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LOCAL COHOMOLOGY MODULES AND THEIR PROPERTIES
ЛОКАЛЬНІ КОГОМОЛОГІЧНІ МОДУЛІ ТА ЇХНІ ВЛАСТИВОСТІ

Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and let M be a generalized Cohen-Macaulay R-module of dimension $d\geq 2$. We show that

$$D\left(H^d_{\mathfrak{m}}\Big(D\big(H^d_{\mathfrak{m}}(D_{\mathfrak{m}}(M))\big)\Big)\right)\approx D_{\mathfrak{m}}(M),$$

where $D=\mathrm{Hom}(-,E)$ and $D_{\mathfrak{m}}(-)$ is the ideal transform functor. Also, assuming that I is a proper ideal of a local ring R, we obtain some results on the finiteness of Bass numbers, cofinitness, and cominimaxness of local cohomology modules with respect to I.

Нехай $(R,\mathfrak{m})$ — повне локальне нетерове кільце, а M — узагальнений R-модуль Коена – Маколея, що має розмірність $d\geq 2$. Доведено, що

$$D\left(H^d_{\mathfrak{m}}\Big(D\big(H^d_{\mathfrak{m}}(D_{\mathfrak{m}}(M))\big)\Big)\right)\approx D_{\mathfrak{m}}(M),$$

де $D=\mathrm{Hom}(-,E)$ і $D_{\mathfrak{m}}(-)$ — функтор перетворення ідеалу. Також якщо I є нетривіальним ідеалом локального кільця R, отримано деякі результати щодо фінітності чисел Басса, кофінітності та комінімаксності локальних модулів когомології відносно I.

1. Introduction. Throughout this paper, let R denote a commutative Noetherian local ring (with identity) and I an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H^i_I(M)=\varinjlim_{n\geq 1}\mathrm{Ext}^i_R(R/I^n,M).$$

For more details about local cohomology modules see [2, 4]. We shall refer to D_I as the I-transform functor. Note that this functor is left exact. For an R-module M, we call $D_I(M)= =\varinjlim_{n\geq 1}\mathrm{Hom}_R(I^n,M)$ the ideal transform of M with respect to I, or, alternatively, the I-transform of M.

Let $(R,\mathfrak{m})$ be a Noetherian local ring and let M be a non-zero finitely generated R-module of dimension $n>0$. We say that M is a generalized Cohen – Macaulay R-module precisely when $H^i_I(M)$ is finitely generated for all $i\neq n$. (Such modules were called "quasi-Cohen – Macaulay modules" by P. Schenzel in [11].)

Also we shall use D to denote the exact, contravariant, R-linear functor $\mathrm{Hom}_R(-,E)$, where $E:=E(R/\mathfrak{m})$ is the injective envelope of the simple R-module $R/\mathfrak{m}$. For each R-module L, we denote set $\{\mathfrak{p}\in\mathrm{Ass}_R L\colon \dim R/\mathfrak{p}=\dim L\}$ by $\mathrm{Assh}_R L$. Also, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x\in R\colon x^n\in\mathfrak{b}$ for some $n\in\mathbb{N}\}$ and we denote $\{\mathfrak{p}\in\mathrm{Spec}(R):\mathfrak{p}\supseteq\mathfrak{b}\}$ by $V(\mathfrak{b})$. For any unexplained notation and terminology we refer the reader to [2, 3, 6].

2. Ideal transform and local cohomology.

Lemma 2.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let M be an R-module of dimension $d\geq 2$. Then $D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(M)\right)\right)\right)\approx D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}\left(D_{\mathfrak{m}}(M)\right)\right)\right)\right)$, where $D=\operatorname{Hom}(-,E)$ and $D_{\mathfrak{m}}(-)$ is the ideal transform.*

Proof. Consider the exact sequence

$$0\longrightarrow\frac{M}{\Gamma_{\mathfrak{m}}(M)}\longrightarrow D_{\mathfrak{m}}\left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right)\longrightarrow H^1_{\mathfrak{m}}\left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right)\longrightarrow 0$$

or

$$0\longrightarrow\frac{M}{\Gamma_{\mathfrak{m}}(M)}\longrightarrow D_{\mathfrak{m}}(M)\longrightarrow H^1_{\mathfrak{m}}(M)\longrightarrow 0.$$

