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L. Kargın, M. Cenkci (Akdeniz Univ., Antalya, Turkey)

RECURRENCES AND CONGRUENCES FOR HIGHER ORDER GEOMETRIC POLYNOMIALS AND RELATED NUMBERS

РЕКУРЕНТНІ ТА КОНГРУЕНТНІ СПІВВІДНОШЕННЯ ДЛЯ ГЕОМЕТРИЧНИХ ПОЛІНОМІВ ВИЩОГО ПОРЯДКУ І ВІДПОВІДНИХ ЧИСЕЛ

We obtain new recurrence relations, an explicit formula, and convolution identities for higher order geometric polynomials. These relations generalize known results for geometric polynomials, and lead to congruences for higher order geometric polynomials, in particular, for p-Bernoulli numbers.

Отримано нові рекурентні співвідношення, точну формулу та тотожності згортки для геометричних поліномів вищого порядку. Ці співвідношення узагальнюють відомі результати для геометричних поліномів і дають можливість отримати конгруентності для геометричних поліномів вищого порядку, зокрема для p-чисел Бернуллі.

1. Introduction. For a complex variable y, the geometric polynomials $w_n(y)$ of degree n are defined by [31]

$$w_n(y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! y^k, \tag{1.1}$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind [15]. These polynomials have been studied from analytic, combinatoric, and number theoretic points of view. Analytically, they are used in evaluating geometric series of the form [4]

$$\sum_{k=0}^{\infty} k^n y^k$$

with

$$\left(y \frac{d}{dy} \right)^n \frac{1}{1-y} = \sum_{k=0}^{\infty} k^n y^k = \frac{1}{1-y} w_n \left(\frac{y}{1-y} \right)$$

for every $|y| < 1$ and every $n \in \mathbb{Z}$, $n \geq 0$. Combinatorially, they are related to the total number of preferential arrangements of n objects

$$w_n(1) := w_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!,$$

that is, the number of partitions of an n-element set into k nonempty distinguishable subsets (c.f. [10]). Number theoretic studies on the geometric polynomials are mostly originated from their exponential generating function

$$\sum_{n=0}^{\infty} w_n(y)\frac{t^n}{n!} = \frac{1}{1-y(e^t-1)}.$$

For example, setting $y = -\frac{1}{2}$ gives

$$w_n\left(-\frac{1}{2}\right) = \frac{2}{n+1}\left(1-2^{n+1}\right)B_{n+1} = -\frac{T_n}{2^n},$$

where B_n are Bernoulli numbers and T_n are tangent numbers. Bernoulli numbers also occur in integrals involving geometric polynomials, namely, we have [24]

$$\int_0^1 w_n(-y)dy = B_n, \quad n > 0.$$

Moreover, we note that [21]

$$\int_0^1 (1-y)^p w_n(-y)dy = \frac{1}{p+1}B_{n,p},$$

where $B_{n,p}$ are p-Bernoulli numbers [30] (see Section 2 for definitions). The congruence identities of geometric numbers is also one of the subjects studied. Gross [16] showed that

$$w_{n+4} = w_n \pmod{10},$$

which was generalized by Kauffman [19] later. Mező [27] also gave an elementary proof for Gross' identity. Moreover, Diagana and Maïga [11] used p-adic Laplace transform and p-adic integration to give some congruences for geometric numbers. We refer to the papers [5–7, 12, 20, 29] and the references therein for more on geometric numbers and polynomials.

In the literature, there are numerous studies for the generalization of geometric polynomials (see, e.g., [13, 14, 22, 23]). One of the natural extension of geometric polynomials is the higher order geometric polynomials [4]

$$w_n^{(r)}(y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (r)_k y^k, \quad r > 0, \tag{1.2}$$

where $(x)_n$ is the Pochhammer symbol defined by $(x)_n = x(x+1)\ldots(x+n-1)$ with $(x)_0 = 1$. It is evident that $w_n^{(1)}(y) = w_n(y)$. The polynomials $w_n^{(r)}(y)$ have the property [4]

$$\left(y\frac{d}{dy}\right)^n \frac{1}{(1-y)^{r+1}} = \sum_{k=0}^{\infty} \binom{k+r}{k} k^n y^k = \frac{1}{(1-y)^{r+1}} w_n^{(r+1)}\left(\frac{y}{1-y}\right) \tag{1.3}$$

for any $n, r = 0, 1, 2, \ldots$, and may be defined by means of the exponential generating function [4]

$$\sum_{n=0}^{\infty} w_n^{(r)}(y)\frac{t^n}{n!} = \left(\frac{1}{1-y(e^t-1)}\right)^r.$$

On the other hand, the higher order geometric polynomials and exponential (or single variable Bell) polynomials

$$\varphi_n(y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} y^k$$

are connected by

$$w_n^{(r)}(y) = \frac{1}{\Gamma(r)} \int_0^\infty \lambda^{r-1} \varphi_n(y\lambda) e^{-\lambda} d\lambda \tag{1.4}$$

(c.f. [4, 8]). According to this integral representation, several generating functions and recurrence relations for higher order geometric polynomials were obtained in [8]. Namely, $w_{n+m}^{(r)}(y)$ admits a recurrence relation according to the family $\left\{ y^j w_n^{(r+j)}(y) \right\}$ as follows:

$$w_{n+m}^{(r)}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \left\{ {m \atop j} \right\} (r)_j j^{n-k} y^j w_k^{(r+j)}(y). \tag{1.5}$$

Setting $y = 1$ in (1.2), we have higher order geometric numbers $w_n^{(r)}$. The higher order geometric numbers and geometric numbers are connected with $w_n^{(1)} = w_n$ and the formula

$$w_n^{(r)} = \frac{1}{r! 2^r} \sum_{k=0}^{r} \left[{r+1 \atop k+1} \right] w_{n+k}, \tag{1.6}$$

which was proved by a combinatorial method in [1] (Theorem 2). Here, $\left[{n \atop k} \right]$ is the Stirling number of the first kind [15]. Moreover, some congruence identities for the higher order geometric numbers can also be found in the recent work [11].

