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G. Dwivedi (Birla Inst. Technology and Sci. Pilani, Rajasthan, India)

GENERALIZED PICONE IDENTITY FOR FINSLER p-LAPLACIAN AND ITS APPLICATIONS*

УЗАГАЛЬНЕНА ТОТОЖНІСТЬ ПІКОНЕ ДЛЯ p-ЛАПЛАСІАНА ФІНСЛЕРА ТА ЇЇ ЗАСТОСУВАННЯ

We prove a generalized Picone-type identity for Finsler p-Laplacian and use it to establish some qualitative results for some boundary-value problems involving Finsler p-Laplacian.

Доведено узагальнену тотожність типу Піконе для p-лапласіана Фінслера, яку потім використано для отримання деяких якісних результатів для граничних задач, що включають p-лапласіан Фінслера.

1. Introduction. In this paper, we establish a generalized Picone identity for the class of operators

$$\Delta_{H,p}u := \operatorname{div}(H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)), \tag{1.1}$$

where $p > 1,\ H : \mathbb{R}^n \to [0, \infty),\ n \geq 2,$ is a strictly convex, twice differentiable function which is positively homogeneous of degree 1, Δ and Δ_ξ denote the usual gradient operators with respect to variable x and ξ, respectively. The operators of the form (1.1) are called Finsler p-Laplacian or anisotropic p-Laplacian. A prototype function H is given by

$$H(\xi) = \|\xi\|_r = \left(\sum_{i=1}^{n} |\xi_i|^r\right)^{1/r}, \quad r > 1.$$

For this choice of H, the operator (1.1) reduces to

$$\Delta_{H,p}v = \operatorname{div}\left(\|\nabla v\|_r^{p-2}\,\nabla^r v\right), \tag{1.2}$$

where $\nabla^r v = \left(|v_{x_1}|^{r-2}v_{x_1}, \ldots, |v_{x_n}|^{r-2}v_{x_n}\right)$. (1.2) reduces to p-Laplacian if $r = 2$ and $p \in (1, \infty)$, while it reduces to pseudo p-Laplacian if $r = p > 1$. In case of $r = p = 2$, we get standard Laplace operator from (1.2).

Finsler p-Laplacian has been studied by several authors. V. Ferone and B. Kawohl [18] proved some properties of Finsler p-Laplacian such as existence of fundamental solution, maximum principle, comparison principle, mean value property etc. Belloni et al. [6] obtained positivity and simplicity of the first eigenvalue and Faber – Krahn inequality for Finsler p-Laplacian with Dirichlet boundary conditions. They also established symmetry of positive solutions to

$$-\Delta_{H,n}u = f(u) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a smooth and bounded domain in $\mathbb{R}^n$. G. Wang and C. Xia [26] obtained a lower bound for the first eigenvalue for Finsler p-Laplacian with Neumann boundary conditions. F. Della Pietra and

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N. Gavitone [9, 10] discussed existence and properties of the first eigenvalue of Finsler p-Laplacian with Dirichlet and Robin boundary conditions. G. Wang and C. Xia [27] studied blow up analysis for the problem $-\Delta_{H,2}u = V(x)e^u$ in dimension 2. We refer to [8, 22, 24, 29] and reference cited therein for some further existence and qualitative results involving Finsler p-Laplacian.

Next, let us recall some historical developments in Picone identity. The classical Picone identity [23] says that if u and v are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0. \tag{1.3}$$

(1.3) has an enormous applications to second-order elliptic equations and systems (see, for instance, [1, 2, 21] and the references therein). In order to apply (1.3) to p-Laplace equations, W. Allegretto and Y. X. Huang [3] extended (1.3) as follows.

Theorem 1.1 [3]. *Let $v > 0$ and $u \geq 0$ be differentiable functions in a domain Ω of $\mathbb{R}^n$. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla u\,\nabla v,$$

$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v) = R(u,v)$. Moreover, $L(u,v) \geq 0$ and $L(u,v) = 0$ a.e. in Ω if and only if $\nabla\left(\frac{u}{v}\right) = 0$ a.e. in Ω.

A nonlinear analogue of Theorem 1.1 was proved by J. Tyagi [25] in case of $p = 2$ and by K. Bal [4] in general case. The results of K. Bal [4] was further generalized by T. Feng [16] as follows.

