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VOLTERRA-TYPE OPERATOR ON THE SUBCLASSES OF UNIVALENT FUNCTIONS

ОПЕРАТОРИ ТИПУ ВОЛЬТЕРРА НА ПІДКЛАСАХ УНІВАЛЕНТНИХ ФУНКЦІЙ

In this article, we examine the necessary and sufficient conditions for a member to belong to the class of starlike and convex functions of complex order b $(b \neq 0)$ and spirallike functions of type λ $\left(-\dfrac{\pi}{2} < \lambda < \dfrac{\pi}{2}\right)$ with the complex order b $(b \neq 0)$. We obtain sharp estimates for the coefficient of the second term in the Taylor series of functions belonging to the mentioned classes.

In the main part of this paper, we obtain the necessary and sufficient conditions of boundedness for the image of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ under the action of a Volterra-type operator and the product of the composition operator and Volterra-type operator in the space of univalent functions and its subspace. Finally, we obtain an estimate of the Schwartzian norm of the above operators in these spaces.

Вивчаються необхідні та достатні умови належності елемента до класу зіркоподібних та опуклих функцій комплексного порядку b $(b \neq 0)$ і спіралеподібних функцій типу λ $\left(-\dfrac{\pi}{2} < \lambda < \dfrac{\pi}{2}\right)$ комплексного порядку b $(b \neq 0)$. Встановлено точні оцінки для коефіцієнта другого члена ряду Тейлора для функцій із вказаних класів.

У основній частині роботи отримано необхідні та достатні умови обмеженості образу відкритого одиничного диска $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ під дією оператора типу Вольтерра і добутку оператора композиції та оператора типу Вольтерра у просторі унівалентних функцій та його підпросторі. Насамкінець встановлено оцінки для норми Шварца згаданих операторів у цих просторах.

1. Introduction. The convolution or Hadamard product of two power series functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Let $\mathcal{H}(\mathbb{D})$ denote the class of all functions holomorphic in the unit disk $\mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, of the complex plan $\mathbb{C}$. Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{D}$. Thus the class $\mathcal{A}$ is a subclass of $\mathcal{H}(\mathbb{D})$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic in $\mathbb{D}$. If $p(z) \in \mathcal{P}$ satisfies $\Re\,(p(z)) > 0$, $z \in \mathbb{D}$, then we say that $p(z)$ is the Caratheodory function (see [4]).

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

for some α, $0 \leq \alpha < 1$, then $f(z)$ is said to be starlike of order α in $\mathbb{D}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$, which are starlike of order α in $\mathbb{D}$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{D}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

for some $\alpha,\ 0 \le \alpha < 1$.

As usual, in the present investigation, we write

$$\mathcal{S}^* = \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} = \mathcal{K}(0).$$

Moreover, for some non-zero complex numbers b, we consider the subclasses $\mathcal{S}_b^*$ and $\mathcal{K}_b$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_b^* = \left\{f(z) \in \mathcal{A} \colon \Re\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0,\ z \in \mathbb{D}\right\}$$

and

$$\mathcal{K}_b = \left\{f(z) \in \mathcal{A} \colon \Re\left[1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in \mathbb{D}\right\}.$$

Then we can see that

$$\mathcal{S}_{1-\alpha}^* = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha).$$

Let f be a function analytic and locally univalent in the unit disk $\mathbb{D}$,

$$S_f = (f''/f')' - 1/2(f''/f')^2$$

denote its Schwarzian derivative and

$$\|S_f(z)\| = \sup_{z\in\mathbb{D}} (1 - |z|^2)^2 |S_f|$$

denote its Schwarzian norm.