In this paper, dealing with two-variable geometric polynomials defined in [25] by

$$\sum_{n=0}^{\infty} w_n^{(r)}(x;y) \frac{t^n}{n!} = \left(\frac{1}{1 - y(e^t - 1)} \right)^r e^{xt}, \tag{1.7}$$

we obtain new recurrence relations, an explicit formula, and a result generalizing (c.f. [8])

$$\sum_{k=0}^{n} \binom{n}{k} w_k^{(r)}(y) w_{n-k}(y) = \frac{w_{n+1}^{(r)}(y) + r w_n^{(r)}(y)}{r(1+y)}$$

for higher order geometric polynomials. We particularly use the explicit formula to obtain an integral representation similar to (1.4) involving r-Bell polynomials, which are defined in [26] as

$$\varphi_{n,r}(y) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r y^k, \tag{1.8}$$

where $\left\{ {n+r \atop k+r} \right\}_r$ are r-Stirling numbers of the second kind [3]. The resulting integral representation enables us to utilize some properties of r-Bell polynomials for higher order geometric polynomials. In particular, we evaluate the infinite sum

$$\sum_{k=0}^{\infty}(k+r)^n \binom{k+r-1}{k}$$

in terms of higher order geometric polynomials, obtain an ordinary generating function for higher order geometric polynomials, introduce a new recurrence for $w_{n+m}^{(r)}(y)$, and generalize (1.6). We also give an integral representation relating the higher order geometric polynomials and p-Bernoulli numbers, and express properties of p-Bernoulli numbers originating from those for the higher order geometric polynomials. Besides, using some of theses results, we prove congruences for higher order geometric polynomials and p-Bernoulli numbers. Particularly, we state a von Staudt – Clausen-type congruence for p-Bernoulli numbers.

This paper is organized as follows. In Section 2, we summarize known results that we need throughout the paper. We state and prove aforementioned results for higher order geometric polynomials and p-Bernoulli numbers in Section 3. In Section 4, we deal with some congruences for higher order geometric polynomials and p-Bernoulli numbers.

2. Preliminaries. The Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ can be defined by means of

$$x(x+1)\ldots(x+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

or by the generating function

$$\big(-\log(1-x)\big)^k = k! \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^k}{k!}$$

(c.f. [9, 15]). It follows from either of these definitions that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{2.1}$$

with

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 0, \quad \text{if} \quad n > 0, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = 0, \quad \text{if} \quad k > n \quad \text{or} \quad k < 0.$$

We note the following special values which will be used in the sequel:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = 1, \qquad \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \quad \text{if} \quad n > 0,$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}, \qquad \begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3}, \qquad \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{2}\binom{n}{4}.$$

Many properties of $\begin{bmatrix} n \\ k \end{bmatrix}$ can be found in [9, p. 214 – 219]. In particular, we have

$$k \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{i=k-1}^{n-1} \binom{n}{i} (n-i-1)! \begin{bmatrix} i \\ k-1 \end{bmatrix}.$$

This equality can be used to obtain some congruences for $\begin{bmatrix} n \\ k \end{bmatrix}$. For example, if we take $n = q$, where q is a prime number, then

$$\begin{bmatrix} q \\ k \end{bmatrix} \equiv 0 \pmod q, \quad k = 2, 3, \ldots, q-1, \tag{2.2}$$

since

$$\binom{q}{i} \equiv 0 \pmod q, \quad i = 1, 2, \ldots, q-1.$$

The Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ can be defined by means of

$$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x(x-1)\ldots(x-k+1),$$

or by the generating function

$$(\mathrm{e}^x - 1)^k = k! \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!}$$

(c.f. [9, 15]). It follows from the generating function that

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$$

with

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0, \quad \text{if} \quad n > 0, \qquad \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0, \quad \text{if} \quad k > n \quad \text{or} \quad k < 0,$$

$$\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1, \qquad \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1, \quad \text{if} \quad n > 0.$$

We note the following known identity for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ for future reference:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n. \tag{2.3}$$

Performing the product of two generating functions for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, we obtain the convolution formula [18]

$$\binom{k_1+k_2}{k_1}\left\{{n \atop k_1+k_2}\right\} = \sum_{m=0}^{n}\binom{n}{m}\left\{{m \atop k_1}\right\}\left\{{n-m \atop k_2}\right\}.$$

Letting $k = k_1 + k_2$ and $n = q$, a prime number, we deduce that

$$\left\{{q \atop k}\right\} \equiv 0 \ (\operatorname{mod} q), \quad k = 2, 3, \ldots, q-1, \tag{2.4}$$

since again

$$\binom{q}{i} \equiv 0 \ (\operatorname{mod} q), \quad i = 1, 2, \ldots, q-1,$$

and $1 < k < q$.

Stirling numbers have been generalized in many ways. One of them is called r-Stirling numbers (or weighted Stirling numbers). r-Stirling numbers of the second kind $\left\{{n \atop k}\right\}_r$ can be defined by means of the generating function (see [3])

$$(\mathrm{e}^x - 1)^k \mathrm{e}^{rx} = k! \sum_{n=k}^{\infty} \left\{{n \atop k}\right\}_r \frac{x^n}{n!}. \tag{2.5}$$

The Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

or by the equivalent recursion

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} = 0 \quad \text{for} \quad n \geq 2.$$

The first values are

$$B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42},$$

and $B_{2k+1} = 0$ for $k \geq 1$. The denominators of the Bernoulli numbers can be completely determined due to von Staudt – Clausen theorem: for any integer $n \geq 1$, B_{2n} can be written as

$$B_{2n} = A_{2n} - \sum_{q \,:\, (q-1)|2n} \frac{1}{q},$$

where A_{2n} is an integer and the sum runs over all the prime numbers such that $(q-1)|2n$. It can be stated equivalently as

$$qB_{2n} \equiv \begin{cases} 0 \ (\operatorname{mod} q), & \text{if} \quad (q-1) \nmid 2n, \\ -1 \ (\operatorname{mod} q), & \text{if} \quad (q-1) \mid 2n. \end{cases} \tag{2.6}$$

We note that this classification is also valid for B_1.