Theorem 1.2 [16]. *Let $v > 0$ and $u \geq 0$ be differentiable functions in a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$. Assume that differentiable functions $g(u)$ and $f(v)$ satisfy that for $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\frac{g(u)f'(v)|\nabla v|^p}{[f(v)]^2} \geq \frac{p}{q}\left[\frac{g'(u)|\nabla v|^{p-1}}{pf(v)}\right]^q,$$

where $g(u),\ g'(u) > 0$ for $u > 0$; $g(u),\ g'(u) = 0$ for $u = 0$, and $f(v),\ f'(v) > 0$. Denote

$$L(u,v) = |\nabla u|^p - \frac{g'(u)|\nabla v|^{p-2}\nabla v \nabla u}{f(v)} + \frac{g(u)f'(v)|\nabla v|^p}{[f(v)]^2},$$

$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{g(u)}{f(v)}\right)|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v) = R(u,v) \geq 0$. Moreover, $L(u,v) = 0$ a.e. in Ω if and only if

$$\nabla\left(\frac{u}{v}\right) = 0,\ |\nabla u|^p = \left[\frac{g'(u)|\nabla v|^{p-1}}{pf(v)}\right]^q, \qquad \frac{p}{q}\left[\frac{g'(u)|\nabla v|^{p-1}}{pf(v)}\right]^q = \frac{g(u)f'(v)|\nabla v|^p}{[f(v)]^2}$$

a.e. in Ω.

There are several other interesting articles dealing with Picone identity in different contexts. For instance, for a Picone-type identity to higher order half linear differentiable operators, we refer to [20] and the references therein, for Picone identities to half-linear elliptic operators with $p(x)$-Laplacians, we refer to [28] for Picone-type identity to pseudo p-Laplacian with variable power, we refer to [7] and for Picone identity for biharmonic operators and applications, we refer to [11 – 14, 17]. J. Jaroš [19] proved a Picone identity for the class of operators (1.1). Their main result is follows.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and H be an arbitrary norm in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$. Assume that $u, v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\Omega)$ with $v(x) \neq 0$ in Ω and denote*

$$\Phi(u,v) := H(\nabla u)^p + (p-1)\frac{|u|^p}{|v|^p} H(\nabla v)^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle \nabla u, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\rangle.$$

Then

$$H(\nabla u)^p - \left\langle \nabla\left(\frac{|u|^p}{|v|^{p-2}v}\right),\ H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle = \Phi(u,v)$$

and $\Phi(u,v) \geq 0$ a.e. in Ω. If, in addition, $H(\xi)^p$ is strictly convex in $\mathbb{R}^n$, then $\Phi(u,v) = 0$ a.e. in Ω if and only if u is a constant multiple of v in Ω.

Bal et al. [5] generalized Theorem 1.3 as follows.

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For $u,\ v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\Omega)$ with $u \geq 0$ and $v > 0$, define*

$$A(u,v) = H(\nabla u)^p - p\frac{u^{p-1}}{f(v)}\left\langle \nabla u, H(\nabla v)^{p-1}\nabla_\xi(\nabla v)\right\rangle + \frac{u^p f'(v)}{(f(v))^2} H(\nabla v)^p =$$

$$= H(\nabla u)^p - \left\langle \nabla\left(\frac{u^p}{f(v)}\right), H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle \geq 0$$

for $f \in \mathbb{M} := \left\{f\colon (0,\infty) \to (0,\infty) f\colon f'(y) \geq (p-1)f(y)^{\frac{p-1}{p-2}}\right\} \subset C^1((0,\infty))$. Moreover, $A(u,v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω, where c is a constant.

In this paper, we prove a new nonlinear Picone-type identity, which is a generalization of Theorem 1.4. The main result of this paper is follows.

