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EMBEDDING THEOREMS AND MAXIMAL SUBSEMIGROUPS OF SOME LINEAR TRANSFORMATION SEMIGROUPS WITH RESTRICTED RANGE

ТЕОРЕМИ ПРО ВКЛАДЕННЯ ТА МАКСИМАЛЬНІ ПІДНАПІВГРУПИ ДЕЯКИХ НАПІВГРУП ЛІНІЙНИХ ПЕРЕТВОРЕНЬ З ОБМЕЖЕНИМ ОБРАЗОМ

Let V be a vector space over a field and let $T(V)$ denote the semigroup of all linear transformations from V into V. For a fixed subspace W of V, let $F(V,W)$ be the subsemigroup of $T(V)$ consisting of all linear transformations α from V into W such that $V\alpha \subseteq W\alpha$. In this paper, we prove that any regular semigroup S can be embedded in $F(V,W)$ with $\dim(V) = |S^1|$ and $\dim(W) = |S|$, and determine all the maximal subsemigroups of $F(V,W)$ when W is a finite dimensional subspace of V over a finite field.

Нехай V — векторний простір над деяким полем, а $T(V)$ — напівгрупа всіх лінійних перетворень з V у V. Для фіксованого підпростору W простору V нехай $F(V,W)$ — піднапівгрупа напівгрупи $T(V)$, яка складається з усіх лінійних перетворень α з V у W таких, що $V\alpha \subseteq W\alpha$. Доведено, що будь-яку регулярну напівгрупу S можна вкласти у $F(V,W)$ з $\dim(V) = |S^1|$ і $\dim(W) = |S|$, та визначено всі максимальні піднапівгрупи з $F(V,W)$, якщо W — скінченновимірний підпростір V над скінченним полем.

1. Introduction. Let $T(X)$ be the set of all full transformations from a nonempty set X into itself. It is well-known that $T(X)$ is a regular semigroup under composition of functions. The properties of $T(X)$ have been widely studied. In 1959, Hall (see [5], Theorem 1.10) showed that every semigroup S can be embedded in a full transformation semigroup $T(S^1)$ by using the extended right regular representation of S. In [3] (Theorem 8.5) showed that any right cancellative, right simple semigroup S without idempotents can be embedded in a Bear – Levi semigroup of type (p,p) where $p = |S|$. In [2] (Theorem 1.20) proved that any inverse semigroup S can be embedded in the symmetric inverse semigroup $I(S)$ of all injective partial transformations of S.

If $X = \{1, 2, \ldots, n\}$ with $n \in \mathbb{Z}^+$, we write T_n instead of $T(X)$. In 1966 Bayramov [1] characterized all the maximal subsemigroups of T_n, which is either the union of a maximal subgroup of the symmetric group S_n and $T_n \setminus S_n$ or it is the union of the set of all transformations $\alpha \in T_n$ with $|X\alpha| \leq n-2$ and S_n. Later in 2002, You [20] determined all the maximal regular subsemigroups of all ideals of T_n. In 2004, Yang and Yang [19] completely described the maximal subsemigroups of ideals of T_n. And in 2015, East, Michell and Péresse [4] classified the maximal subsemigroups of $T(X)$ when X is an infinite set containing certain subgroups of the symmetric group on X.

For a fixed nonempty subset Y of a set X, let

$$T(X,Y) = \{\alpha \in T(X) \colon X\alpha \subseteq Y\},$$

where $X\alpha$ denotes the image of α. Then $T(X,Y)$ is a subsemigroup of $T(X)$. In 1975, Symons [18] described all the automorphisms of $T(X,Y)$. He also determined when $T(X_1,Y_1)$ is isomorphic to $T(X_2,Y_2)$. In 2005, Nenthein, Youngkhong and Kemprasit [8] characterized the regular elements of $T(X,Y)$. In 2008, Sanwong and Sommanee [12] defined