Many generalizations of Bernoulli numbers appear in the literature. One generalization is the p-Bernoulli numbers $B_{n,p}$, which are due to Rahmani [30], defined by means of the generating function

$$\sum_{n=0}^{\infty} B_{n,p}\frac{x^n}{n!} = {}_2F_1\big(1,1;p+2,1-e^t\big),$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

p-Bernoulli numbers are related to Bernoulli numbers in that $B_{n,0}=B_n$ and

$$\sum_{k=0}^{p}\begin{bmatrix}p\\k\end{bmatrix}(-1)^k B_{n+k}=\frac{p!}{p+1}B_{n,p} \quad \text{for} \quad n,p\geq 0, \tag{2.7}$$

and satisfy an explicit formula of the form

$$B_{n,p}=\frac{p+1}{p!}\sum_{k=0}^{n}\begin{Bmatrix}n+p\\k+p\end{Bmatrix}_p\frac{(-1)^k(k+p)!}{k+p+1}. \tag{2.8}$$

3. Recurrence relations. From the generating function for higher order two-variable geometric polynomials (1.7), we have

$$w_n^{(r)}(x;y)=\sum_{k=0}^{n}\binom{n}{k}w_k^{(r)}(y)x^{n-k}. \tag{3.1}$$

Then it is obvious that

$$w_n^{(r)}(0;y)=w_n^{(r)}(y), \qquad w_n^{(1)}(x;y)=w_n(x;y),$$

$$w_n^{(r)}(0;1)=w_n^{(r)} \quad \text{and} \quad w_n^{(1)}(0;1)=w_n.$$

Setting $x+r$ instead of x in (1.7), we have

$$w_n^{(r)}(x+r;y)=(-1)^n w_n^{(r)}(-x;-y-1) \quad \text{for} \quad n\geq 0.$$

Then, for $x=0$, we conclude that

$$w_n^{(r)}(r;y)=(-1)^n w_n^{(r)}(-y-1), \tag{3.2}$$

a relationship between two-variable and single variable higher order geometric polynomials.

Proposition 3.1. *For $n\geq 0$ and $r>0$, we have the following recurrence formulae*:

$$\sum_{k=0}^{n}\binom{n}{k}w_k^{(r)}(y)r^{n-k}=(-1)^n w_n^{(r)}(-y-1) \tag{3.3}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}w_k^{(r+1)}(y)=\frac{1}{ry}w_{n+1}^{(r)}(y). \tag{3.4}$$

Proof. Combining (3.1) and (3.2), we have (3.3). Furthermore, taking $x = 1$ in (3.1) and using the recurrence relation presented in [25] (Theorem 3.4)

$$w_{n+1}^{(r)}(x;y) = w_n^{(r)}(x;y) + ryw_n^{(r+1)}(x+1;y)$$

for $x = 1$, we obtain (3.4).

Theorem 3.1. *For every* $n \geq 0$ *and every* $r_1, r_2 > 0$, *we have the convolution identity*

$$\sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(y) w_{n-k}^{(r_2)}(y) = \frac{w_{n+1}^{(r_1+r_2-1)}(y) + (r_1+r_2-1) w_n^{(r_1+r_2-1)}(y)}{(r_1+r_2-1)(1+y)}. \tag{3.5}$$

Proof. We first note that

$$yr \frac{e^{(x+1)t}}{\big(1 - y(e^t - 1)\big)^{r+1}} = \frac{d}{dt}\left(\left(\frac{1}{1 - y(e^t-1)}\right)^r e^{xt}\right) - \frac{xe^{xt}}{\big(1 - y(e^t - 1)\big)^r}.$$

Let $x = x_1 + x_2 - 1$ and $r = r_1 + r_2 - 1$. Then by (1.7), product of two infinite series, and formal differentiation under summation, we obtain

$$yr \frac{e^{(x+1)t}}{\big(1 - y(e^t - 1)\big)^{r+1}} = y(r_1 + r_2 - 1) \frac{e^{\left(x_1+x_2\right)t}}{\big(1 - y(e^t - 1)\big)^{r_1+r_2}} =$$

$$= y(r_1+r_2-1) \sum_{n=0}^{\infty} w_n^{(r_1)}(x_1;y) \frac{t^n}{n!} \sum_{n=0}^{\infty} w_n^{\left(r_2\right)}(x_2;y) \frac{t^n}{n!} =$$

$$= y(r_1+r_2-1) \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(x_1;y) w_{n-k}^{(r_2)}(x_2;y) \right] \frac{t^n}{n!},$$

$$\frac{xe^{xt}}{\big(1 - y(e^t - 1)\big)^r} = (x_1 + x_2 - 1) \sum_{n=0}^{\infty} w_n^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) \frac{t^n}{n!},$$

and

$$\frac{d}{dt}\left(\left(\frac{1}{1 - y(e^t-1)}\right)^r e^{xt}\right) = \sum_{n=0}^{\infty} w_{n+1}^{(r_1+r_2-1)}(x_1 + x_2 - 1; y) \frac{t^n}{n!}.$$

Equating coefficients of $\dfrac{t^n}{n!}$ on both sides, we derive

$$\sum_{k=0}^{n} \binom{n}{k} w_k^{(r_1)}(x_1;y) w_{n-k}^{(r_2)}(x_2;y) =$$

$$= \frac{1}{y(r_1+r_2-1)} \left[w_{n+1}^{(r_1+r_2-1)}(x_1+x_2-1;y) - (x_1+x_2-1) w_n^{(r_1+r_2-1)}\big(x_1+x_2-1;y\big) \right].$$

Setting $x_1 = r_1$, $x_2 = r_2$ and using (3.2), we obtain the convolution formula (3.5).