Theorem 1.5. *Let H be an arbitrary norm in $\mathbb{R}^n$ which is of class $C^1(\mathbb{R}^n\backslash\{0\})$. Assume that $u, v \in W^{1,p}_{\rm loc}(\Omega) \cap C(\Omega)$ with $u \geq 0$ and $v > 0$, where $\Omega \subseteq \mathbb{R}^n$ is a domain. Assume that g and f are twice differentiable functions satisfying*

$$\frac{g(u) f'(v) H(\nabla v)^p}{(f(v))^2} \geq \frac{p}{q}\left(\frac{g'(u) H(\nabla v)^{p-1}}{p f(v)}\right)^q,$$

where $g(u),\ g'(u) > 0$ for $u > 0$, $g(u),\ g'(u) = 0$ if $u = 0$, $f(v),\ f'(v) > 0$. Denote

$$L(u,v) = H(\nabla u)^p + \frac{g(u) f'(v)}{(f(v))^2} H(\nabla v)^p - \left\langle \frac{g'(u)}{f(v)}\nabla u, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle,$$

$$R(u,v) = H(\nabla u)^p - \left\langle \nabla\left(\frac{g(u)}{f(v)}\right), H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle.$$

Then (i) $L(u,v) = R(u,v) \geq 0$; (ii) $L(u,v) = 0$ *a.e. in Ω if and only if*

$$\nabla u = \left(\frac{g'(u)}{pf(v)}\right)^{1/p-1} \nabla v, \qquad \frac{g(u)f'(v)H(\nabla v)^p}{(f(v))^2} \geq \frac{p}{q}\left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q, \tag{1.4}$$

$$H(\nabla u)^p = \left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q. \tag{1.5}$$

***Remark* 1.1.** 1. If we choose $g(u) = u^p$ in Theorem 1.5, then we obtain Picone identity of Bal et al. [5].

2. If we choose $g(u) = u^p$ and $f(v) = v^{p-1}$ in Theorem 1.5, then we obtain Picone identity of J. Jaroš [19].

3. If we choose $H(\xi) = \left(\sum_{i=1}^n |\xi_i|^p\right)^{1/p}$ in Theorem 1.5, then we obtain Picone identity of T. Feng [16].

This paper is organized as follows. In Section 2, we state some elementary properties of an arbitrary norm. In Section 3, we prove Theorem 1.5 and Section 4 deals with some applications of Theorem 1.5.

2. Preliminaries. In this section, we recall some elementary properties of an arbitrary norm on $\mathbb{R}^n$. For further details, we refer to [19] and references therein. Let $H : \mathbb{R}^n \to [0, \infty)$ be any arbitrary norm in $\mathbb{R}^n$, i.e., a strictly convex, twice differentiable function such that:

(i) $H(\xi) > 0$ for any $\xi \neq 0$,

(ii) $H(t\xi) = |t|H(\xi)$ for all $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

(iii) if H is $C^1(\mathbb{R}^n\backslash\{0\})$, then $\nabla_\xi H(t\xi) = \operatorname{sgn} t \nabla_\xi H(\xi)$ for all $\xi \neq 0$ and $t \neq 0$,

(iv) $\langle \xi, \nabla_\xi H(\xi)\rangle = H(\xi)$ for all $\xi \in \mathbb{R}^n$, where the left-hand side is zero for $\xi = 0$,

(v) there exist constant $0 < c_1 \leq c_2$ such that $c_1|x| \leq H(x) \leq c_2|x|$.

Next, we define the dual norm H_0 of H by

$$H_0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi\rangle}{H(\xi)},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$.

Any norm H of class C^1 for $\xi \neq 0$ and its dual H_0 satisfy the following properties:

(i) $H_0(\nabla H(\xi)) = 1$ for $\xi \in \mathbb{R}^n\backslash\{0\}$,

(ii) $H(\nabla H_0(x)) = 1$ for $x \in \mathbb{R}^n\backslash\{0\}$,

(iii) $H[H_0(x)\nabla H_0(x)]\nabla_\xi[H_0(x)\nabla H_0(x)] = x$,

(iv) $H_0[H(\xi)\nabla_\xi H(\xi)]\nabla H_0[H(\xi)\nabla_\xi H(\xi)] = \xi$,

where (ii), (iii) hold for all $x, \xi \in \mathbb{R}^n$, $H(0)\nabla_\xi H(0)$ and $H_0(0)\nabla H_0(0)$ are defined to be 0.

The Hölder-type inequality for norm H as follows:

$$H(\xi)H_0(x) \geq \langle x, \xi\rangle, \tag{2.1}$$

and equality holds if and only if $H_0(x) = H(\eta)$.

