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UNIVALENCE CRITERIA AND QUASICONFORMAL EXTENSION OF A GENERAL INTEGRAL OPERATOR*

УНІВАЛЕНТНІ КРИТЕРІЇ ТА КВАЗІКОНФОРМНЕ РОЗШИРЕННЯ ІНТЕГРАЛЬНОГО ОПЕРАТОРА ЗАГАЛЬНОГО ВИГЛЯДУ

We give some sufficient conditions of analyticity and univalence for functions defined by an integral operator. Next, we refine the result to a quasiconformal extension criterion with the help of the Becker's method. Further, new univalence criteria and the significant relationships with other results are given. A number of known univalence conditions would follow upon specializing the parameters involved in main results.

Запропоновано достатні умови аналітичності та унівалентності для функцій, що визначаються деяким інтегральним оператором. Цей результат зводиться до критерію квазіконформного розширення за допомогою методу Бекера. Далі отримано нові критерії унівалентності та вказано важливі зв'язки з іншими результатами. Також з основного результату при різних значеннях параметрів, які задіяні у формулюванні цього результату, випливають деякі відомі умови унівалентності.

1. Introduction. Denote by $\mathcal{U}_r = \left\{z \in \mathbb{C} \colon |z| < r\right\}$, $0 < r \leq 1$, the disk of radius r and let $\mathcal{U} = \mathcal{U}_1$. Let $\mathcal{A}$ denote the class of *analytic functions* in the open unit disk $\mathcal{U}$ which satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the functions f which are *univalent* in $\mathcal{U}$. Also, let $\mathcal{P}$ denote the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_k z^k$ that satisfy the condition $\Re\, p(z) > 0$ $(z \in \mathcal{U})$, and Ω be a class of functions w which are analytic in $\mathcal{U}$ and such that $|w(z)| < 1$ for $z \in \mathcal{U}$. These classes have been one of the important subjects of research in geometric function theory for a long time (see [34]).

We say that a sense-preserving homeomorphism f of a plane domain $G \subset \mathbb{C}$ is k-*quasiconformal*, if f is absolutely continuous on almost all lines parallel to coordinate axes and $|f_{\overline{z}}| \leq k|f_z|$, almost everywhere in G, where $f_{\overline{z}} = \partial f/\partial\overline{z}$, $f_z = \partial f/\partial z$ and k is a real constant with $0 \leq k < 1$. For the general defnition of quasiconformal mappings see [1].

Univalence of complex functions is an important property but, in many cases is impossible to show directly that a certain function is univalent. For this reason, many authors found different sufficient conditions of univalence. Two of the most important are the well-known criteria of Becker [3] and Ahlfors [1]. Becker and Ahlfors' works depend upon a ingenious use of the theory of the Loewner chains and the generalized Loewner differential equation. Extensions of these two criteria were given by Ruscheweyh [30], Singh and Chichra [33], Kanas and Lecko [14, 15] and Lewandowski [18]. The recent investigations on this subject are due to Raducanu et al. [29] and Deniz and Orhan [8, 9]. Furthermore, Pascu [24] and Pescar [25] obtained some extensions of Becker's univalence criterion for an integral operator, while Ovesea [23] obtained a generalization of Ruscheweyh's univalence criterion for an integral operator.

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In the present paper, we formulate a new criteria for univalence of the functions defined by an integral operator G_α, considered in [23], and improve obtained there results. Also, we obtain a refinement to a quasiconformal extension criterion of the main result. In the special cases, our univalence conditions contain the results obtained by some of the authors cited in references. Our considerations are based on the theory of Loewner chains.

2. Loewner chains and quasiconformal extension. The method of Loewner chains will prove to be crucial in our later consideration therefore we present a brief summary of that method.

Let $\mathcal{L}(z,t) = a_1(t)z+a_2(t)z^2+\ldots,\ a_1(t)\neq 0$, be a function defined on $\mathcal{U}\times I$, where $I := [0,\infty)$ and $a_1(t)$ is a complex-valued, locally absolutely continuous function on I. Then $\mathcal{L}(z,t)$ is said to be *Loewner chain* if $\mathcal{L}(z,t)$ has the following conditions:

(i) $\mathcal{L}(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t\in I$;

(ii) $\mathcal{L}(z,t)\prec\mathcal{L}(z,s)$ for all $0\le t\le s<\infty$,

where the symbol $\prec$ stands for subordination. If $a_1(t)=e^t$, then we say that $\mathcal{L}(z,t)$ is a *standard Loewner chain*.