The above exact sequences induce an exact sequence

$$H^1_{\mathfrak{m}}\left(H^1_{\mathfrak{m}}(M)\right)\longrightarrow H^2_{\mathfrak{m}}\left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right)\longrightarrow H^2_{\mathfrak{m}}(D_{\mathfrak{m}}(M))\longrightarrow H^2_{\mathfrak{m}}\left(H^1_{\mathfrak{m}}(M)\right)\longrightarrow\ldots.$$

Since $H^1_{\mathfrak{m}}\left(H^1_{\mathfrak{m}}(M)\right)=H^2_{\mathfrak{m}}\left(H^1_{\mathfrak{m}}(M)\right)=0$, hence $H^2_{\mathfrak{m}}\left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right)\approx H^2_{\mathfrak{m}}(D_{\mathfrak{m}}(M))$. Now, for all $i\geq 2$ and, in particular, for $i=d$, we have

$$H^i_{\mathfrak{m}}(M)=H^i_{\mathfrak{m}}\left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right)\approx H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(M)),\quad H^d_{\mathfrak{m}}(M)\approx H^d_{\mathfrak{m}}(D_{\mathfrak{m}}(M)).$$

Consequently, $D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(M)\right)\right)\right)\approx D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}\left(D_{\mathfrak{m}}(M)\right)\right)\right)\right)$.

Lemma 2.2. *Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and let M be a generalized Cohen–Macaulay R-module of dimension $d\geq 2$. Let $x_1,\ldots,x_d$ be an $\mathfrak{m}$-filter regular sequence for $D_{\mathfrak{m}}(M)$ and for $0\leq i\leq d$, we set $L_i=H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(M))$, $K_i=H^i_{(x_1,\ldots,x_i)}(D_{\mathfrak{m}}(M))$ and $T_i=\left(\frac{K_i}{L_i}\right)_{x_{i+1}}$. Then*

$$H^d_{\mathfrak{m}}(D(L_d))\approx H^{d-1}_{\mathfrak{m}}\left(D\left(\frac{K_{d-1}}{L_{d-1}}\right)\right),$$

$$H^{d-1}_{\mathfrak{m}}(D(K_{d-1}))\approx H^{d-2}_{\mathfrak{m}}\left(D\left(\frac{K_{d-2}}{L_{d-2}}\right)\right).$$

Proof. Let $N=D_{\mathfrak{m}}(M)$. Since $\Gamma_{\mathfrak{m}}(N)=H^1_{\mathfrak{m}}(N)=0$, so N is a generalized Cohen–Macualay module of dimension $d\geq 2$. Now, let $x_1,\ldots,x_d$ be an $\mathfrak{m}$-filter regular sequence for N. Consider the following exact sequences:

$$0\longrightarrow N\longrightarrow T_0\longrightarrow K_1\longrightarrow 0,$$

$$0\longrightarrow K_1\longrightarrow T_1\longrightarrow K_2\longrightarrow 0,$$

$$0\longrightarrow\frac{K_2}{L_2}\longrightarrow T_2\longrightarrow K_3\longrightarrow 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 \longrightarrow \frac{K_{d-2}}{L_{d-2}} \longrightarrow T_{d-2} \longrightarrow K_{d-1} \longrightarrow 0,$$

$$0 \longrightarrow \frac{K_{d-1}}{L_{d-1}} \longrightarrow T_{d-1} \longrightarrow L_d \longrightarrow 0,$$

where, by [10] (Corollary 2.6), $K_2 \approx H^1_{Rx_2}(K_1)$ and also we have the following:

$$L_2 = H^2_{\mathfrak{m}}(N) \approx \Gamma_{Rx_3}\left(H^2_{Rx_1+Rx_2}(N)\right) \approx \Gamma_{\mathfrak{m}}\left(H^2_{(x_1,x_2)}(N)\right) \approx \Gamma_{\mathfrak{m}}(K_2).$$