Recall first that if f maps the disk conformally onto a convex region, then the function

$$g(z) = 1 + \frac{zf''(z)}{f'(z)}$$

has positive real part in $\mathbb{D}$ (see, for instance, [3]). Since $g(0) = 1$, this say that g is subordinate to the half-plan mapping $\mathcal{L}(z) = (1+z)/(1-z)$, so that $g(z) = \mathcal{L}\left(\varphi(z)\right)$ for some Schwarz functions φ. In other words,

$$\frac{zf''(z)}{f'(z)} = \frac{1+\varphi(z)}{1-\varphi(z)} - 1 = \frac{2\varphi(z)}{1-\varphi(z)},$$

where φ is analytic and has the property $|\varphi(z)| \le |z|$ in $\mathbb{D}$. With the notation $\psi(z) = \dfrac{\varphi(z)}{z}$ this gives the representation

$$\frac{f''(z)}{f'(z)} = \frac{2\psi(z)}{1 - z\psi(z)} \tag{1}$$

for the pre-Schwarzian, where ψ is analytic and satisfies $|\psi(z)| \leq 1$ in $\mathbb{D}$. Straight forward calculation now gives the Schwarzian of f in the form

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{2\psi'(z)}{(1 - z\psi(z))^2}.$$

But $|\psi'(z)| \leq \left(1 - |\psi(z)|^2\right)/(1-|z|^2)$ by the invariant form of the Schwarz lemma, so we conclude that

$$|S_f(z)| \leq 2\frac{1 - |\psi(z)|^2}{(1 - |z|^2)\,(1 - |z\psi(z)|)^2} \leq \frac{2}{(1 - |z|^2)^2}. \tag{2}$$

In other words, the inequality (2) says that the Schwarzian norm

$$\|S_f(z)\| = \sup_{z\in\mathbb{D}} (1 - |z|^2)^2|S_f|,$$

of convex mapping is no large than 2. The bound is best possible since the parallel strip mapping

$$L(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

has Schwarzian $S_L(z) = 2(1 - z^2)^{-2}$. Nehari [12] also stated that $\|S_f(z)\| < 2$ if the convex mapping f is bounded.

The composition operators C_φ are defined on $\mathcal{H}(\mathbb{D})$ as follows:

$$C_\varphi(f) = f \circ \varphi, \qquad f \in \mathcal{H}(\mathbb{D}),$$

for some self-maps $\varphi\colon \mathbb{D} \to \mathbb{D}$ of the unit disk $\mathbb{D}$.

For $g \in \mathcal{H}(\mathbb{D})$, the integral operator I_g

$$I_g h(z) = \int\limits_0^z h'(\xi)g(\xi)d\xi, \qquad h \in \mathcal{H}(\mathbb{D}),$$

was introduced in [14] and is called the Volterra-type operator.

In this paper, we introduced some new subclasses of $\mathcal{H}(\mathbb{D})$ as follows:

$$P(\beta, b) := \left\{p(z) \in \mathcal{P} : \Re\left[\frac{1}{b}\left(\frac{zp'(z)}{p(z)}\right)\right] \geq \beta\right\}$$

and

$$P'(\beta, b) := \left\{p(z) \in \mathcal{P} : \Re\left[\frac{1}{b}\left(\frac{zp'(z)}{p(z)}\right)\right] \leq \beta\right\}$$

for some real numbers β and non-zero complex numbers b.

Example. Let

$$p(z) = \frac{1}{1-z} = 1 + z + z^2 + \ldots \in P(-1/2, 1),$$

$$p(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 \pm \ldots \in P(-1/2, 1),$$

$$p(z) = 1 + z \in P'(1/2, 1), \qquad p(z) = 1 - z \in P'(1/2, 1).$$

Moreover, for some non-zero complex numbers b and real λ, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$, we define the subclass $\hat{\mathcal{S}}_\lambda(b)$ of $\mathcal{A}$ as follows:

$$\hat{\mathcal{S}}_\lambda(b) := \left\{ f(z) \in \mathcal{A} \colon \Re \left[e^{i\lambda} \left(1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right) \right] > 0, \ z \in \mathbb{D} \right\}.$$

If a function $f(z)$ belong to the class $\hat{\mathcal{S}}_\lambda(b)$, we say that $f(z)$ is spirallike of type λ with the complex order b, $b \neq 0$.

In this paper, we get some properties for the functions in $\mathcal{S}_b^*$, $\mathcal{K}_b$ and $\hat{\mathcal{S}}_\lambda(b)$. Also we study the Volterra-type operator I_g on $\mathcal{K}$ and $\mathcal{K}_b$. Furthermore, we get necessary and sufficient condition such that $I_g h(\mathbb{D})$ is bounded, moreover, we obtain the sufficient condition such that $\|S_f(z)\| < 2$.