In order to prove main results we need the following theorem due to Pommerenke [27] (see also [28]). This theorem is often used to find out univalency for an analytic function, apart from the theory of Loewner chains.

Theorem 2.1 [28]. *Let $\mathcal{L}(z,t)=a_1(t)z+a_2(t)z^2+\ldots$ be analytic in $\mathcal{U}_r$ for all $t\in I$. Suppose that*:

(i) *$\mathcal{L}(z,t)$ is a locally absolutely continuous function in the interval I, and locally uniformly with respect to $\mathcal{U}_r$*;

(ii) *$a_1(t)$ is a complex valued continuous function on I such that $a_1(t)\neq 0,\ |a_1(t)|\to\infty$ for $t\to\infty$ and*

$$\left\{\frac{\mathcal{L}(z,t)}{a_1(t)}\right\}_{t\in I}$$

forms a normal family of functions in $\mathcal{U}_r$;

(iii) *there exists an analytic function $p\colon\mathcal{U}\times I\to\mathbb{C}$ satisfying $\Re p(z,t)>0$ for all $z\in\mathcal{U},\ t\in I$ and*

$$z\frac{\partial\mathcal{L}(z,t)}{\partial z}=p(z,t)\frac{\partial\mathcal{L}(z,t)}{\partial t},\qquad z\in\mathcal{U}_r,\quad t\in I. \tag{2.1}$$

Then, for each $t\in I,\ \mathcal{L}(\cdot,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$ and $\mathcal{L}(z,t)$ is a Loewner chain.

The equation (2.1) is called the *generalized Loewner differential equation*.

The following strengthening of Theorem 2.1 leads to the method of constructing quasiconformal extension, and is based on the result due to Becker (see [3 – 5]).

Theorem 2.2 [3 – 5]. *Suppose that $\mathcal{L}(z,t)$ is a Loewner chain for which $p(z,t)$, defined in* (2.1), *satisfies the condition*

$$p(z,t)\in U(k):=\left\{w\in\mathbb{C}\colon\left|\frac{w-1}{w+1}\right|\le k\right\}=$$

$$=\left\{w\in\mathbb{C}\colon\left|w-\frac{1+k^2}{1-k^2}\right|\le\frac{2k}{1-k^2}\right\},\quad 0\le k<1,$$

for all $z\in\mathcal{U}$ and $t\in I$. Then $\mathcal{L}(z,t)$ admits a continuous extension to $\overline{\mathcal{U}}$ for each $t\in I$ and the function $F(z,\bar z)$ defined by

$$F(z,\bar{z}) = \begin{cases} \mathcal{L}(z,0) & \text{for} \quad |z| < 1, \\ \mathcal{L}\left(\dfrac{z}{|z|}, \log|z|\right) & \text{for} \quad |z| \geq 1, \end{cases}$$

is a k-quasiconformal extension of $\mathcal{L}(z,0)$ to $\mathbb{C}$.

Detailed information about Loewner chains and quasiconformal extension criterion can be found in [1, 2, 6, 7, 17, 26]. For a recent account of the theory we refer the reader to [12, 13]. One can also see the following studies [10, 11, 16, 31, 32] dealing with local and boundary behavior of conformal, quasiconformal and quasiregular mappings, as well as their generalizations.

3. Univalence criteria. The first theorem is our glimpse of the role of the generalized Loewner chains in univalence results for an operator G_α, studied in [23]. The theorem formulates the conditions under which such an operator is analytic and univalent.

Theorem 3.1. *Let α, c and s be complex numbers such that $c \notin [0,\infty)$; $s = a + ib$, $a > 0$, $b \in \mathbb{R}$; $m > 0$ and $f, g \in \mathcal{A}$. If there exists a function h, analytic in $\mathcal{U}$ and such that $h(0) = h_0$, $h_0 \in \mathbb{C}$, $h_0 \notin (-\infty, 0]$, and the inequalities*

$$\left|\alpha - \frac{m}{2a}\right| < \frac{m}{2a}, \tag{3.1}$$

$$\left|\frac{c}{h(z)} + \frac{m}{2\alpha}\right| < \frac{m}{2|\alpha|}, \tag{3.2}$$

and

$$\left|\frac{-c\alpha}{ah(z)}|z|^{m/a} + \left(1 - |z|^{m/a}\right)\left[(\alpha-1)\frac{zg'(z)}{g(z)} + 1 + \frac{zf''(z)}{f'(z)} + \frac{zh'(z)}{h(z)}\right] - \frac{m}{2a}\right| \leq \frac{m}{2a} \tag{3.3}$$

hold true for all $z \in \mathcal{U}$, then the function

$$G_\alpha(z) = \left[\alpha \int_0^z g^{\alpha-1}(u) f'(u)\,du\right]^{1/\alpha} \tag{3.4}$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. We first note that G_α is well defined and analytic in the unit disk. We rewrite G_α in the form