In the following theorem we give a new explicit expression for higher order geometric polynomials and numbers.

Theorem 3.2. *For* $n \geq 0$,

$$w_n^{(r)}(y) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r (r)_k (-1)^{n+k} (y+1)^k. \tag{3.6}$$

In particular,

$$w_n^{(r)} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r (r)_k (-1)^{n+k} 2^k.$$

Proof. Writing $x = r$ in (1.7), employing the generalized binomial formula, and using the generating function of r-Stirling numbers (2.5), we have

$$\sum_{n=0}^{\infty} w_n^{(r)}(r;y) \frac{t^n}{n!} = \left(\frac{1}{1 - y(e^t - 1)} \right)^r e^{rt} = \sum_{k=0}^{\infty} (r)_k y^k \frac{\left(e^t - 1\right)^k}{k!} e^{rt} =$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r (r)_k y^k \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r (r)_k y^k \right] \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$w_n^{(r)}(r;y) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r (r)_k y^k.$$

Using (3.2) and replacing y with $-(y+1)$, we reach the desired equation.

Now, with use of Theorem 3.2, we connect higher order geometric polynomials and r-Bell polynomials in the following lemma which will be useful for the subsequent results.

Lemma 3.1. *For every* $n \geq 0$ *and every* $r > 0$, *we have the integral representation*

$$(-1)^n w_n^{(r)}(-y-1) = \frac{1}{\Gamma(r)} \int_0^{\infty} \lambda^{r-1} \varphi_{n,r}(y\lambda) e^{-\lambda} d\lambda. \tag{3.7}$$

Proof. By (1.8) we have

$$\int_0^{\infty} \lambda^{r-1} \varphi_{n,r}(y\lambda) e^{-\lambda} d\lambda = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r y^k \int_0^{\infty} \lambda^{r+k-1} e^{-\lambda} d\lambda = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \Gamma(r+k) y^k.$$

Using (3.6) in the above yields the desired equation.

Higher order geometric polynomials are seen in the evaluation of the infinite series (1.3). If we apply Lemma 3.1 to the Dobinski's formula for r-Bell polynomials

$$\varphi_{n,r}(y) = \frac{1}{e^y} \sum_{n=0}^{\infty} \frac{(k+r)^n}{k!} x^k,$$

we can evaluate a new infinite series in terms of higher order geometric polynomials.

Theorem 3.3. *For every $n \geq 0$ and every $r > 0$, $|y| < 1$,*

$$\sum_{k=0}^{\infty}(k+r)^n\binom{k+r-1}{k}y^k = \frac{(-1)^n}{(1-y)^r}w_n^{(r)}\left(\frac{1}{y-1}\right).$$

Next we introduce ordinary generating function for higher order geometric polynomials.

Theorem 3.4. *For real $y < -\dfrac{1}{2}$, the higher order geometric polynomials have the generating function*

$$\sum_{n=0}^{\infty}w_n^{(r)}(y)t^n = \frac{(-1)^r}{(1+rt)y^r}\,{}_2F_1\left(\frac{rt+1}{t}, r; \frac{rt+t+1}{t}; \frac{y+1}{y}\right).$$

Proof. We start by observing the ordinary generating function for r-Bell polynomials [26] (Theorem 3.2)

$$\sum_{n=0}^{\infty}\varphi_{n,r}(y)t^n = \frac{-1}{rt-1}\frac{1}{e^y}\,{}_1F_1\left(\frac{rt-1}{t}; \frac{rt+t-1}{t}; y\right).$$

In light of the equation (3.7), this equation can be written as

$$\sum_{n=0}^{\infty}(-1)^n w_n^{(r)}(-y-1)t^n = \frac{1}{(1-rt)\Gamma(r)}\int\limits_0^{\infty}\lambda^{r-1}e^{-(y+1)\lambda}\,{}_1F_1\left(\frac{rt-1}{t}; \frac{rt+t-1}{t}; y\lambda\right)d\lambda =$$

$$= \frac{1}{(1-rt)\Gamma(r)}\sum_{k=0}^{\infty}\frac{\left(\dfrac{rt-1}{t}\right)_k}{\left(\dfrac{rt+t-1}{t}\right)_k}\frac{y^k}{k!}\int\limits_0^{\infty}\lambda^{r+k-1}e^{-(y+1)\lambda}d\lambda =$$

$$= \frac{1}{(1-rt)(1+y)^r}\sum_{k=0}^{\infty}\frac{\left(\dfrac{rt-1}{t}\right)_k(r)_k}{\left(\dfrac{rt+t-1}{t}\right)_k k!}\left(\frac{y}{1+y}\right)^k =$$

$$= \frac{1}{(1-rt)(1+y)^r}\,{}_2F_1\left(\frac{rt-1}{t}, r; \frac{rt+t-1}{t}; \frac{y}{1+y}\right).$$

We then replace $-(y+1)$ with y and $-t$ with t to obtain the desired equation.

Now, we give an alternative representation for $w_{n+m}^{(r)}(y)$, which also generalizes (1.6) in the following theorem.

