1 q_1^3 q_2)/64 + (7p_1^2 q_1 q_3)/128 +$$

$$+ (21p_1^2 p_2 q_1^2)/256 - (21p_1^2 q_1^2 q_2)/256) - (3p_2^2 q_1^2)/32 - (p_2 p_3 q_1)/8 - (q_1 q_2 q_3)/8) +$$

$$+ ((255p_1^2(p_2 - q_2)^2)/1024 + (15p_1 p_2 - 15p_1 q_2)^2/4608 - (97p_1^3 p_3)/192 + (97p_1^3 q_3)/192 +$$

$$+ (p_1^3(8p_3 - 8q_3 + 8p_2 q_1 + 8q_1 q_2 - 4q_1^3))/288 + (85p_1^3(p_3/12 - q_3/12 + (p_2 q_1)/12 + (q_1 q_2)/12 -$$

$$- q_1^3/24))/16 - (323p_1^2 p_2^2)/1024 - (323p_1^2 q_2^2)/1024 + (97p_1^3 q_1^3)/384 + (323p_1^2 p_2 q_2)/512 -$$

$$- (97p_1^3 p_2 q_1)/192 - (97p_1^3 q_1 q_2)/192)B_1^4 + B_1(p_6/24 - q_6/24 + (q_1^2 q_2^2)/32 - (p_2 p_4)/24 +$$

$$+ (p_5 q_1)/24 + (q_1 q_5)/24 + (q_2 q_4)/24 + (5p_2 q_1^4)/384 + (p_3 q_1^3)/48 + (p_4 q_1^2)/32 - (q_1^2 q_4)/32 +$$

$$+ (q_1^3 q_3)/48 - (5q_1^4 q_2)/384 + p_2^3/96 - p_3^2/48 - q_2^3/96 + q_3^2/48 - (p_2^2 q_1^2)/32 - (p_2 p_3 q_1)/16 -$$

$$- (q_1q_2q_3)/16) + ((p_1^3(15p_1p_2 - 15p_1q_2))/288 + (85p_1^3((5p_1p_2)/32 - (5p_1q_2)/32))/16 -$$
$$- (971p_1^4p_2)/512 + (971p_1^4q_2)/512 + (255p_1^4(p_2 - q_2))/256)B_1^5 + B_1^2((3q_1^2q_2^2)/128 +$$
$$+ (8p_3 - 8q_3 + 8p_2q_1 + 8q_1q_2 - 4q_1^3)^2/4608 + (p_1p_5)/10 + (p_2p_4)/32 - (p_1q_5)/10 -$$
$$- (p_2q_4)/32 - (p_4q_2)/32 + (q_2q_4)/32 + (p_2q_2^2)/64 + (p_2^2q_2)/64 - (p_1q_1^5)/80 - p_2^3/64 -$$
$$- q_2^3/64 + (3p_2^2q_1^2)/128 - (3p_1p_2^2q_1)/40 + (p_1p_2q_1^3)/20 + (3p_1p_3q_1^2)/40 - (3p_1q_1q_2^2)/40 -$$
$$- (3p_1q_1^2q_3)/40 + (p_1q_1^3q_2)/20 - (3p_2q_1^2q_2)/64 - (p_1p_2p_3)/10 + (p_1p_4q_1)/10 + (p_2p_3q_1)/32 +$$
$$+ (p_1q_1q_4)/10 + (p_1q_2q_3)/10 + (p_2q_1q_3)/32 - (p_3q_1q_2)/32 - (q_1q_2q_3)/32) + ((8p_2q_1 + 8q_1q_2 -$$
$$- 8q_1^3)^2/4608)B_2^2 + ((5p_2q_1^4)/384 - (5q_1^4q_2)/384 - (B_1p_1q_1^5)/80)B_5 + ((5q_1^4q_2)/96 -$$
$$- (5p_2q_1^4)/96 - (p_3q_1^3)/48 - (q_1^3q_3)/48 - (q_1^2q_2^2)/32 - B_1((p_1p_2q_1^3)/20 - (p_1q_1^5)/20 +$$
$$+ (p_1q_1^3q_2)/20) + (p_2^2q_1^2)/32)B_4.$$

After applying modulus on both sides of the equations and applying the Lemma 2.1, we obtain

$$|a_2| \le B_1,$$
$$|a_3| \le B_1^2 + B_1/2,$$
$$|a_4| \le 2B_1^3/3 + 5B_1^2/4 + 4B_1/3 + 4|B_2|/3 + |B_3|/3,$$
$$|a_5| \le 7B_1/4 + |B_3|(B_1 + 3/4) + 35B_1^2/8 + 5B_1^3/4 + |B_2|(4B_1 + 9/4),$$
$$|a_6| \le 16B_1/5 + 8|B_4|/5 + |B_5|/5 + |B_3|(B_1^2 + 77B_1/24 + 24/5) + 63B_1^2/8 + 19B_1^3/4 +$$
$$+ 7B_1^4/12 + B_1^5/5 + |B_2|(4B_1^2 + 77B_1/8 + 32/5),$$
$$|a_7| \le 31B_1/6 + |B_3|(10B_1^3/9 + 115B_1^2/24 + 2023B_1/90 + 16|B_2|/9 + 35/3) +$$
$$+ 1787B_1^2/90 + 231B_1^3/16 + 32|B_2|^2/9 + 5909B_1^4/36 + 2|B_3|^2/9 + 2899B_1^5/12 +$$
$$+ 49B_1^6/45 + |B_5|(4B_1/5 + 5/6) + |B_4|(32B_1/5 + 5) + |B_2|(1124B_1^3/9 +$$
$$+ 385B_1^2/24 + 3349B_1/90 + 25/2).$$

The theorem is proved.

Considering the function ϕ to be

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots, \quad 0 < \alpha \le 1, \tag{2.3}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$, for the class of strongly starlike functions.

Also, if we consider

$$\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \ldots, \quad 0 \le \beta < 1, \tag{2.4}$$

then we have $B_1 = B_2 = 2(1 - \beta)$.

***Remark* 2.1.** By choosing $\phi(z)$ of the form (2.3) and (2.4), we can easily obtain the initial seven Taylor coefficient estimates $|a_2|, |a_3|, \ldots, |a_7|$ based on the result discussed in the Theorem 2.1, of which the higher estimates like $|a_5|$, $|a_6|$ and $|a_7|$ are obtained for the first time without any assumption in the estimation process.

***Remark* 2.2.** Based on the coefficient estimation procedure discussed in the above remark, one can easily notice that the initial estimates $|a_2|$ and $|a_3|$ leads to the well-known results given earlier by Brannan and Taha [4].

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