In addition to the references referred to in this article, to better understand the properties of the spaces and operators mentioned in this work, we can study references [1, 2, 6, 13].

2. Some of the new properties of the functions belong to the subclasses of starlike and convex functions. In this section, we examine some of the properties of the functions belong to the subclasses of starlike and convex functions of complex order b, $b \neq 0$, and spirallike functions of type λ, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$, with the complex order b, $b \neq 0$. Among the items of interest are the necessary and sufficient conditions for a member to belong to the classes $\mathcal{S}_b^*$, $\mathcal{K}_b$ and $\hat{\mathcal{S}}_\lambda(b)$. In the following, we obtain the sharp estimates for the second coefficients of the Taylor series of functions belonging to the mentioned classes.

Theorem 2.1. *Let* $b \in \mathbb{C}$, $b \neq 0$, $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *and* $\beta = e^{-i\lambda}\cos\lambda$. *Then* f *belongs to* $\hat{\mathcal{S}}_\lambda(b)$ *if and only if there is* $g \in \mathcal{S}_b^*$ *such that*

$$f(z) = z\left(\frac{g(z)}{z}\right)^{\beta}. \tag{3}$$

The branch of the power function is choosen such that $\left(\frac{g(z)}{z}\right)^{\beta}\Big|_{z=0} = 1.$

Proof. First assume $f \in \hat{\mathcal{S}}_\lambda(b)$. Cleary the relation (3) is equivalent to

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{\cos\lambda}}, \quad z \in \mathbb{D},$$

we choose the branch of the power function such that $\left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{\cos\lambda}}\Big|_{z=0} = 1$. A simple computation yields the relation

$$1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right) = (1 + i\tan\lambda)\left(1 + \frac{zf'(z)}{bf(z)}\right) - \frac{1 + i\tan\lambda}{b} - i\tan\lambda.$$

Therefore,

$$\Re\left[1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right)\right] = \frac{1}{\cos\lambda}\Re\left[e^{i\lambda}\left(1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right)\right)\right].$$

Since $f \in \hat{\mathcal{S}}_\alpha(b)$, consequently g is starlike of complex order b.

Conversely, if $g \in \mathcal{S}_b^*$, then in view of the above relation and the fact that $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, one deduces that

$$\Re\left[e^{i\lambda}\left(1+\frac{1}{b}\left(\frac{zg'(z)}{g(z)}-1\right)\right)\right] > 0, \quad z \in \mathbb{D}.$$

Thus, f is spirallike of type λ with complex order b.

Theorem 2.1 is proved.

Theorem 2.2. *Let $b \in \mathbb{C}$ and $b \neq 0$. Then we have the equality*

$$\mathcal{S}_b^* = \left\{zh'(z) \colon h \in \mathcal{K}_b\right\}.$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_b^*$. It's obvious that

$$f(z) = z\left(1+\sum_{n=2}^{\infty} a_n z^{n-1}\right).$$

We put

$$h'(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1},$$

therefore,

$$h(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n.$$

Then $f(z) = zh'(z)$. Applying that $h(z) \in \mathcal{K}_b$ if and only if $zh'(z) \in \mathcal{S}_b^*$, we deduced that $h(z)$ belongs to $\mathcal{K}_b$.

Theorem 2.2 is proved.

Theorem 2.3. *For the function $f(z) \in \mathcal{A}$, it follows that*

$$f(z) \in \mathcal{S}_b^* \Longleftrightarrow z\left(\frac{f(z)}{z}\right)^{\frac{1}{b}} \in \mathcal{S}^*, \quad b \in \mathbb{C}, \quad b \neq 0.$$

Proof. Let $f(z)$ be a starlike of complex order b. By using Theorem 2.2, there is $h \in \mathcal{K}_b$ such that $f(z) = zh'(z)$. Since $h \in \mathcal{K}_b$, then, by using Theorem 1.2 in [5], we have $z\left(h'(z)\right)^{\frac{1}{b}} \in \mathcal{S}^*$.