Theorem 3.5. *For all nonnegative integers n, m, r and p, we have*

$$w_{n+m}^{(r)}(y) = \sum_{k=0}^{m}\left\{\begin{matrix} m+r \\ k+r \end{matrix}\right\}_r (r)_k(-1)^{m+k}(y+1)^k w_n^{(r+k)}(y) \tag{3.8}$$

and

$$w_n^{(r+p)}(y) = \frac{1}{(r)_p(1+y)^p}\sum_{k=0}^{p}\begin{bmatrix} p+r \\ k+r \end{bmatrix}_r w_{n+k}^{(r)}(y). \tag{3.9}$$

Proof. We prove (3.8) first. Using the following property of r-Bell polynomials presented in [28] (Eq. (8))

$$\varphi_{n+m,r}(y) = \sum_{k=0}^{m} \left\{ {m+r \atop k+r} \right\}_r y^k \varphi_{n,r+k}(y)$$

in (3.7), we have

$$(-1)^{n+m} w_{n+m}^{(r)}(-y-1) = \sum_{k=0}^{m} \left\{ {m+r \atop k+r} \right\}_r y^k \frac{\Gamma(k+r)}{\Gamma(r)} \frac{1}{\Gamma(k+r)} \int\limits_0^\infty \lambda^{r+k-1} \varphi_{n,r+k}(y\lambda) e^{-\lambda} d\lambda =$$

$$= \sum_{k=0}^{m} \left\{ {m+r \atop k+r} \right\}_r (r)_k y^k (-1)^n w_n^{(r)}(-y-1),$$

which is equal to (3.8).

To prove (3.9), we use the formula

$$y^p \varphi_{n,r+p}(y) = \sum_{k=0}^{p} \left[{p+r \atop k+r} \right]_r (-1)^{p-k} \varphi_{n+k,r}(y)$$

[28] (Eq. (11)) in (3.7).

Using (3.3) in (3.8), we obtain the following result similar which is slightly different from (1.5).

***Corollary* 3.1.** *We have*

$$w_{n+m}^{(r)}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ {m+r \atop j+r} \right\}_r \binom{n}{k} (j+r)^{n-k} (-1)^{n+m+j} (r)_j (y+1)^j w_k^{(r+j)}(-y-1).$$

We note that it is also possible to derive this result by applying (1.4) and (3.7) in

$$\varphi_{n+m,r}(y) = \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \left\{ {m+r \atop j+r} \right\}_r (j+r)^{n-k} y^j \varphi_k(y),$$

a formula given in [28] (Eq. (9)). Moreover, for $r=1$, (3.9) can be written as

$$(1+y)^p w_n^{(p+1)}(y) = \frac{1}{p!} \sum_{k=0}^{p} \left[{p+1 \atop k+1} \right] w_{n+k}(y), \qquad (3.10)$$

which is also polynomial extension of (1.6). Replacing y by $-y$ and integrating both sides with respect to y from 0 to 1, we have

$$\int\limits_0^1 (1-y)^p w_n^{(p+1)}(-y) dy = \frac{1}{p!} \sum_{k=0}^{p} \left[{p+1 \atop k+1} \right] B_{n+k}.$$

Then using (2.7), we obtain the following integral representation for p-Bernoulli numbers.

Theorem 3.6. *For $n \geq 1$ and $p \geq 0$,*

$$\int\limits_0^1 (1-y)^p w_n^{(p+1)}(-y) dy = (-1)^{n-1} \frac{p+1}{p+2} B_{n-1,p+1}. \qquad (3.11)$$

The explicit formula (2.8) for p-Bernoulli numbers can be also deduced using this integral representation in (3.6).

The following theorem generalizes the identities (2.8) and (2.7).

Theorem 3.7. *For $n, p, m \geq 0$, we have*

$$B_{n+m,p} = (p+1)\sum_{k=0}^{m} \left\{ {m+p \atop k+p} \right\}_p \frac{(-1)^k (p+1)_k}{k+p+1} B_{n,p+k}. \qquad (3.12)$$

For $n, r \geq 1$ and $p \geq 0$, we get

$$B_{n,p+r} = \frac{r(p+r+1)}{(r+1)(p+r)(r)_p}\sum_{k=0}^{p} \left[{p+r \atop k+r} \right]_r (-1)^k B_{n+k,r}. \qquad (3.13)$$

Proof. Firstly, we replace y with $-y$ in (3.9), multiply both sides by $(1-y)^{r-1}$, and integrate with respect to y from 0 to 1. The result is

$$\int_0^1 (1-y)^{p+r-1} w_n^{(r+p)}(-y)dy = \frac{1}{(p)_r}\sum_{k=0}^{p} \left[{p+r \atop k+r} \right]_r \int_0^1 (1-y)^{r-1} w_{n+k}^{(r)}(-y)dy.$$

From (3.11) this equation turns into

$$B_{n-1,p+r} = \frac{r(p+r+1)}{(r+1)(p+r)(r)_p}\sum_{k=0}^{p} \left[{p+r \atop k+r} \right]_r (-1)^k B_{n+k-1,r}.$$

Replacing n with $n+1$ in the above equation completes the proof (3.13).

Applying the same method to the identity (3.8) gives (3.12).

4. Congruences. In this section, we first consider congruences modulo a prime number q for higher order geometric polynomials. We start with two auxiliary results.

Lemma 4.1. *Let q be an odd prime and y be an integer. Then we have*

$$w_q(y) \equiv y \pmod{q}.$$

Proof. From (1.1), we obtain

$$w_q(y) = \left\{ {q \atop 0} \right\} + \left\{ {q \atop 1} \right\} y + \left\{ {q \atop q} \right\} q! y^q + \sum_{k=2}^{q-1} \left\{ {q \atop k} \right\} k! y^k.$$

Since

$$\left\{ {q \atop 0} \right\} = 0, \qquad \left\{ {q \atop 1} \right\} = \left\{ {q \atop q} \right\} = 1$$

and by (2.4)

$$k! \left\{ {q \atop k} \right\} \equiv 0 \pmod{q}, \quad k = 2, 3, \ldots, q-1,$$

we get the desired result

$$w_q(y) \equiv y + q! y^q \equiv y \pmod{q}.$$

Lemma 4.2. *Let q be a prime and y be an integer. Then, for all $n \geq 1$, we have*

$$w_{q+n-1}(y) \equiv w_n(y) \pmod q.$$

Proof. If $q = 2$, then, by (1.1), we obtain

$$w_{n+1}(y) - w_n(y) = \sum_{k=0}^{n+1} \left\{ {n+1 \atop k} \right\} k! y^k - \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! y^k$$

$$= (n+1)! y^{n+1} + \sum_{k=2}^{n} \left(\left\{ {n+1 \atop k} \right\} - \left\{ {n \atop k} \right\} \right) k! y^k \equiv 0 \pmod 2,$$

since $\left\{ {n \atop k} \right\} = 0$ and $\left\{ {n \atop 1} \right\} = 1$ for $n > 0$.