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{g(u)}{u}\right)^{\alpha-1} f'(u)\,du\right]^{1/\alpha},$$

where singularity of $g(z)/z$ at $z = 0$ is removed. Because $g \in \mathcal{A}$, the function $g(z)/z = 1 + \ldots$ is analytic in $\mathcal{U}$, and then there exists a disc $\mathcal{U}_{r_1}$, $0 < r_1 \leq 1$, in which $g(z)/z \neq 0$ for all $z \in \mathcal{U}_{r_1}$. By changing the variable, we next have

$$G_\alpha(z) = z\left[\alpha \int_0^1 w^{\alpha-1}\left(\frac{g(zw)}{zw}\right)^{\alpha-1} f'(zw)\,dw\right]^{1/\alpha} = z + c_2 z^2 + \ldots,$$

so that G_α is analytic in some neighbourhood of the origin.

Next, we prove that there exists a real number $r \in (0,1]$ such that the function $\mathcal{L}(\cdot,t)$, defined formally by

$$\mathcal{L}(z,t) =$$

$$= \left(\alpha \int\limits_0^{e^{-st}z} g^{\alpha-1}(u)f'(u)du - \frac{a}{c}\left(e^{mt}-1\right)e^{-st}zg^{\alpha-1}(e^{-st}z)f'(e^{-st}z)h(e^{-st}z)\right)^{1/\alpha}, \qquad (3.5)$$

is analytic in $\mathcal{U}_r$ for all $t \in [0,\infty) = I$.

Denoting $\phi(z) = g(z)/z$ we define now a function

$$\phi_1(z,t) = \alpha \int\limits_0^{e^{-st}z} u^{\alpha-1}\phi(u)f'(u)du = e^{-st\alpha}z^\alpha + \ldots,$$

so that ϕ_1 can be rewritten in the form

$$\phi_1(z,t) = z^\alpha \phi_2(z,t),$$

where ϕ_2 is analytic in $\mathcal{U}_{r_1}$. Hence, the function

$$\phi_3(z,t) = \phi_2(z,t) - \frac{a}{c}\left(e^{mt}-1\right)e^{-st\alpha}\phi(e^{-st}z)f'(e^{-st}z)h(e^{-st}z)$$

is analytic in $\mathcal{U}_{r_1}$ and

$$\phi_3(0,t) = e^{-st\alpha}\left[\left(1+\frac{a}{c}h_0\right) - \frac{a}{c}h_0e^{mt}\right].$$

Now, we prove that $\phi_3(0,t) \neq 0$ for all $t \in I$. It is easy to see that $\phi_3(0,0) = 1$. Suppose that there exists $t_0 > 0$ such that $\phi_3(0,t_0) = 0$. Then the equality $e^{mt_0} = \dfrac{c+ah_0}{ah_0}$ holds. Since $h_0 \notin (-\infty,0]$, this equality implies that $c > 0$, which contradicts $c \notin [0,\infty)$. From this we conclude that $\phi_3(0,t) \neq 0$ for all $t \in I$. Therefore, there is a disk $\mathcal{U}_{r_2}$, $r_2 \in (0,r_1]$, in which $\phi_3(z,t) \neq 0$ for all $t \in I$. Thus, we can choose a principal branch of $\left[\phi_3(z,t)\right]^{1/\alpha}$ analytic in $\mathcal{U}_{r_2}$. By the construction of $\mathcal{L}(z,t)$ and (3.5) we have that

$$\mathcal{L}(z,t) = z\left[\phi_3(z,t)\right]^{1/\alpha} = a_1(t)z + a_2(t)z^2 + \ldots$$

and, consequently, the function $\mathcal{L}(z,t)$ is analytic in $\mathcal{U}_{r_2}$.

We note that

$$a_1(t) = e^{t\left(\frac{m}{\alpha}-s\right)}\left[\left(1+\frac{a}{c}h_0\right)e^{-mt} - \frac{a}{c}h_0\right]^{1/\alpha},$$

for which we consider the principal branch equal to 1 at the origin.