$\Bigg($Another proof for this theorem. We set $F(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{b}}$. Therefore,

$$\Re\left[\frac{zF'(z)}{F(z)}\right] = \Re\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right] > 0,$$

and then $f \in \mathcal{S}_b^*$.$\Bigg)$

By using Theorems 2.1 and 2.2, we have the following theorem.

Theorem 2.4. *Let $b \in \mathbb{C}$ and $b \neq 0$, $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\beta = e^{i\lambda}\cos\lambda$. Then f belongs to $\hat{\mathcal{S}}_\lambda(b)$ if and only if there is $h \in \mathcal{K}_b$ such that*

$$f(z) = z\left(h'(z)\right).$$

Theorem 2.5. *Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be a starlike function of complex order b, $b \neq 0$. Then $|a_2| \leq 2|b|$. This bound is sharp. Equality is attained for $f_b(z) = \dfrac{z}{(1-z)^{2b}}$.*

Proof. Let $f(z)$ belongs to $\mathcal{S}_b^*$. By using Theorem 2.3, we have $g(z) = z\left(\dfrac{f(z)}{z}\right)^{\frac{1}{b}} \in \mathcal{S}^*$. Let $g(z) = z + b_2z^2 + b_3z^3 + \ldots$, therefore $b_2 = \dfrac{1}{b}a_2$. So, by using Bieberbach theorem, we have $|a_2| = |b||b_2| \leq 2|b|$. Since

$$\frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{n}(j + 2(b-1))}{(n-1)!} z^n,$$

then it is obvious that equality is attained for f_b.

Theorem 2.5 is proved.

Theorem 2.6. *Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be a spirallike function of type λ, $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with complex order b, $b \neq 0$. Then $|a_2| \leq 2|b|\cos\lambda$.*

Proof. Let $f(z) \in \hat{\mathcal{S}}_\lambda(b)$. By Theorem 2.1, there is $g(z) = z + b_2z^2 + b_3z^3 + \ldots \in \mathcal{S}_b^*$ such that $f(z) = z\left(\dfrac{g(z)}{z}\right)^{\beta}$, $\beta = e^{-i\lambda}\cos\lambda$. Then $a_2 = b_2\beta$, so $|a_2| = |b_2|\cos\lambda$. By using Theorem 2.5, we have $|a_2| \leq 2|b|\cos\lambda$.

Remark. Since $\hat{\mathcal{S}}_\lambda(1) = \hat{\mathcal{S}}_\lambda$, we obtain Corollary 2.4.12 in [7] as a result of the above theorem.

3. Volterra-type operator on subclasses of convex functions. Here, first, we express and prove two lemmas that are widely used in proving the main theorems of this paper.

Lemma 3.1. 1. *Let $b \in \mathbb{C}$, $b \neq 0$ and $\beta \geq 0$. If g belongs to $P(\beta, b)$, then I_g is an operator on $\mathcal{K}_b$.*

2. *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$ and $0 \leq \alpha + \beta < 1$. If g belongs to $P(\beta, 1)$, then I_g is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}(\alpha + \beta)$.*

Proof. Due to the similarity of the proof of parts 1 and 2, so we only do the proof of part 1.

Let $h \in \mathcal{K}_b$, then

$$\Re\left[1 + \frac{1}{b}\left(\frac{z\left(I_gh\right)''(z)}{\left(I_gh\right)'(z)}\right)\right] = \Re\left[1 + \frac{1}{b}\left(\frac{zh''(z)g(z) + zh'(z)g'(z)}{h'(z)g(z)}\right)\right] =$$

$$= \Re\left[1 + \frac{1}{b}\left(\frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}\right)\right] =$$

$$= \Re\left[1 + \frac{1}{b}\left(\frac{zh''(z)}{h'(z)}\right)\right] + \Re\left[\frac{1}{b}\left(\frac{zg'(z)}{g(z)}\right)\right]. \qquad (4)$$

By hypothesis of this lemma and relation (4), we have

$$\Re\left[1+\frac{1}{b}\left(\frac{z\,(I_g h)''\,(z)}{(I_g h)'\,(z)}\right)\right]>0,$$

therefore, $I_g h$ belongs to $\mathcal{K}_b$ for each $h\in\mathcal{K}_b$.