Now, suppose that q is an odd prime and let $n \geq q-1$. Then again by (1.1) we write

$$w_{q+n-1}(y) - w_n(y) = \sum_{k=0}^{q+n-1} \left\{ {q+n-1 \atop k} \right\} k! y^k - \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! y^k =$$

$$= \sum_{k=0}^{q-1} \left(\left\{ {q+n-1 \atop k} \right\} - \left\{ {n \atop k} \right\} \right) k! y^k +$$

$$+ \sum_{k=q}^{q+n-1} \left\{ {q+n-1 \atop k} \right\} k! y^k - \sum_{k=q}^{n} \left\{ {n \atop k} \right\} k! y^k \equiv$$

$$\equiv \sum_{k=0}^{q-1} \left(\left\{ {q+n-1 \atop k} \right\} - \left\{ {n \atop k} \right\} \right) k! y^k \pmod q.$$

By using (2.3), we obtain

$$w_{q+n-1}(y) - w_n(y) \equiv \sum_{k=0}^{q-1} k! y^k \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \left(j^{q-1} - 1\right) \equiv 0 \pmod q,$$

since $(j, q) = 1$ and $j^{q-1} - 1 \equiv 0 \pmod q$.

If $1 \leq n < q-1$, then we write

$$w_{q+n-1}(y) - w_n(y) = \sum_{k=0}^{q+n-1} \left\{ {q+n-1 \atop k} \right\} k! y^k - \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! y^k =$$

$$= \sum_{k=0}^{q-1} \left(\left\{ {q+n-1 \atop k} \right\} - \left\{ {n \atop k} \right\} \right) k! y^k +$$

$$+ \sum_{k=q}^{q+n-1} \left\{ {q+n-1 \atop k} \right\} k! y^k - \sum_{k=n}^{q-1} \left\{ {n \atop k} \right\} k! y^k + \sum_{k=n+1}^{q-1} \left\{ {n \atop k} \right\} k! y^k \equiv$$

$$\equiv \sum_{k=0}^{q-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n (j^{q-1} - 1) \equiv 0 \pmod q,$$

since $\left\{ {n \atop k} \right\} = 0$ when $k > n$.

Therefore, for $n \geq 1$, $w_{q+n-1}(y) \equiv w_n(y) \pmod q$.

We note that a more general result can be found in [2] for Fubini numbers.

Theorem 4.1. *Let q be an odd prime. If $1 + y$ is not a multiple of q, then $w_q^{(q)}(y) \equiv 0 \pmod q$.*

Proof. We set $p = q - 1$ and $n = q$ in (3.10) to obtain

$$(1 + y)^{q-1} (q - 1)! w_q^{(q)}(y) = \sum_{k=1}^{q} \left[{q \atop k} \right] w_{q+k-1}(y) =$$

$$= \left[{q \atop 1} \right] w_q(y) + \left[{q \atop q} \right] w_1(y) + \sum_{k=2}^{q-1} \left[{q \atop k} \right] w_{q+k-1}(y) =$$

$$= (q - 1)! w_q(y) + y + \sum_{k=2}^{q-1} \left[{q \atop k} \right] w_{q+k-1}(y).$$

By Lemmas 4.1 and 4.2, we find that

$$(1 + y)^{q-1} (q - 1)! w_q^{(q)}(y) \equiv (-1)y + y + \sum_{k=2}^{q-1} \left[{q \atop k} \right] w_k(y) \equiv 0 \pmod q,$$

since by (2.2) $\left[{q \atop k} \right] \equiv 0 \pmod q$ for $2 \leq k \leq q - 1$ and $w_k(y)$ is an integer when y is an integer. The result now follows from Fermat's and Wilson's theorems.

It is obvious from (1.2) that if y is an integer which is a multiple of q, then $w_n^{(r)}(y) \equiv 0 \pmod q$, since $\left\{ {n \atop k} \right\} (r)_k$ is an integer. We note that Theorem 4.1 is a special case which can be drawn from the following result.

Theorem 4.2. *If y is an integer that is not a multiple of q, then $w_n^{(r)}(y) \equiv 0 \pmod q$ for $n \geq 1$ and $r \equiv 0 \pmod q$.*

Proof. Let $r = tq$ for some integer t. By (1.2), we have

$$w_n^{(r)}(y) = \sum_{k=0}^{n} k! \binom{tq + k - 1}{k} \left\{ {n \atop k} \right\} y^k.$$

Since

$$k! \binom{tq + k - 1}{k} = (tq + k - 1)(tq + k - 2) \ldots (tq + 1)(tq) \equiv 0 \pmod q,$$

we have the result.