The above exact sequences induces the following exact sequences:

$$0 \longrightarrow D(L_d) \longrightarrow D(T_{d-1}) \longrightarrow D\left(\frac{K_{d-1}}{L_{d-1}}\right) \longrightarrow 0,$$

$$0 \longrightarrow D(K_{d-1}) \longrightarrow D(T_{d-2}) \longrightarrow D\left(\frac{K_{d-2}}{L_{d-2}}\right) \longrightarrow 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 \longrightarrow D(K_3) \longrightarrow D(T_2) \longrightarrow D\left(\frac{K_2}{L_2}\right) \longrightarrow 0,$$

$$0 \longrightarrow D(K_2) \longrightarrow D(T_1) \longrightarrow D(K_1) \longrightarrow 0,$$

$$0 \longrightarrow D(K_1) \longrightarrow D(T_0) \longrightarrow D(N) \longrightarrow 0.$$

Since for $x_{i+1} \in \mathfrak{m}$, the map $T_i \xrightarrow{x_{i+1}} T_i$ is an isomorphism, so, for all $j \geq 0$, the map

$$H^j_{\mathfrak{m}}(D(T_i)) \xrightarrow{x_{i+1}} H^j_{\mathfrak{m}}(D(T_i))$$

is an isomorphism. On the other hand, $H^j_{\mathfrak{m}}(D(T_i))$ is $\mathfrak{m}$-torsion and so is an Rx_{i+1}-torsion. It follows that $H^j_{\mathfrak{m}}(D(T_i)) = 0$ for all $j \geq 0$. Therefore, we have

$$H^d_{\mathfrak{m}}(D(L_d)) \approx H^{d-1}_{\mathfrak{m}}\left(D\left(\frac{K_{d-1}}{L_{d-1}}\right)\right),$$

$$H^{d-1}_{\mathfrak{m}}(D(K_{d-1})) \approx H^{d-2}_{\mathfrak{m}}\left(D\left(\frac{K_{d-2}}{L_{d-2}}\right)\right), \tag{2.1}$$

$$H^2_{\mathfrak{m}}(D(K_2)) \approx H^1_{\mathfrak{m}}(D(K_1)),$$

$$H^1_{\mathfrak{m}}(D(K_1)) \approx D(N).$$

Theorem 2.1. *Let the situation and notation be as in Lemma* 2.2. *Then*

$$D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(D_{\mathfrak{m}}(M))\right)\right)\right) \approx D_{\mathfrak{m}}(M).$$

Proof. Note that, for all $2 \leq i \leq d-1$, the R-module L_i is of finite length and so is $D(L_i)$. Hence, for all $j \geq 1$ and $2 \leq i \leq d-1$, $H^j_{\mathfrak{m}}(D(L_i)) = 0$. By the notation of previous lemma, the exact sequence

$$0 \longrightarrow L_i \longrightarrow K_i \longrightarrow \frac{K_i}{L_i} \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow D\left(\frac{K_i}{L_i}\right) \longrightarrow D(K_i) \longrightarrow D(L_i) \longrightarrow 0.$$

Now we can write

$$H^j_{\mathfrak{m}}\left(D\left(\frac{K_i}{L_i}\right)\right) \approx H^j_{\mathfrak{m}}(D(K_i)) \tag{2.2}$$

for $j \geq 2$. Finally, from (2.1) and (2.2), we have the following:

$$H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(N)\right)\right) \approx H^{d-1}_{\mathfrak{m}}\left(D\left(\frac{K_{d-1}}{L_{d-1}}\right)\right) \approx H^{d-1}_{\mathfrak{m}}(K_{d-1}) \approx H^{d-2}_{\mathfrak{m}}\left(\frac{K_{d-2}}{L_{d-2}}\right) \approx$$

$$\approx H^{d-2}_{\mathfrak{m}}(K_{d-2}) \approx \dots \approx H^2_{\mathfrak{m}}(D(K_2)) \approx H^1_{\mathfrak{m}}(D(K_1)) \approx D(N).$$

Consequently, $H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(N)\right)\right) \approx D(N)$. Since R is complete, it follows that

$$D\left(H^d_{\mathfrak{m}}\left(D\left(H^d_{\mathfrak{m}}(N)\right)\right)\right) \approx D(D(N)) \approx N = D_{\mathfrak{m}}(M).$$

***Corollary* 2.1.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and let M be a generalized Cohen–Macaulay R-module of dimension $d \geq 2$. Then $\left(D(H^d_{\mathfrak{m}}(M)\right)$ is Cohen–Macaulay iff $D_{\mathfrak{m}}(M)$ is Cohen–Macaulay and this is equivalent to the following*:

$$\left\{i \in \mathbb{N}_0 : H^i_{\mathfrak{m}}(M) \neq 0\right\} \subseteq \{0, 1, d\}.$$

Proof. The assertion follows immediately from above theorem.