Lemma 3.2. *The function f is convex of complex order b, $b\neq 0$ in $\mathbb{D}$ if and only if*

$$f'*\frac{z\left(\dfrac{1+x}{2b}\right)+1-z}{(1-z)^2}\neq 0,\quad z\in\mathbb{D},\quad |x|=1.$$

Proof. The function f is convex of complex order b if and only if

$$\Re\left[1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right]>0,\quad z\in\mathbb{D}. \tag{5}$$

By Lemma 1 in [8], relation (5) is equivalent to

$$1+\frac{1}{b}\left(\frac{(zf'(z))'}{f'(z)}-1\right)\neq\frac{x-1}{x+1},\quad z\in\mathbb{D},\quad |x|=1,\quad x\neq -1,$$

which simplifies to

$$(1+x)\left(zf'(z)\right)'+(2b-x-1)f'(z)\neq 0.$$

We have

$$(1+x)\left(zf'(z)\right)'=f'(z)*\frac{1+x}{(1-z)^2}$$

and

$$(2b-x-1)f'(z)=f'(z)*\frac{2b-x-1}{1-z},$$

so that

$$(1+x)\left(zf'(z)\right)'+(2b-x-1)f'(z)=f'(z)*\frac{1+x}{(1-z)^2}+f'(z)*\frac{2b-x-1}{1-z}=$$

$$=f'(z)*\frac{(1+x)+(1-z)(2b-x-1)}{(1-z)^2}=$$

$$=f'(z)*\frac{(1+x-2b)z+2b}{(1-z)^2}\neq 0.$$

Since $b\neq 0$, we get

$$f'*\frac{z\left(\dfrac{1+x-2b}{2b}\right)+1}{(1-z)^2}\neq 0,\quad z\in\mathbb{D},\quad |x|=1,\quad x\neq -1.$$

The case $x=1$ in the convolution condition is equivalent to stating $f'\neq 0$ for each $z\in\mathbb{D}$, which is a necessary condition for univalence.

We now obtain the necessary and sufficient conditions to boundedness the image of the open unit disk $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ under the effect of the Volterra-type operator on the space consisting of univalent functions and its subspace, and finally we get an estimate of the Schwartzian norm for the above-mentioned operator on class $\mathcal{A}$ and subclass $\mathcal{K}(\alpha)$ of it, where $0\leq\alpha<1$.

Theorem 3.1. *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$, $\alpha + \beta \geq 0$ and $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. Then the image $(I_g h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\longrightarrow 1} (1 - |z|) \left| 2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)} \right| < 1. \tag{6}$$

Proof. By using Lemma 3.1 (part 2), we have that $I_g h$ belongs to $\mathcal{K}$. In relation (1), we put $f = I_g h$. Then there is an analytic function ψ such that

$$\frac{(I_g h)''}{I_g h)'} = \frac{2\psi(z)}{1 - z\psi(z)},$$

therefore

$$\psi(z) = \frac{(I_g h)''(z)/(I_g h)'(z)}{2 + z(I_g h)''(z)/(I_g h)'(z)} = \frac{g'(z)/g(z) + h''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)}.$$

We have

$$\frac{1 - |z|}{|1 - z\psi(z)|} = \frac{1 - |z|}{\left| 1 - \dfrac{zg'(z)/g(z) + zh''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)} \right|} =$$

$$= \frac{1 - |z|}{\dfrac{2}{2 + zg'(z)/g(z) + zh''(z)/h'(z)}} =$$

$$= \frac{1}{2}(1 - |z|) \left| 2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)} \right|.$$

By Theorem 2 in [3], we get that the image $(I_g h)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z|\longrightarrow 1} \frac{1 - |z|}{|1 - z\psi(z)|} < \frac{1}{2}.$$

By relation (6), the proof is complete.

Now we want to state a theorem that is similar to the previous theorem in proof. The main difference is the use of Lemma 3.2 instead of Lemma 3.1 in proof.

Theorem 3.2. *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$ such that $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. Then the image $(I_g h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\longrightarrow 1} (1 - |z|) \left| 2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)} \right| < 1.$$

By using Lemma 3.1 (part 2), we have the following corollary.