Next, we state an elementary lemma. For a proof we refer to [19].

Lemma 2.1. *Let H be a norm in $\mathbb{R}^n$ such that $H \in C^1(\mathbb{R}^n\backslash\{0\})$ and H^p, $1 < p < \infty$, is strictly convex. If*

$$H(\xi)^p + (p-1)H(\eta)^p - p\langle \xi, H(\eta)^{p-1}\nabla H(\eta)\rangle = 0$$

for some $\xi, \eta \in \mathbb{R}^n$, $\eta \neq 0$, and $H(\xi) = H(\eta)$, then $\xi = \eta$.

Lemma 2.2 (Young's inequality). *If a and b are two nonnegative real numbers and p and q are such that $\frac{1}{p}+\frac{1}{q}=1$, then equality*

$$ab \le \frac{a^p}{p}+\frac{b^q}{q} \tag{2.2}$$

holds if and only if $a^p = b^q$.

Proof. For a proof, we refer to [15].

3. Proof of Theorem 1.5. It is easy to see that

$$\nabla\left(\frac{g(u)}{f(v)}\right) = \frac{f(v)g'(u)\nabla u - g(u)f'(v)\nabla v}{(f(v))^2},$$

$$R(u,v) = H(\nabla u)^p - \left\langle \nabla\left(\frac{g(u)}{f(v)}\right), H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle =$$

$$= H(\nabla u)^p - \left\langle \frac{g'(u)\nabla u}{f(v)}, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle + \left\langle \frac{g(u)f'(v)\nabla v}{(f(v))^2}, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle =$$

$$= H(\nabla u)^p - \left\langle \frac{g'(u)\nabla u}{f(v)}, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle + \frac{g(u)f'(v)}{(f(v))^2}H(\nabla v)^p =$$

$$= H(\nabla u)^p + \frac{g(u)f'(v)}{(f(v))^2}H(\nabla v)^p - \frac{g'(u)H(\nabla v)^{p-1}H(\nabla u)}{f(v)} + \frac{g'(u)H(\nabla v)^{p-1}H(\nabla u)}{f(v)} -$$

$$- \left\langle \frac{g'(u)\nabla u}{f(v)}, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle =$$

$$= \underbrace{p\left[\frac{H(\nabla u)^p}{p} + \frac{1}{q}\left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q\right] - \frac{g'(u)H(\nabla v)^{p-1}H(\nabla u)}{f(v)}}_{\text{(I)}} +$$

$$+ \underbrace{\frac{g(u)f'(v)H(\nabla v)^p}{(f(v))^2} - \frac{p}{q}\left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q}_{\text{(II)}} +$$

$$+ \underbrace{\frac{g'(u)H(\nabla v)^{p-1}H(\nabla u)}{f(v)} - \left\langle \frac{g'(u)\nabla u}{f(v)}, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right\rangle}_{\text{(III)}}.$$

Now, we plan to show that (I), (II), (III) are nonnegative. Take $a = H(\nabla u)$ and $b = \frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}$ in (2.2), we get

$$\frac{g'(u)H(\nabla v)^{p-1}H(\nabla u)}{pf(v)} \le \frac{H(\nabla u)^p}{p} + \frac{1}{q}\left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q,$$

which implies (I) ≥ 0. By using (1.4), we obtain (II) ≥ 0. To show (III) ≥ 0, let us rewrite

$$(\text{III}) = H(\nabla u)H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1} -$$

$$-\left\langle \nabla u,\, H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1}\nabla_\xi H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)\right\rangle.$$

Take $\xi = \nabla u,\ x = H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1}\nabla_\xi H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)$ in (2.1), we obtain

$$\left\langle H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1}\nabla_\xi H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right), \nabla u\right\rangle \le$$

$$\le H(\nabla u)H_0\left(H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1}\nabla_\xi H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)\right) =$$

$$= H(\nabla u)H\left(\left(\frac{g'(u)}{f(v)}\right)^{1/p-1}\nabla v\right)^{p-1} =$$

$$= \frac{g'(u)}{f(v)}H(\nabla u)H(\nabla v)^{p-1} \ge 0.$$

The equality in (I) holds if and only if

$$H(\nabla u)^p = \left(\frac{g'(u)H(\nabla v)^{p-1}}{pf(v)}\right)^q = \left(\frac{g'(u)}{pf(v)}\right)^q H(\nabla v)^p,$$

$$H(\nabla u) = H\left(\left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v\right) \quad \text{a.e. in} \quad \Omega.$$