3. Cofiniteness and cominimaxness of local cohomology modules.

Lemma 3.1. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Let $\mathcal{M}(R, I)_{\text{com}}$ denote the category of I-cominimax modules over R. Then $\mathcal{M}(R, I)_{com}$ forms an Abelian subcategory of the category of all R-modules. That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cominimax modules, then* $\ker f$ *and* $\operatorname{coker} f$ *are I-cominimax.*

Proof. See [5] (Theorem 2.6).

***Remark* 3.1.** For Noetherian local ring $(R, \mathfrak{m})$ of dimension $d \geq 1$ and proper ideal I of R, we set

$$T_1 = \{p \in \operatorname{Assh}(R) \,|\, \operatorname{Rad}(p + I) = \mathfrak{m}\},$$

$$T_2 = \operatorname{Ass}_R(R) \backslash T_1.$$

Let $0 = \bigcap_{p_i \in \operatorname{Ass}(R)} q_i$ be a minimal primary decomposition for the zero ideal of R such that q_i is p_i-primary. If $L_1 = \bigcap_{q_i \in T_1} q_i$ and $L_2 = \bigcap_{q_i \in T_2} q_i$, then $\operatorname{Ass} \dfrac{R}{L_1} = T_1$ and $\operatorname{Ass} \dfrac{R}{L_2} = T_2$. By [2] (Theorem 8.2.1), $H^d_I\left(\dfrac{R}{L_2}\right) = 0$ and $H^i_I\left(\dfrac{R}{L_1}\right) \approx H^i_{\mathfrak{m}}\left(\dfrac{R}{L_1}\right)$ for all $i \geq 0$. Thus, $H^i_I\left(\dfrac{R}{L_1}\right)$ is Artinian for all $i \geq 0$ and $\operatorname{cd}\left(I, \dfrac{R}{L_2}\right) \leq d - 1$.

On the other hand, $\operatorname{Ann}(L_2) \subseteq \operatorname{Ann}(L_2 M)$, so $\operatorname{Supp}(L_2 M) \subseteq \operatorname{Supp}(L_2)$. Therefore, $H^i_I(L_2 M) \approx \approx H^i_{\mathfrak{m}}(L_2 M)$ for all $i \geq 0$ and $H^i_I(L_2 M)$ is Artinian for $i \geq 0$.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and I be a proper ideal of R. Then the following statements are equivalent:*

(1) *the Bass-numbers of $H_I^{d-1}(R)$ are finite;*

(2) *for any finitely generated R-module M, the Bass numbers of $H_I^{d-1}(M)$ are finite;*

(3) *$H_I^{d-1}(R)$ is I-cominimax;*

(4) *for any finitely generated R-module M, the R-module $H_I^{d-1}(M)$ is I-cominimax.*

Proof. $1 \leftrightarrow 3$. Follows from [1] (Theorem 2.12).

$2 \rightarrow 1$. Is clear.

$2 \leftrightarrow 4$. If $\dim M = d$ the assertion follows from [8] (Proposition 5.1). If $\dim M \leq d-1$, then $H_I^{d-1}(M)$ is Artinian.

$1 \rightarrow 2$. Let M be a finitely generated R-module. If $\dim M < d-1$, then $H_I^{d-1}(M) = 0$ and the result follows. If $\dim M = d-1$, then $H_I^{d-1}(M)$ is Artinian and by [8] (Proposition 5.1).

Therefore we assume that $\dim M = d$. By [1] (Theorem 2.12), the Bass numbers of $H_I^{d-1}(M)$ are finite iff $\operatorname{Hom}_R\left(\dfrac{R}{\mathfrak{m}}, H_I^{d-1}(M)\right)$ be a finitely generated. Therefore with out lose of generality, we may assume that $(R, \mathfrak{m})$ is a complete Noetherian local ring. By notation in Remark 3.1, from the exact sequence

$$0 \longrightarrow L_2 \longrightarrow \frac{R}{L_1} \longrightarrow \frac{R}{L_1 + L_2} \longrightarrow 0,$$