$$F(X,Y) = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$$

and showed that $F(X,Y)$ is the largest regular subsemigroup of $T(X,Y)$. This semigroup plays a crucial role in characterization of Green's relations on $T(X,Y)$. Moreover, they determined a class of maximal inverse subsemigroups of $T(X,Y)$. In 2011, Sanwong [11] described Green's relations, ideals and all the maximal regular subsemigroups of $F(X,Y)$. Also, the author proved that every regular semigroup S can be embedded in $F(S^1,S)$. Later in 2013, Sommanee and Sanwong [15] computed the rank of $F(X,Y)$ when X is a finite set. Furthermore, they obtained the rank and idempotent rank of its ideals. Recently in 2018, Sommanee [13] described the maximal inverse subsemigroups of $F(X,Y)$ and completely determined all the maximal regular subsemigroups of its ideals.

For a vector space V over a field F, let $T(V)$ be the set of all linear transformations from V into V. It is known that $T(V)$ is a regular semigroup under composition of functions (see [2, p. 57]). In 2004, Mendes-Gonçalves and Sullivan [7] (Theorem 3.12) proved that any right simple, right cancellative semigroup S without idempotents can be embedded in some $GS(m,m)$, the linear Baer – Levi semigroup on V. After that in 2012, Sullivan [16] (Theorem 3) proved that any semigroup S can be embedded in $T(V)$ for some vector space V with dimension $|S^1|$.

For a fixed subspace W of a vector space V, let

$$T(V,W) = \{\alpha \in T(V) : V\alpha \subseteq W\}.$$

Then $T(V,W)$ is a subsemigroup of $T(V)$. In 2007, Nenthein and Kemprasit [9] proved that $\alpha \in$ $\in T(V,W)$ is a regular element of $T(V,W)$ if and only if $V\alpha = W\alpha$. As a consequence, they showed that $T(V,W)$ is regular if and only if either $V = W$ or $W = \{0\}$. Later in 2008, Sullivan [17] proved that the set

$$F(V,W) = \{\alpha \in T(V,W) : V\alpha \subseteq W\alpha\},$$

consisting of all regular elements in $T(V,W)$, is the largest regular subsemigroup of $T(V,W)$. He characterized Green's relations on $T(V,W)$ and showed that the semigroup $F(V,W)$ is always a right ideal of $T(V,W)$. The author also described all the ideals of $F(V,W)$ and $T(V,W)$. Recently in 2017, Sommanee and Sangkhanan [14] determined the maximal regular subsemigroups of $F(V,W)$ when W is a finite dimensional subspace of V over a finite field F. Moreover, they computed the rank and the idempotent rank of $F(V,W)$ when V is a finite dimensional vector space over a finite field F.

Here, we prove that any regular semigroup S can be embedded in $F(V,W)$ where $\dim(V) =$ $= |S^1|$ and $\dim(W) = |S|$, and determine all the maximal subsemigroups of $F(V,W)$ when W is a finite dimensional subspace of V over a finite field F.

2. Preliminaries and notations. Let S be a semigroup. We call $a \in S$ a *regular element* if $a = axa$ for some $x \in S$, and S is said to be a *regular semigroup* if every element of S is regular. An element $e \in S$ is called an *idempotent* if $e^2 = e$. A nonempty subset A of S is said to be an *ideal* if $SA \subseteq A$ and $AS \subseteq A$. A proper (regular) subsemigroup M of S is a *maximal* (*regular*) *subsemigroup* of S if, whenever $M \subseteq T \subseteq S$ for some a (regular) subsemigroup T of S, then $M = T$ or $T = S$.

Let a and b be elements of a semigroup S. The Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ on S are defined as follows: $a\mathcal{L}b$ if $S^1a = S^1b$, $a\mathcal{R}b$ if $aS^1 = bS^1$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $a\mathcal{J}b$ if $S^1aS^1 = S^1bS^1$, where S^1 is a semigroup S with the identity adjoined, if necessary. For each $a \in S$, we denote $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class and $\mathcal{J}$-class containing a by L_a, R_a, H_a and J_a, respectively.

A semigroup S is said to be *embedded* in a semigroup T if there exists an injective function $\varphi\colon S \to T$ such that $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in S$.