If $\left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v \ne 0$ for some $x_0 \in S := \{x \in \Omega \colon R(u,v) = 0\}$, then

$$(\text{I}) = H(\nabla u)^p + (p-1)\left(\frac{g'(u)}{pf(v)}\right)^q H(\nabla v)^p - \frac{g'(u)}{f(v)}H(\nabla v)^{p-1}H(\nabla u) =$$

$$= H(\nabla u)^p + (p-1)H\left(\left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v\right)^p -$$

$$-\,p\left\langle \nabla u, H\left(\left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v\right)^{p-1}\nabla_\xi H\left(\left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v\right)\right\rangle.$$

When $(\text{I}) = 0$, by Lemma 2.1,

$$\nabla u = \left(\frac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v.$$

If $\left(\dfrac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v = 0$ for some subset S_0 of S, then $\nabla u = 0$ a.e. in S_0 which implies $\nabla u =$ $= \left(\dfrac{g'(u)}{pf(v)}\right)^{1/p-1}\nabla v.$

Theorem 1.5 is proved.

4. Applications of Theorem 1.5. In this section, we use Theorem 1.5 to prove some qualitative result. We assume that Ω is a smooth and bounded domain in $\mathbb{R}^n$ and the functions f and g satisfy assumption of Theorem 1.5. First, we prove a Hardy-type inequality.

Theorem 4.1 (Hardy-type inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and $v \in C_c^\infty(\Omega)$ be such that*

$$-\Delta_{H,p} v \geq \lambda k(x) f(v), \quad v > 0 \quad \textit{in} \qquad \Omega$$

for some $\lambda > 0$ and nonnegative function $k \in L^\infty(\Omega)$. Then, for any $u \in C_c^\infty(\Omega)$, $u \geq 0$ and $g(u) \in C_c^\infty(\Omega)$,

$$\int\limits_\Omega H(\nabla u)^p dx \geq \lambda \int\limits_\Omega k(x) g(u) dx.$$

Proof. Take $\phi \in C_c^\infty(\Omega)$, $\phi > 0$. By Theorem 1.5, we have

$$0 \leq \int\limits_\Omega L(\phi, v)\, dx =$$

$$= \int\limits_\Omega R(\phi, v)\, dx = \int\limits_\Omega H(\nabla\phi)^p dx - \Delta\left(\frac{g(\phi)}{f(v)}\right) H(\nabla v)^{p-1}\nabla_\xi H(\nabla v) dx =$$

$$= \int\limits_\Omega H(\nabla\phi)^p dx + \int\limits_\Omega \frac{g(\phi)}{f(v)}\Delta_{H,p} v\, dx \leq$$

$$\leq \int\limits_\Omega H(\nabla\phi)^p dx - \lambda\int\limits_\Omega g(\phi) k(x)\, dx,$$

and letting $\phi \to u$, we get

$$\int\limits_\Omega H(\nabla u)^p dx \geq \lambda \int\limits_\Omega k(x) g(u) dx.$$

Theorem 4.1 is proved.

Next, we prove a comparison result.

Theorem 4.2. *Let $k_1(x)$ and $k_2(x)$ be two continuous functions such that $k_1(x) < k_2(x)$ on a bounded domain $\Omega \subset \mathbb{R}^n$. If there exists a function $u \in C^2(\Omega)$ satisfying*

$$\begin{gathered} -\Delta_{H,p} u = \frac{k_1(x) g(u)}{u} \quad \textit{in} \quad \Omega, \\ u > 0,\ g(u) > 0 \quad \textit{in} \quad \Omega, \\ u = 0 = g(u) \quad \textit{on} \quad \partial\Omega, \end{gathered} \tag{4.1}$$

then any nontrivial solution v *to the equation*

$$-\Delta_{H,p}v = k_2(x)f(v) \quad \textit{in} \quad \Omega \tag{4.2}$$

must change sign.