we have $\operatorname{Supp}(L_2) \subseteq \operatorname{Supp}\left(\dfrac{R}{L_1}\right)$ and $H_I^i(L_2) \approx H_{\mathfrak{m}}^i(L_2)$ for $i \geq 0$. Also, the exact sequence

$$0 \longrightarrow L_2 \longrightarrow R \longrightarrow \frac{R}{L_2} \longrightarrow 0$$

induces the following exact sequence:

$$H_I^{d-1}(L_2) \longrightarrow H_I^{d-1}(R) \longrightarrow H_I^{d-1}\left(\frac{R}{L_2}\right) \longrightarrow H_I^d(L_2) \longrightarrow \ldots,$$

which implies that $H_I^{d-1}\left(\dfrac{R}{L_2}\right)$ is I-cominimax. Since $\operatorname{cd}\left(I, \dfrac{R}{L_2}\right) \leq d-1$, it follows that

$$H_I^{d-1}\left(\frac{R}{L_2}\right) \otimes M \approx H_I^{d-1}\left(\frac{R}{L_2} \otimes M\right) \approx H_I^{d-1}\left(\frac{M}{L_2 M}\right).$$

Thus, by Lemma 3.1, $H_I^{d-1}\left(\dfrac{M}{L_2 M}\right)$ is I-cominimax (for this we consider a free resolution of M). Also, from the exact sequence

$$0 \longrightarrow L_2 M \longrightarrow M \longrightarrow \frac{M}{L_2 M} \longrightarrow 0$$

we have the following exact sequence:

$$H_I^{d-1}(L_2 M) \longrightarrow H_I^{d-1}(M) \longrightarrow H_I^{d-1}\left(\frac{M}{L_2 M}\right) \longrightarrow H_I^d(L_2 M).$$

By Remark 3.1, it follows that $H_I^{d-1}(M)$ is I-cominimax.

Lemma 3.2. *Let R be a Noetherian ring, I an ideal of R and M an R-module such that $\dim M \leq 1$. Then for all $n \geq 0$ and all finitely generated R-module K, the R-module $\operatorname{Tor}_n^R(K, M)$ is I-cofinite.*

Proof. See [9] (Lemma 3.3).

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module such that $\sqrt{I + \operatorname{Ann} M} = \mathfrak{m}$. Then R-module $H_I^n(M)$ is Artinian and I-cofinite, for all $n \geq 0$.*

Proof. We have the following relations:

$$H_I^n(M) \simeq H^n_{I+\operatorname{Ann} M/\operatorname{Ann} M}(M) = H^n_{\mathfrak{m}/\operatorname{Ann} M}(M) \simeq H^n_{\mathfrak{m}}(M).$$

So the R-module $H_I^n(M)$ is Artinian. On the other hand, we have

$$\operatorname{Hom}_R(R/I, H_I^n(M)) \simeq \operatorname{Hom}_R\left(R/I, \operatorname{Hom}_R\left(R/\operatorname{Ann} M, H_I^n(M)\right)\right) \simeq$$

$$\simeq \operatorname{Hom}_R\left(R/\operatorname{Ann} M + I, H_I^n(M)\right).$$

Now, since the R-module $\operatorname{Hom}_R\left(R/I + \operatorname{Ann} M, H_I^n(M)\right)$ is of finite length and $H_I^n(M)$ is Artinian, so $H_I^n(M)$ is I-cofinite by [8] (Proposition 4.1).

***Remark* 3.2.** Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of dimension $d \geq 1$ and let I be an ideal of R. If $\sup\{n \in \mathbb{N}_0 \colon H_I^n(M) \neq 0\} = d$, since R is complete, then from Lichtenbaum – Hartshorn vanishing theorem, the set $A = \left\{p \in \operatorname{Ass} R | \sqrt{p + I} = \mathfrak{m}\right\}$ is non empty. Set $J = \bigcap_{p \in A} p$. Also we have $m \operatorname{Ass} M/\Gamma_J(M) \subseteq \operatorname{Ass} M \backslash V(J) \subseteq \operatorname{Spec}(R) \backslash A$. Then by Lichtenbaum – Hartshorn vanishing theorem, $H_I^d(R/\Gamma_J(R)) = 0$. Since $M/\Gamma_J(M)$ is an $R/\Gamma_J(R)$-module, it follows that $H_I^d(M/\Gamma_J(M)) = 0$.