Let V be a vector spaces over a field F. A function $\alpha\colon V \to V$ is a *linear transformation on* V if

$$(u+v)\alpha = u\alpha + v\alpha \text{ and } (au)\alpha = a(u\alpha)$$

for all vectors $u, v \in V$ and scalar $a \in F$. The set $T(V)$ of all linear transformations from V into V is a semigroup with respect to the composition operation. This semigroup is called a *linear transformation semigroup*. We denote by Θ_V the zero map in $T(V)$, that is, $\Theta_V : V \to \{0\}$.

For a fixed subspace W of a vector space V, let

$$T(V,W) = \{\alpha \in T(V)\colon V\alpha \subseteq W\} \text{ and } F(V,W) = \{\alpha \in T(V,W)\colon V\alpha \subseteq W\alpha\}.$$

Then $T(V,W)$ is a subsemigroup of $T(V)$ and $F(V,W)$ is the largest regular subsemigroup of $T(V,W)$.

For any set A, $|A|$ means the cardinality of the set A.

In this paper, a subspace of a vector space V over a field F generated by a linearly independent subset $\{e_i : i \in I\}$ of V is denoted by $\langle e_i \rangle$. If we write $U = \langle e_i \rangle$ when U is a subspace of V, it means the set $\{e_i : i \in I\}$ is a basis of U with $\dim(U) = |I|$. Let $\{u_i : i \in I\}$ be a subset of V. Then the notation $\sum a_i u_i$ means the linear combination

$$a_{i_1}u_{i_1} + a_{i_2}u_{i_2} + \ldots + a_{i_n}u_{i_n}$$

for some $n \in \mathbb{Z}^+$, $u_{i_1}, u_{i_2}, \ldots, u_{i_n} \in \{u_i : i \in I\}$ and scalars $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in F$.

A construction of a map $\alpha \in T(V)$, we first choose a basis $\{e_i : i \in I\}$ for a vector space V and a subset $\{u_i : i \in I\}$ of V, and then let $e_i\alpha = u_i$ for each $i \in I$ and extending this action by linearity to the whole of V. To shorten this process, we simply say, given $\{e_i : i \in I\}$ and $\{a_i : i \in I\}$ within the context. Then $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

Let $S_1, S_2, \ldots, S_n$ be subspaces of a vector space V where $n \geq 2$. We call V the *internal direct sum* of $S_1, S_2, \ldots, S_n$, and we write

$$V = S_1 \oplus S_2 \oplus \ldots \oplus S_n,$$

if $V = S_1 + S_2 + \ldots + S_n = \{s_1 + s_2 + \ldots + s_n : s_i \in S_i, 1 \leq i \leq n\}$ and $S_i \cap (S_1 + \ldots + S_{i-1} + + S_{i+1} + \ldots + S_n) = \{0\}$ for all $1 \leq i \leq n$. We note that if U is a subspace of V, then there exists a subspace T of V such that $V = U \oplus T$ (see [10], Theorem 1.4).

The *external direct sum* of a family of rings $\{R_i : i \in I\}$, denoted $\sum_{i \in I} R_i$, is the set of all sequences (r_i) where $r_i \in R_i$ and at most finitely many r_i are non-zero.

3. Embedding theorems. In 2011, Sanwong proved that every regular semigroup S can be embedded in $F(S^1, S)$ (see [11], Theorem 3). Here, we prove a linear version of that result.

***Remark* 3.1** ([6, p. 182], Remark (c))**.** Let X be any nonempty set and R a ring with identity. Let V be the external direct sum $\sum R_i$ with the copies of R indexed by the set X. Then V is a free R-module on the set X such that X is a basis of V. In particular, if R is a field, then $V = \sum R_i$ is a vector space over R with dimension $|X|$.