Since $\left|a\alpha - \dfrac{m}{2}\right| < \dfrac{m}{2}$ is equivalent to $\Re\left\{\dfrac{m}{\alpha}\right\} > a = \Re(s)$, we get

$$\lim_{t\to\infty}\left|a_1(t)\right| = \infty.$$

Moreover, $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $\mathcal{L}(z,t)$ in $\mathcal{U}_{r_2}$, it follows that there exist a number r_3 such that $0 < r_3 < < r_2$ and a constant $K = K(r_3)$ such that

$$\left|\frac{\mathcal{L}(z,t)}{a_1(t)}\right| < K, \qquad z \in \mathcal{U}_{r_3}, \quad t \in I.$$

By the Montel's theorem [22], $\left\{\dfrac{\mathcal{L}(z,t)}{a_1(t)}\right\}_{t\in I}$ forms a normal family in $\mathcal{U}_{r_3}$. From the analyticity of $\dfrac{\partial\mathcal{L}(z,t)}{\partial t}$, it may be concluded that for all fixed numbers $T > 0$ and $r_4,\ 0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (that depends on T and r_4) such that

$$\left|\frac{\partial\mathcal{L}(z,t)}{\partial t}\right| < K_1, \qquad z \in \mathcal{U}_{r_4}, \quad t \in [0,T].$$

Therefore, the function $\mathcal{L}(z,t)$ is locally absolutely continuous in I, locally uniform with respect to $\mathcal{U}_{r_4}$.

Let $p : \mathcal{U}_r \times I \to \mathbb{C}$ denote a function

$$p(z,t) = z\frac{\partial\mathcal{L}(z,t)}{\partial z}\Big/\frac{\partial\mathcal{L}(z,t)}{\partial t}\,,$$

that is, analytic in $\mathcal{U}_r,\ 0 < r < r_4$, for all $t \in I$ (the singularity at $z = 0$ is removable). If the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} = \frac{\dfrac{z\partial\mathcal{L}(z,t)}{\partial z} - \dfrac{\partial\mathcal{L}(z,t)}{\partial t}}{\dfrac{z\partial\mathcal{L}(z,t)}{\partial z} + \dfrac{\partial\mathcal{L}(z,t)}{\partial t}} \tag{3.6}$$

is analytic in $\mathcal{U} \times I$ and $\left|w(z,t)\right| < 1$ for all $z \in \mathcal{U}$ and $t \in I$, then $p(z,t)$ has an analytic extension with positive real part in $\mathcal{U}$ for all $t \in I$. According to (3.6), we have

$$w(z,t) = \frac{(1+s)A(z,t) - m}{(1-s)A(z,t) + m}, \tag{3.7}$$

where

$$\begin{aligned} A(z,t) = \frac{-c\alpha}{ah(e^{-st}z)}e^{-mt} + \left(1 - e^{-mt}\right)\Bigg[(\alpha-1)\frac{e^{-st}zg'(e^{-st}z)}{g(e^{-st}z)} + \\ +1 + \frac{e^{-st}zf''(e^{-st}z)}{f'(e^{-st}z)} + \frac{e^{-st}zh'(e^{-st}z)}{h(e^{-st}z)}\Bigg] \end{aligned} \tag{3.8}$$

for $z \in \mathcal{U}$ and $t \in I$. Hence, the inequality $\left|w(z,t)\right| < 1$ is equivalent to

$$\left|A(z,t) - \frac{m}{2a}\right| < \frac{m}{2a}, \qquad a = \Re(s), \quad z \in \mathcal{U}, \quad t \in I.$$

Define now

$$B(z,t) = A(z,t) - \frac{m}{2a}, \qquad z \in \mathcal{U}, \quad t \in I.$$

From (3.1), (3.2) and (3.8) it follows that

$$\big|B(z,0)\big| = \left|\frac{c\alpha}{ah(z)} + \frac{m}{2a}\right| < \frac{m}{2a} \tag{3.9}$$

and

$$\big|B(0,t)\big| = \frac{1}{a}\left|\frac{c\alpha e^{-mt}}{h_0} - a\alpha\big(1-e^{-mt}\big) + \frac{m}{2}\right| =$$

$$= \frac{1}{a}\left|\left(\frac{c\alpha}{h_0} + \frac{m}{2}\right)e^{-mt} + \left(\frac{m}{2} - a\alpha\right)\left(1-e^{-mt}\right)\right| < \frac{m}{2a}. \tag{3.10}$$

Since $|e^{-st}z| \le |e^{-st}| = e^{-at} < 1$ for all $z \in \overline{\mathcal{U}} = \big\{z \in \mathbb{C} : |z| \le 1\big\}$ and $t > 0$, we conclude that for each $t > 0$ $B(z,t)$ is an analytic function in $\overline{\mathcal{U}}$. Using the maximum modulus principle it follows that for all $z \in \mathcal{U} \setminus \{0\}$ and each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$\big|B(z,t)\big| < \lim_{|z|=1}\big|B(z,t)\big| = \left|B(e^{i\theta},t)\right| \tag{3.11}$$

for all $z \in \mathcal{U}$ and $t \in I$.