Theorem 4.3. *If y is an integer such that y and $1 + y$ are not multiples of an odd prime q, then $w_{q-1}^{(r)}(y) \equiv 0 \pmod q$ for $r \equiv 1 \pmod q$.*

Proof. Let $r = 1 + tq$ for some integer t. By (1.2), we have

$$w_{q-1}^{(r)}(y) = \sum_{k=0}^{q-1} k! \binom{tq+k}{k} \left\{ {q-1 \atop k} \right\} y^k.$$

Since

$$\binom{tq+k}{k} = \frac{(tq+k)(tq+k-1)\ldots(tq+1)}{k!} \equiv \frac{k(k-1)\ldots 1}{k!} = 1(\operatorname{mod} q),$$

we deduce that

$$w_{q-1}^{(r)}(y) \equiv \sum_{k=0}^{q-1} k! \left\{ {q-1 \atop k} \right\} y^k \ (\operatorname{mod} q).$$

It follows from (2.3) that

$$k! \left\{ {q-1 \atop k} \right\} \equiv (-1)^{k-1} \ (\operatorname{mod} q)$$

for $1 \leq k \leq q-1$. Since $\left\{ {q-1 \atop 0} \right\} = 0$, then we have

$$w_{q-1}^{(r)}(y) \equiv \sum_{k=0}^{q-1} (-1)^{k-1} y^k = 1 - \sum_{k=0}^{q-1} (-1)^k y^k = 1 - \frac{1+y^q}{1+y} \ (\operatorname{mod} q),$$

which implies

$$(1+y) w_{q-1}^{(r)}(y) \equiv 1 + y - 1 + y^q \equiv 0 \ (\operatorname{mod} q),$$

and the result.

These results and their proofs are direct generalizations of the corresponding congruences for higher order geometric numbers given in [11] (Corollary 4.2).

We conclude the study of congruences for higher order geometric polynomials by a similar result.

Theorem 4.4. *If* y *is an integer that is not a multiple of an odd prime* q*, then* $w_{q+1}^{(r)}(y) \equiv 0$ $(\operatorname{mod} q)$ *for* $r \equiv 0 \ (\operatorname{mod} q)$ *and* $w_{q+1}^{(r)}(y) \equiv -y \ (\operatorname{mod} q)$ *for* $r \equiv -1 \ (\operatorname{mod} q)$.

Proof. For a prime q and nonnegative integer m, we have

$$\left\{ {q+m \atop k} \right\} \equiv \left\{ {m+1 \atop k} \right\} + \left\{ {m \atop k-q} \right\} \ (\operatorname{mod} q).$$

This result was given by Howard in [17], and can be easily verified by induction on m. It then follows that $\left\{ {q+1 \atop k} \right\} \equiv 0 \ (\operatorname{mod} q)$ for $k = 3, 4, \ldots, q$ and $\left\{ {q+1 \atop 2} \right\} \equiv 1 \ (\operatorname{mod} q)$.

Now, we write (1.2) as

$$w_{q+1}^{(r)}(y) = \sum_{k=0}^{q+1} k! \binom{r+k-1}{k} \left\{ {q+1 \atop k} \right\} y^k = ry + r(r+1) \left\{ {q+1 \atop 2} \right\} y^2 +$$

$$+(q+1)! \binom{r+q}{q} y^{q+1} + \sum_{k=3}^{q} k! \binom{r+k-1}{k} \left\{ {q+1 \atop k} \right\} y^k \equiv ry + r(r+1) y^2 \ (\operatorname{mod} q),$$

from which the results follow.

In the rest of this section we consider congruences for p-Bernoulli numbers. In particular, the following theorem states a von Staudt – Clausen-type result for p-Bernoulli numbers.

Theorem 4.5. *Let n be a positive integer. Then we have $4B_{2n,2} \equiv -1 \pmod 2$ and if $(q-1) \mid 2n$ for an odd prime q, then $qB_{2n,q} \equiv -\dfrac{1}{2} \pmod q$.*

Proof. First, we take $p = 2$ in (2.6). This gives

$$\frac{2}{3}B_{2n,2} = B_{2n+2},$$

or, equivalently,

$$4B_{2n,2} = 3 \cdot 2B_{2n+2}.$$

The result then follows from the von Staudt – Clausen theorem.

Next, let q be an odd prime. Then we replace n by $2n$ and p by q in (2.6) and obtain

$$\frac{q!}{q+1}B_{2n,q} = \sum_{k=0}^{q} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k B_{2n+k} =$$

$$= \begin{bmatrix} q \\ 0 \end{bmatrix} B_{2n} - \begin{bmatrix} q \\ 1 \end{bmatrix} B_{2n+1} + \begin{bmatrix} q \\ q-1 \end{bmatrix} B_{2n+q-1} - \begin{bmatrix} q \\ q \end{bmatrix} B_{2n+q} + \sum_{k=2}^{q-2} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k B_{2n+k}.$$

Since

$$\begin{bmatrix} q \\ 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} q \\ 1 \end{bmatrix} = (q-1)!, \qquad \begin{bmatrix} q \\ q-1 \end{bmatrix} = \frac{q(q-1)}{2}, \qquad \begin{bmatrix} q \\ q \end{bmatrix} = 1$$

and $B_{2n+1} = 0,\ n \geq 1$, the above equality turns into

$$\frac{q!}{q+1}B_{2n,q} = \frac{q-1}{2}qB_{2n+q-1} + \sum_{k=2}^{q-2} \frac{1}{q} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k qB_{2n+k}.$$

If $(q-1) \mid 2n$, then $(q-1) \mid 2n+q-1$, so $qB_{2n+q-1} \equiv -1 \pmod q$ by (2.6). We also have $(q-1) \nmid 2n+k$ for $k = 2, 3, \ldots, q-2$, so $qB_{2n+k} \equiv 0 \pmod q$ again by (2.6). Noting that $\begin{bmatrix} q \\ k \end{bmatrix} \equiv 0 \pmod q$ for $2 \leq k \leq q-2$, we observe that the sum vanishes modulo q. Thus,

$$\frac{q!}{q+1}B_{2n,q} \equiv -\frac{q-1}{2} \pmod q,$$

or, equivalently,

$$qB_{2n,q} \equiv -\frac{1}{2} \pmod q,$$

by Wilson's theorem.