***Corollary* 3.1.** *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$, $\alpha + \beta \geq 0$. If $g \in P_\beta$ and $h \in \mathcal{K}(\alpha)$, then*

$$\|S_{I_g h}\| \leq 2.$$

By using Lemma 3.2, we obtain the following corollary.

***Corollary* 3.2.** *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$. If $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$, then*

$$\|S_{I_g h}\| \leq 2.$$

By using Theorem 3.1, we get the following corollary.

***Corollary* 3.3.** *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$, $\alpha + \beta \geq 0$ and $g \in P_\beta$ and $h \in \mathcal{K}(\alpha)$. If*

$$\limsup_{|z| \longrightarrow 1} (1-|z|) \left| 2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)} \right| < 1,$$

then

$$\|S_{I_g h}\| < 2.$$

By using Theorem 3.2, we have the following corollary.

***Corollary* 3.4.** *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$ such that $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. If*

$$\limsup_{|z| \longrightarrow 1} (1-|z|) \left| 2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)} \right| < 1,$$

then

$$\|S_{I_g h}\| < 2.$$

4. Product of composition operators and Volterra-type operator on subclasses of convex functions. Products of composition operators and integral-type operators have been recently introduced by S. Li and S. Stević in [9–11]. Here, we shall be interested in studing the product of composition operators and Volterra-type integral operators, which are defined by

$$(C_\sigma(I_g h))(z) = \int_0^{\sigma(z)} h'(\xi) g(\xi) d\xi, \quad z \in \mathbb{D},$$

on subclasses of $\mathcal{A}$, where $g \in \mathcal{A}$ and σ is an analytic self-map of the unit disk. In this section, we assume that $\sigma(z)$ be the Mobius automorphism $\sigma(z) = \dfrac{z + z_0}{1 + \bar{z}_0 z}$ on $\mathbb{D}$, where, z_0 be the fixed point in $\mathbb{D}$.

In fact, in the final section of this paper, we intend to examine a similar discussion of the previous section for the said operator.

Lemma 4.1. *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$ and $0 \leq \alpha + \beta < 1$. If g belongs to $P(\beta, 1)$, then $C_\sigma I_g$ is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}$.*

Proof. By hypothesis of this lemma and by Lemma 3.1 (part 2), it is obvious that I_g is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}$. Therefore $I_g h$ is a convex map. Let $f = I_g h$. By Lemma 1 in [7], we have $fo\sigma$ is a convex mapping of $\mathbb{D}$. We know that

$$fo\sigma(z) = f(\sigma(z)) = (I_g h)(\sigma(z)) = \int_0^{\sigma(z)} h'(\xi) g(\xi) d\xi = (C_\sigma I_g h)(z).$$

Lemma 4.2. *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$. If*

$$\left(h'g\right)(z) * \frac{(x-1)z}{2(1-z)^2} \neq 0,$$

then $C_\sigma I_g h \in \mathcal{K}$.

By using Lemma 3.2 and Lemma 1 in [3], we have the following theorem.

Theorem 4.1. *Let $\beta \in \mathbb{R}$, $0 \leq \alpha < 1$, $\alpha + \beta \geq 0$ and $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. Then the image $(C_\sigma I_g h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\longrightarrow 1} (1-|z|) \left| \frac{(\sigma(z) - z_0) A(z) + 2}{(\sigma(z) - z_0 + \bar{z}_0 z \sigma(z) - z) A(z) + 2\bar{z}_0 z + 2} \right| < \frac{1}{2}, \tag{7}$$

where $A(z) = \dfrac{g'(\sigma(z))}{g(\sigma(z))} + \dfrac{h''(\sigma(z))}{h'(\sigma(z))}$.