Proof. Assume that v does not change sign and $\varepsilon > 0$. Since $g(u) = 0$ on $\partial\Omega$, $\dfrac{g(u)}{f(v)+\varepsilon} \in$ $\in W_0^{1,p}(\Omega)$. By Theorem 1.5,

$$0 \leq \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx =$$

$$= \int_\Omega H(\nabla u)^p dx - \int_\Omega \left\langle \nabla\left(\frac{g(u)}{f(v)+\varepsilon}\right), H(\nabla v)^{p-1}\nabla_\xi(\nabla v)\right\rangle dx =$$

$$= \int_\Omega H(\nabla u)^p dx + \int_\Omega \frac{g(u)}{f(v)+\varepsilon}\Delta_{H,p}v dx.$$

As $\varepsilon \to 0$, we obtain

$$\int_\Omega H(\nabla u)^p dx + \int_\Omega \frac{g(u)}{f(v)}\Delta_{H,p}v dx \geq 0. \tag{4.3}$$

On using (4.1) and (4.2) in (4.3), we have

$$\int_\Omega (k_1(x) - k_2(x))g(u)dx \geq 0,$$

which is a contradiction because $k_1(x) < k_2(x)$ and $g(u) > 0$.

Theorem 4.2 is proved.

Finally, we establish a qualitative result concerning a system of equations involving Finsler p-Laplacian.

Theorem 4.3. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $(u,v) \in C^2(\Omega) \times C^2(\Omega)$ *be a positive solution to the elliptic system*

$$-\Delta_{H,p}u = f(v) \quad \textit{in} \quad \Omega,$$

$$-\Delta_{H,p}v = \frac{(f(v))^2 u}{g(u)} \quad \textit{in} \quad \Omega,$$

$$u > 0, v > 0,\ g(u), f(v) > 0 \quad \textit{in} \quad \Omega,$$

$$u = 0 = g(u) \quad \textit{on} \quad \partial\Omega.$$

Then $\nabla u = \left(\dfrac{g'(u)}{pf(v)}\right)^{1/p-1} \nabla v.$

Proof. For any $\phi_1, \phi_2 \in W_0^{1,p}(\Omega)$, we get

$$\int_\Omega H(\nabla u)^{p-1} \nabla_\xi H(\nabla u) \nabla \phi_1 dx = \int_\Omega f(v) \phi_1 dx,$$

$$\int_\Omega H(\nabla v)^{p-1} \nabla_\xi H(\nabla v) \nabla \phi_2 dx = \int_\Omega \frac{(f(v))^2 u}{g(u)} \phi_2 dx.$$

Let $\varepsilon > 0$. Since $g(u) = 0$ on $\partial\Omega$, $\dfrac{g(u)}{f(v)+\varepsilon} \in W_0^{1,p}(\Omega)$. On choosing $\phi_1 = u$, $\phi_2 = \dfrac{g(u)}{f(v)+\varepsilon}$, we obtain

$$\int_\Omega H(\nabla u)^{p-1} \nabla_\xi H(\nabla u) \nabla u dx = \int_\Omega f(v) u dx, \tag{4.4}$$

$$\int_\Omega H(\nabla v)^{p-1} \nabla_\xi H(\nabla v) \nabla \left(\frac{g(u)}{f(v)+\varepsilon} \right) dx = \int_\Omega u f(v) dx. \tag{4.5}$$

On using (4.4) and (4.5), we have

$$\int_\Omega H(\nabla v)^{p-1} \nabla_\xi H(\nabla v) \nabla \left(\frac{g(u)}{f(v)+\varepsilon} \right) dx = \int_\Omega u f(v) dx =$$

$$= \int_\Omega H(\nabla u)^{p-1} \nabla_\xi H(\nabla u) \nabla u dx =$$

$$= \int_\Omega H(\nabla u)^p dx.$$

As $\varepsilon \to 0$, we obtain

$$\int_\Omega L(u,v) dx = \int_\Omega R(u,v) dx = 0$$

and, by Theorem 1.5, $\nabla u = \left(\dfrac{g'(u)}{f(v)} \right)^{1/p-1}$ a.e. in Ω.

Theorem 4.3 is proved.

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