In the following theorem, we give a congruence for $B_{q,q}$, where $q > 3$ is a prime.

Theorem 4.6. *For a prime $q > 3$, we have*

$$qB_{q,q} \equiv \frac{1}{12} - B_{q+1} \pmod q.$$

Proof. Let $q > 3$ be a prime. Writing $n = p = q$ in (2.7) gives

$$\frac{q!}{q+1}B_{q,q} = \begin{bmatrix} q \\ 0 \end{bmatrix} B_q - \begin{bmatrix} q \\ 1 \end{bmatrix} B_{q+1} + \begin{bmatrix} q \\ q-1 \end{bmatrix} B_{2q-1} - \begin{bmatrix} q \\ q \end{bmatrix} B_{2q} -$$

$$- \begin{bmatrix} q \\ q-2 \end{bmatrix} B_{2q-2} + \sum_{k=2}^{q-3} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k B_{q+k} =$$

$$= -(q-1)!B_{q+1} - B_{2q} - \begin{bmatrix} q \\ q-2 \end{bmatrix} B_{2q-2} + \sum_{k=2}^{q-3} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k B_{q+k},$$

or, equivalently,

$$q!B_{q,q} = -(q-1)!(q+1)B_{q+1} - (q+1)B_{2q} -$$

$$-\frac{(q+1)(3q-1)(q-1)(q-2)}{24} qB_{2q-2} + (q+1) \sum_{k=2}^{q-3} \frac{1}{q} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k qB_{q+k},$$

since

$$\begin{bmatrix} q \\ q-2 \end{bmatrix} = \frac{(3q-1)q(q-1)(q-2)}{24}.$$

Now $(q-1) \nmid (q+k)$ for $k = 2, 3, \ldots, q-3$. So by the von Staudt–Clausen theorem $qB_{q+k} \equiv 0 \pmod q$. Moreover, $\begin{bmatrix} q \\ k \end{bmatrix} \equiv 0 \pmod q$ for $k = 2, 3, \ldots, q-3$, hence the sum above vanishes modulo q. The von Staudt–Clausen theorem also implies that $qB_{q+1} \equiv 0 \pmod q$, $qB_{2q} \equiv 0 \pmod q$ and $qB_{2q-2} \equiv -1 \pmod q$. All these and the Wilson's theorem give

$$qB_{q,q} \equiv \frac{1}{12} - B_{q+1} + B_{2q} \pmod q.$$

The result readily follows by employing Adam's theorem which states that $q \mid n$ implies $B_n \equiv 0 \pmod q$ for primes $(q-1) \nmid n$.

Finally, we give a congruence for $B_{q,q+1}$, where $q > 3$ is a prime.

Theorem 4.7. *For a prime $q > 3$, we have*

$$qB_{q,q+1} \equiv 0 \pmod q.$$

Proof. Letting $n = p = q$ and $r = 1$ in (3.13) and using $B_{n,1} = -2B_{n+1}$, we get

$$(q+1)!B_{q,q+1} = (q+2) \sum_{k=0}^{q} \begin{bmatrix} q+1 \\ k+1 \end{bmatrix} (-1)^{k-1} B_{q+k+1} =$$

$$= q(q+2) \sum_{k=0}^{q} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^k B_{q+k} + (q+2) \sum_{k=0}^{q} \begin{bmatrix} q \\ k \end{bmatrix} (-1)^{k-1} B_{q+k+1}.$$

Equation (2.7) enable us to write the first sum as

$$q(q+2)\sum_{k=0}^{q}\begin{bmatrix}q\\k\end{bmatrix}(-1)^k B_{q+k}=\frac{(q+2)q!}{(q+1)}qB_{q,q}.$$

By Theorem 4.6, we conclude that

$$q(q+2)\sum_{k=0}^{q}\begin{bmatrix}q\\k\end{bmatrix}(-1)^k B_{q+k}\equiv\frac{(q+2)q!}{(q+1)}\left(\frac{1}{12}-B_{q+1}\right)\equiv 0 \pmod q,$$

by the von Staudt – Clausen theorem.

Now, we seperate the terms of the second sum as

$$(q+2)\sum_{k=0}^{q}\begin{bmatrix}q\\k\end{bmatrix}(-1)^{k-1}B_{q+k+1}=-(q+2)\begin{bmatrix}q\\q-1\end{bmatrix}B_{2q}-(q+2)\begin{bmatrix}q\\q-3\end{bmatrix}B_{2q-2}+$$

$$+\frac{q+2}{q}\sum_{k=2}^{q-4}\begin{bmatrix}q\\k\end{bmatrix}(-1)^{k-1}qB_{q+k+1}=$$

$$=-(q+2)\frac{q-1}{2}qB_{2q}-q+2\binom{q}{4}\frac{q-1}{2}qB_{2q-2}++\frac{q+2}{q}\sum_{k=2}^{q-4}\begin{bmatrix}q\\k\end{bmatrix}(-1)^{k-1}qB_{q+k+1},$$

since $\begin{bmatrix}q\\q-1\end{bmatrix}=\binom{q}{2}$ and $\begin{bmatrix}q\\q-3\end{bmatrix}=\binom{q}{2}\binom{q}{4}$. For $k=2,3,\ldots,q-4$, $(q-1)\nmid(q+k+1)$, so by the von Staudt – Clausen theorem $qB_{q+k+1}\equiv 0 \pmod q$. Moreover, $\begin{bmatrix}q\\k\end{bmatrix}\equiv 0 \pmod q$ in the same range, so we conclude that the sum above vanishes modulo q. The result now follows by noting $\binom{q}{4}\equiv 0 \pmod q$, $qB_{2q-2}\equiv -1 \pmod q$ and $qB_{2q}\equiv 0 \pmod q$.

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