On the other hand, $\operatorname{Ass} \Gamma_J(R) = \operatorname{Ass} R \cap V(J) = A$. So, $\sqrt{\operatorname{Ann} \Gamma_J(R) + I} = \mathfrak{m}$. In particular, by Lemma 3.3, the R-module $H_I^i(\Gamma_J(R))$ is Artinian and I-cofinite for each i.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of dimension $d \geq 1$ and let I be an ideal of R. Then the following statements are equivalent:*

(i) $H_I^{d-1}(R)$ *is I-cofinite;*

(ii) *for every finitely generated R-module M, the R-module $H_I^{d-1}(M)$ is I-cofinite.*

Proof.

(ii)→(i) is clear.

(i)→(ii) If $\dim M < d-1$, then $H_I^{d-1}(M) = 0$. If $\dim M = d-1$, then by [8] (Proposition 5.1), $H_I^{d-1}(M)$ is I-cofinite.

Now, let $\dim M = d$ and $\sup\left\{n \in \mathbb{N}_0 \colon H_I^n(M) \neq 0\right\} = d - 1$. Then by [2] (Excercise 6.1.8), $H_I^{d-1}(M) \simeq H_I^{d-1}(R) \otimes_R M$ and by Lemma 3.1, $H_I^{d-1}(M)$ is I-cofinite. Note that, in view of [7] (Corollary 2.5), $\operatorname{Supp} H_I^{d-1}(R)$ is finite and so its dimension is at most one.

Therefore, we assume that $\sup\{n \in \mathbb{N}_0 \colon H_I^n(M) \neq 0\} = d$. By notation in Remark 3.2, we have the following exact sequence:

$$0 \longrightarrow \Gamma_J(R) \longrightarrow R \longrightarrow R/\Gamma_J(R) \longrightarrow 0,$$

that induces the long exact sequence

$$\ldots \longrightarrow H_I^{d-1}(\Gamma_J(R)) \xrightarrow{f} H_I^{d-1}(R) \xrightarrow{g} H_I^{d-1}(R/\Gamma_J(R)) \xrightarrow{h} H_I^d(\Gamma_J(R)) \longrightarrow \ldots.$$

Note that the category of Artinian I-cofinite modules is a Serre category and so $\mathrm{Im} f$ is I-cofinite. Now from the exact sequence

$$0 \longrightarrow \mathrm{Im}\, f \longrightarrow H_I^{d-1}(R) \longrightarrow \mathrm{Im}\, g \longrightarrow 0,$$

we deduce that $\mathrm{Im} g$ is I-cofinite.

Since $\mathrm{Im}\, h$ is also I-cofinite, it follows from the exact sequence

$$0 \longrightarrow \mathrm{Im}\, g \longrightarrow H_I^{d-1}(R/\Gamma_J(R)) \longrightarrow \mathrm{Im}\, h \longrightarrow 0$$

that $H_I^{d-1}(R/\Gamma_J(R))$ is I-cofinite and of dimension at most one. Now, by Lemma 3.2 and the fact that $H_I^{d-1}(M/\Gamma_J(M)) \simeq H_I^{d-1}(R/\Gamma_J(R)) \otimes_R M$, we deduce that $H_I^{d-1}(M/\Gamma_J(M))$ is I-cofinite. From the exact sequence

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0,$$

we obtain the following long exact sequence:

$$\ldots \longrightarrow H_I^{d-1}(\Gamma_J(M)) \xrightarrow{f_1} H_I^{d-1}(M) \xrightarrow{f_2} H_I^{d-1}(M/\Gamma_J(M)) \xrightarrow{f_3} H_I^{d}(\Gamma_J(M)) \longrightarrow \ldots .$$

For all $n \geq 0$, the R-module $H_I^n(\Gamma_J(M))$ is Artinian and I-cofinite, so by the above long exact sequence, $\mathrm{Im} f_3$ and $\mathrm{Im} f_2$ are I-cofinite. Therefore, from the exact sequence

$$0 \longrightarrow \mathrm{Im} f_1 \longrightarrow H_I^{d-1}(M) \longrightarrow \mathrm{Im} f_2 \longrightarrow 0,$$

we deduce that $H_I^{d-1}(M)$ is also I-cofinite.

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