Denote $u = e^{-st}e^{i\theta}$. Then $|u| = e^{-at}$, and from (3.8) we obtain

$$\left|B(e^{i\theta},t)\right| =$$

$$= \left|\frac{c\alpha}{ah(u)}|u|^{m/a} + \frac{m}{2a} - \left(1 - |u|^{m/a}\right)\left[(\alpha-1)\frac{ug'(u)}{g(u)} + 1 + \frac{uf''(u)}{f'(u)} + \frac{uh'(u)}{h(u)}\right]\right|.$$

Since $u \in \mathcal{U}$, the inequality (3.3) implies that

$$\left|B(e^{i\theta},t)\right| \le \frac{m}{2a},$$

and from (3.9), (3.10) and (3.11), we conclude that

$$\big|B(z,t)\big| = \left|A(z,t) - \frac{m}{2a}\right| < \frac{m}{2a}$$

for all $z \in \mathcal{U}$ and $t \in I$. Therefore, $\big|w(z,t)\big| < 1$ for all $z \in \mathcal{U}$ and $t \in I$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $\mathcal{L}(z,t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$ for all $t \in I$. For $t = 0$, we have $\mathcal{L}(z,0) = G_\alpha(z)$ for $z \in \mathcal{U}$ and, therefore, the function $G_\alpha(z)$ is analytic and univalent in $\mathcal{U}$.

Theorem 3.1 is proved.

Abbreviating (3.3), we can now rephrase Theorem 3.1 in a simpler form.

Theorem 3.2. *Let $f, g \in \mathcal{A}$. Let $m > 0$, the complex numbers α, c, s and the function h be as in Theorem* 3.1. *Moreover, suppose that the inequalities* (3.1) *and* (3.2) *are satisfied. If the inequality*

$$\left|(\alpha-1)\frac{zg'(z)}{g(z)} + 1 + \frac{zf''(z)}{f'(z)} + \frac{zh'(z)}{h(z)} - \frac{m}{2a}\right| \le \frac{m}{2a} \tag{3.12}$$

holds true for all $z \in \mathcal{U}$, then the function G_α defined by (3.4) *is analytic and univalent in $\mathcal{U}$.*

Proof. Making use of (3.2) and (3.12), we obtain

$$\left|\frac{c\alpha}{h(z)}|z|^{m/a}+\frac{m}{2}-a\left(1-|z|^{m/a}\right)\left[(\alpha-1)\frac{zg'(z)}{g(z)}+1+\frac{zf''(z)}{f'(z)}+\frac{zh'(z)}{h(z)}\right]\right|=$$

$$=\left|\left(\frac{c\alpha}{h(z)}+\frac{m}{2}\right)|z|^{m/a}+\right.$$

$$\left.+\left(1-|z|^{m/a}\right)\left[-a\left((\alpha-1)\frac{zg'(z)}{g(z)}+1+\frac{zf''(z)}{f'(z)}+\frac{zh'(z)}{h(z)}\right)+\frac{m}{2}\right]\right|\leq$$

$$\leq|z|^{m/a}\frac{m}{2}+\left(1-|z|^{m/a}\right)\frac{m}{2}=\frac{m}{2},$$

so that the condition (3.3) is satisfied. This finishes the proof, since all the assumption of Theorem 3.1 are satisfied.

The special case of Theorem 3.1, i.e., for $s=\alpha=1$ and $h(z)=-c$, leads to the following result.

***Corollary* 3.1.** *Let* $f\in\mathcal{A}$ *and* $m>1$. *If*

$$\left|\frac{m-2}{2}-\left(1-|z|^{m}\right)\frac{zf''(z)}{f'(z)}\right|\leq\frac{m}{2}$$

holds for $z\in\mathcal{U}$, *then the function* f *univalent in* $\mathcal{U}$.

Corollary 3.1 in turn implies the well-known Becker's univalence citerion [3].

***Remark* 3.1.** Important examples of univalence criteria may be obtained by a suitable choices of f and g, below.

(1) Choose $g_1(z)=z$. Then Theorem 3.1 gives analyticity and univalence of the operator

$$F(z)=\left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{1/\alpha},$$

which was studied by Pascu [24].