Proof. By using Lemma 4.1, we have $(C_\sigma I_g h) \in \mathcal{K}$. In the proof of Theorem 3.1 we saw that

$$\psi(z) = \frac{g'(z)/g(z) + h''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)}.$$

By using Lemma 1 in [3], there is an analytic function λ such that

$$\frac{(C_\sigma I_g h)''}{(C_\sigma I_g h)'} = \frac{2\lambda(z)}{1 - z\lambda(z)},$$

where

$$\lambda(z) = \frac{\psi(\sigma(z)) - \bar{z}_0}{1 - z_0 \psi(\sigma(z))}.$$

We have

$$z\lambda(z) = \frac{\dfrac{zA(z)}{2 + \sigma(z)A(z)} - \bar{z}_0 z}{1 - \dfrac{z_0 A(z)}{2 + \sigma(z)A(z)}} = \frac{(z - \bar{z}_0 z\sigma(z)) A(z) - 2\bar{z}_0 z}{(\sigma(z) - z_0) A(z) + 2},$$

therefore,

$$\frac{1-|z|}{|1 - z\lambda(z)|} = \frac{1-|z|}{\left| 1 - \dfrac{(z - \bar{z}_0 z\sigma(z)) A(z) - 2\bar{z}_0 z}{(\sigma(z) - z_0) A(z) + 2} \right|} =$$

$$= (1-|z|) \left| \frac{(\sigma(z) - z_0) A(z) + 2}{(\sigma(z) - z_0 + \bar{z}_0 z\sigma(z) - z) A(z) + 2\bar{z}_0 z + 2} \right|.$$

By using Theorem 2 in [3], we get the image $(C_\sigma I_g h)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z|\longrightarrow 1} \frac{1-|z|}{|1 - z\lambda(z)|} < \frac{1}{2}.$$

By relation (7), the proof is complete.

In the following, we state a theorem that has a similar proof to the previous theorem. The only major difference is the use of Lemma 4.2 instead of Lemma 4.1 during the proof.

Theorem 4.2. *Let* $g \in \mathcal{P}$ *and* $h \in \mathcal{A}$ *such that* $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. *Then the image* $(C_\sigma I_g h)(\mathbb{D})$ *is bounded if and only if*

$$\limsup_{|z|\longrightarrow 1}(1-|z|)\left|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\right|<\frac{1}{2},$$

where $A(z)=\dfrac{g'(\sigma(z))}{g(\sigma(z))}+\dfrac{h''(\sigma(z))}{h'(\sigma(z))}$.

By using Lemma 4.1, we have the following corollary.

***Corollary* 4.1.** *Let* $\beta \in \mathbb{R},\ 0 \le \alpha < 1,\ \alpha+\beta \ge 0$. *If* $g \in P(\beta, 1)$ *and* $h \in \mathcal{K}(\alpha)$, *then*

$$\|S_{C_\sigma I_g h}\| \le 2.$$

By using of the Lemma 4.2, we obtain the following corollary.

***Corollary* 4.2.** *Let* $g \in \mathcal{P}$ *and* $h \in \mathcal{A}$. *If*

$$\left(h'g\right)(z) * \frac{(x-1)z}{2(1-z)^2} \neq 0,$$

then

$$\|S_{C_\sigma I_g h}\| \le 2.$$

By using Theorem 4.1, we get the following corollary.

***Corollary* 4.3.** *Let* $\beta \in \mathbb{R},\ 0 \le \alpha < 1,\ \alpha+\beta \ge 0$ *and* $g \in P(\beta, 1)$ *and* $h \in \mathcal{K}(\alpha)$. *If*

$$\limsup_{|z|\longrightarrow 1}(1-|z|)\left|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\right|<\frac{1}{2},$$

where $A(z)=\dfrac{g'(\sigma(z))}{g(\sigma(z))}+\dfrac{h''(\sigma(z))}{h'(\sigma(z))}$, *then*

$$\|S_{C_\sigma I_g h}\| < 2.$$

By using Theorem 4.2, we have the following corollary.

***Corollary* 4.4.** *Let* $g \in \mathcal{P}$ *and* $h \in \mathcal{A}$ *such that* $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. *If*

$$\limsup_{|z|\longrightarrow 1}(1-|z|)\left|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\right|<\frac{1}{2},$$

where $A(z)=\dfrac{g'(\sigma(z))}{g(\sigma(z))}+\dfrac{h''(\sigma(z))}{h'(\sigma(z))}$, *then*

$$\|S_{C_\sigma I_g h}\| < 2.$$

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