(2) Setting $f(z)=z$ in Theorem 3.1, we obtain that the operator

$$G(z)=\left[\alpha\int_0^z g^{\alpha-1}(u)du\right]^{1/\alpha}$$

is analytic and univalent in $\mathcal{U}$. The operator G was introduced by Moldoveanu and Pascu [20].

(3) Taking $f'(z)=\dfrac{g(z)}{z}$ in Theorem 3.1, we find that

$$H(z)=\left[\alpha\int_0^z\frac{g^{\alpha}(u)}{u}du\right]^{1/\alpha}$$

is analytic and univalent in $\mathcal{U}$. The operator H was introduced and studied by Mocanu [19].

If we limit a range of parameter a to the case $a \geq 1$, then, applying the Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *Let α, c and s be complex numbers such that $c \notin [0, \infty)$; $s = a + ib$, $a \geq 1$, $b \in \mathbb{R}$, $m > 0$ and $f, g \in \mathcal{A}$. Let the function h be as in Theorem* 3.1. *Moreover, suppose that the inequalities* (3.1) *and* (3.2) *are satisfied. If the inequality*

$$\left| \frac{-c\alpha}{ah(z)} |z|^m + (1 - |z|^m) \left[(\alpha - 1) \frac{zg'(z)}{g(z)} + 1 + \frac{zf''(z)}{f'(z)} + \frac{zh'(z)}{h(z)} \right] - \frac{m}{2a} \right| \leq \frac{m}{2a} \tag{3.13}$$

holds true for all $z \in \mathcal{U}$, then the function $G_\alpha(z)$, defined by (3.4), *is analytic and univalent in $\mathcal{U}$.*

Proof. For $\lambda \in [0, 1]$ define the linear function

$$\phi(z, \lambda) = \lambda k(z) + (1 - \lambda)\, l(z), \qquad z \in \mathcal{U}, \quad t \in I,$$

where

$$k(z) = \frac{c\alpha}{h(z)} + \frac{m}{2}$$

and

$$l(z) = -a \left[(\alpha - 1) \frac{zg'(z)}{g(z)} + 1 + \frac{zf''(z)}{f'(z)} + \frac{zh'(z)}{h(z)} \right] + \frac{m}{2}.$$

For fixed $z \in \mathcal{U}$ and $t \in I$, $\phi(z, \lambda)$ is a point of a segment with endpoints at $k(z)$ and $l(z)$. The function $\phi(z, \lambda)$ is analytic in $\mathcal{U}$ for all $\lambda \in [0, 1]$ and $z \in \mathcal{U}$, satisfies

$$|\phi(z, 1)| = |k(z)| < \frac{m}{2} \tag{3.14}$$

and

$$\left| \phi(z, |z|^m) \right| \leq \frac{m}{2}, \tag{3.15}$$

which follows from (3.2) and (3.13). If λ increases from $\lambda_1 = |z|^m$ to $\lambda_2 = 1$, then the point $\phi(z, \lambda)$ moves on the segment whose endpoints are $\phi(z, |z|^m)$ and $\phi(z, 1)$. Because $a \geq 1$, from (3.14) and (3.15) it follows that

$$\left| \phi(z, |z|^{m/a}) \right| \leq \frac{m}{2}, \quad z \in \mathcal{U}. \tag{3.16}$$

We can observe that the inequality (3.16) is just the condition (3.3), and then Theorem 3.1 now yields that the function $G_\alpha(z)$, defined by (3.4), is analytic and univalent in $\mathcal{U}$.

Theorem 3.3 is proved.

Remark **3.2.** Applying Theorem 3.3 to $m = 2$ and the function $h(z) \equiv 1$, and $g(z) = f(z)$, $\alpha = 1/s$ $\left(\text{or } g(z) = z,\ a = 1,\ c = -\frac{1}{\alpha}, \text{ respectively}\right)$, we obtain the results by Ruscheweyh [30] (or Moldoveanu and Pascu [21], respectively).

Remark **3.3.** Substituting $1/h$ instead of h with $h(0) = 1$ and setting $g(z) = f(z)$, $\alpha = 1/s$, $m = 2$ in Theorem 3.3, we obtain the result due to Singh and Chichra [33].

Remark **3.4.** Setting $g(z) = f(z)$, $s = \alpha = 1$, $c = -1$, $m = 2$ and $h(z) = \dfrac{k(z) + 1}{2}$, where k is an analytic function with positive real part in $\mathcal{U}$ with $k(0) = 1$ in Theorem 3.3, we obtain the result by Lewandowski [18].

Remark **3.5.** For the case when $m = 2$ and $h(0) = h_0 = 1$ Theorems 3.1 and 3.3 reduce to the results by Ovesea [23].

4. Quasiconformal extension criterion. In this section, we will refine the univalence condition given in Theorem 3.1 to a quasiconformal extension criterion.

Theorem 4.1. *Let* α, c *and* s *be complex numbers such that* $c \notin [0, \infty)$, $s = a + ib$, $a > 0$, $b \in \mathbb{R}$, $m > 0$; $k \in [0, 1)$ *and* $f, g \in \mathcal{A}$. *If there exists a function* h, *analytic in* $\mathcal{U}$, *such that* $h(0) = h_0$, $h_0 \in \mathbb{C}$, $h_0 \notin (-\infty, 0]$, *and the inequalities*

$$\left|\alpha - \frac{m}{2a}\right| < \frac{m}{2a},$$

$$\left|\frac{c\alpha}{h(z)} + \frac{m}{2}\right| < k\frac{m}{2} \tag{4.1}$$

and

$$\left|\frac{-c\alpha}{ah(z)}|z|^{m/a} + \left(1 - |z|^{m/a}\right)\left[(\alpha - 1)\frac{zg'(z)}{g(z)} + 1 + \frac{zf''(z)}{f'(z)} + \frac{zh'(z)}{h(z)}\right] - \frac{m}{2a}\right| \le k\frac{m}{2a} \tag{4.2}$$

hold true for all $z \in \mathcal{U}$, *then the function* $G_\alpha(z)$ *given by* (3.4) *has an* K*-quasiconformal extension to* $\mathbb{C}$, *where*

$$K = \begin{cases} k & \textit{for} \quad s = 1, \\ \dfrac{|s-1|^2 + k|\bar{s}^2 - 1|}{|\bar{s}^2 - 1| + k|s-1|^2} & \textit{for} \quad s \neq 1. \end{cases}$$

Proof. In the proof of Theorem 3.1 it has been shown that the function $\mathcal{L}(z, t)$, given by (3.5), is a subordination chain in $\mathcal{U}$. Applying Theorem 2.2 to the function $w(z, t)$ given by (3.7), we obtain that the condition

$$\left|\frac{(1+s)A(z,t) - m}{(1-s)A(z,t) + m}\right| < l, \qquad z \in \mathcal{U}, \quad t \in I, \quad 0 \le l < 1, \tag{4.3}$$

with $A(z, t)$ defined by (3.8), implies l-quasiconformal extensibility of $G_\alpha(z)$. Lengthy, but elementary calculations, show that inequality (4.3) is equivalent to

$$\left|A(z,t) - \frac{m\big((1+l^2) + a(1-l^2) - ib(1-l^2)\big)}{2a(1+l^2) + (1-l^2)(1+|s|^2)}\right| \le \frac{2lm}{2a(1+l^2) + (1-l^2)(1+|s|^2)}. \tag{4.4}$$

Taking into account (4.1) and (4.2), we clearly see that

$$\left|A(z,t) - \frac{m}{2a}\right| \le k\frac{m}{2a}. \tag{4.5}$$

Consider the two disks $\Delta_1(s_1, r_1)$ and $\Delta_2(s_2, r_2)$ defined by (4.4) and (4.5), respectively, where $A(z, t)$ is replaced by a complex variable w. The proof is completed by showing that there exists $l \in [0, 1)$ for which Δ_2 is contained in Δ_1. Equivalently $\Delta_2 \subset \Delta_1$ holds, if $|s_1 - s_2| + r_2 \le r_1$, that is,

$$\left|\frac{m\big((1+l^2) + a(1-l^2)\big) - imb(1-l^2)}{2a(1+l^2) + (1-l^2)\big(1+|s|^2\big)} - \frac{m}{2a}\right| + k\frac{m}{2a} \le \frac{2lm}{2a(1+l^2) + (1-l^2)\big(1+|s|^2\big)}$$

or

$$\frac{(1-l^2)|\bar{s}^2-1|}{2a\Big[2a(1+l^2)+(1-l^2)\big(1+|s|^2\big)\Big]} \le \frac{2l}{2a(1+l^2)+(1-l^2)\big(1+|s|^2\big)} - \frac{k}{2a} \tag{4.6}$$

with the condition

$$\frac{2l}{2a(1+l^2)+(1-l^2)\big(1+|s|^2\big)} - \frac{k}{2a} \ge 0. \tag{4.7}$$

For the case, when $k=0$, the condition (4.7) holds for every l, while (4.6) is satisfied for $l_1 \le l < 1$, where

$$l_1 = \frac{|s-1|^2}{|\bar{s}^2-1|}.$$

If, on the other hand, $s=1$ and $k \in (0,1)$, then (4.7) and (4.6) hold for $k \le l < 1$. Assume now $s \ne 1$ and $k \in (0,1)$. The condition (4.7) reduces to the quadratic inequality

$$l^2\big[k(1+|s|^2)-2ak\big]+4al-k\big[2a+1+|s|^2\big] \ge 0$$

or

$$kl^2|s-1|^2+4al-k|s+1|^2 \ge 0. \tag{4.8}$$

Therefore, we find that (4.7) (or (4.8)) holds for $l_2 \le l < 1$, where

$$l_2 = \frac{\sqrt{4a^2+k^2|\bar{s}^2-1|^2}-2a}{k|s-1|^2}.$$

Similarly, (4.7) may be rewritten as

$$(1-l^2)|\bar{s}^2-1| \le 4al-2ak(1+l^2)-k(1-l^2)(1+|s|^2)$$

or

$$l^2\big[k|s-1|^2+|\bar{s}^2-1|\big]+4al-k|s+1|^2-|\bar{s}^2-1| \ge 0,$$

that is, satisfied for $l_3 \le l < 1$, where

$$l_3 = \frac{|s-1|^2+k|\bar{s}^2-1|}{|\bar{s}^2-1|+k|s-1|^2}.$$

We note that $l_2 \le l_3$. Indeed, it is trivial that

$$\big[|\bar{s}^2-1|+k|s-1|^2\big]\sqrt{4a^2+k^2|\bar{s}^2-1|^2} \le \big[|\bar{s}^2-1|+k|s-1|^2\big]\big[2a+k|\bar{s}^2-1|\big].$$

Moreover, we see at once that

$$\big[|\bar{s}^2-1|+k|s-1|^2\big]\big[2a+k|\bar{s}^2-1|\big] \le \big[|\bar{s}^2-1|+k|s-1|^2\big]\big[2a+k|\bar{s}^2-1|\big]+4ak|s-1|^2.$$

Combining the last two inequalities, we obtain

$$\big[|\bar{s}^2-1|+k|s-1|^2\big]\sqrt{4a^2+k^2|\bar{s}^2-1|^2} \le \big[|\bar{s}^2-1|+k|s-1|^2\big]\big[2a+k|\bar{s}^2-1|\big]+4ak|s-1|^2,$$

which is equivalent to the desired inequality $l_2 \le l_3$. Likewise, it is a simple matter to show that $l_3 < 1$, and the proof is complete, by setting $K := l_3$. We note also, that the case $k=0$ may be included to the last case (i.e., $s \ne 1$).

Theorem 4.1 is proved.

Several similar sufficient conditions for quasiconformal extensions as in the Theorem 4.1 can be derived. Here we select a few example out of a large variety of possibilities. The following is based on the Theorem 2.2.

Theorem 4.2. *Let $\alpha > 0$ and $f, g \in \mathcal{A}$. If*

$$z^{1-\alpha} g(z)^{\alpha-1} f'(z) \in U(k)$$

for all $z \in \mathcal{U}$, then the function $G_\alpha(z)$ can be extended to a k-quasiconformal automorphism of $\mathbb{C}$.

Proof. Set

$$\mathcal{L}(z,t) = \left(\alpha \int\limits_0^z g^{\alpha-1}(u) f'(u) du + (e^{\alpha t} - 1) z^\alpha \right)^{1/\alpha}.$$

An easy computation shows

$$p(z,t) = \frac{1}{e^{\alpha t}} \left(z^{1-\alpha} g(z)^{\alpha-1} f'(z) \right) + \left(1 - \frac{1}{e^{\alpha t}} \right),$$

and the assertion follows by the same methods as in Theorem 4.1, applying Theorems 2.1 and 2.2.

In the same manner, by definition of the suitable Loewner chain, several univalence criterion may by found. For example, the condition

$$\frac{zG'_\alpha(z)}{G_\alpha(z)} \in U(k), \quad \alpha \in \mathbb{C},$$

which is based on the integral operator $G_\alpha(z)$, is given by the Loewner chain

$$\mathcal{L}(z,t) = e^t G_\alpha(z).$$

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