Lemma 3.1 ([16], Theorem 3)**.** *Any semigroup S can be embedded in $T(U)$ where $U = \sum F_i$ is the external direct sum of the copies of a field F indexed by the semigroup S^1 (U is a vector space with dimension $|S^1|$).*

Theorem 3.1. *Let W be a subspace of a vector space V. Then $T(W)$ can be embedded in $F(V, W)$.*

Proof. It is clear that if $V = W$, then $T(W) = F(V, V) = F(V, W)$ and so $T(W) \cong F(V, W)$. But, when $W = \{0\}$, we see that $T(W) = \{\Theta_W\}$ and $F(V, W) = \{\Theta_V\}$. Thus, they are isomorphic.

Now, suppose that $\{0\} \neq W \subsetneq V$. Let $W = \langle w_i \rangle$ and $V = \langle w_i \rangle \oplus \langle v_j \rangle$ for some subspace $\langle v_j \rangle$ of V. Then we have $\{w_i : i \in I\} \neq \varnothing \neq \{v_j : j \in J\}$. Let $\alpha \in T(W)$ and write

$$\alpha = \begin{pmatrix} w_i \\ w_i\alpha \end{pmatrix}.$$

Define $\alpha' \in T(V, W)$ as follows:

$$\alpha' = \begin{pmatrix} w_i & v_j \\ w_i\alpha & 0 \end{pmatrix}.$$

We obtain $V\alpha' \subseteq W\alpha'$, which implies that $\alpha' \in F(V, W)$. For any element $w \in W$, we can write $w = \sum a_i w_i$ and so $w\alpha' = (\sum a_i w_i)\alpha' = \sum a_i(w_i\alpha') = \sum a_i(w_i\alpha) = (\sum a_i w_i)\alpha = w\alpha$. Also, if $\alpha, \beta \in T(W)$ and $w \in W$, then $w\alpha \in W$ and thus $(w\alpha)\beta' = (w\alpha)\beta$. We define

$$\Phi : T(W) \to F(V, W) \text{ by } \alpha\Phi = \alpha' \text{ for all } \alpha \in T(W).$$

We prove that Φ is a monomorphism. Let $\alpha, \beta \in T(W)$. If $\alpha\Phi = \beta\Phi$, then $\alpha' = \beta'$. For $w \in W$, $w = \sum a_i w_i$ and $w\alpha = \big(\sum a_i w_i\big)\alpha = \sum a_i(w_i\alpha) = \sum a_i(w_i\alpha') = \sum a_i(w_i\beta') = \sum a_i(w_i\beta) = \big(\sum a_i w_i\big)\beta = w\beta$. So, $\alpha = \beta$ and hence Φ is injective. Let $v \in V$. Then we can write $v = \sum b_i w_i + \sum c_j v_j$ and $v(\alpha'\beta') = \big(\sum b_i w_i + \sum c_j v_j\big)(\alpha'\beta') = \sum b_i\big(w_i(\alpha'\beta')\big) + \sum c_j\big(v_j(\alpha'\beta')\big) = \sum b_i\big((w_i\alpha')\beta'\big) + \sum c_j\big((v_j\alpha')\beta'\big) = \sum b_i(w_i\alpha)\beta' + \sum c_j(0\beta') = \sum b_i(w_i\alpha)\beta + \sum c_j(0) = \sum b_i\big(w_i(\alpha\beta)\big) + \sum c_j(0) = \sum b_i\big(w_i(\alpha\beta)'\big) + \sum c_j\big(v_j(\alpha\beta)'\big) = \big(\sum b_i w_i + \sum c_j v_j\big)(\alpha\beta)' = v(\alpha\beta)'$. Whence, $(\alpha\beta)' = \alpha'\beta'$, it follows that $(\alpha\beta)\Phi = (\alpha\Phi)(\beta\Phi)$. Thus, Φ is a monomorphism and therefore $T(W)$ can be embedded in $F(V, W)$.

Theorem 3.1 is proved.

By Lemma 3.1, any semigroup S can be embedded in $T(W)$ for some vector space W with dimension $|S^1|$. And by Theorem 3.1, $T(W)$ can be embedded in $F(V, W)$ when V is any vector space which contains W. So, we have the following corollary.

***Corollary* 3.1.** *Any semigroup S can be embedded in $F(V, W)$ for some subspace W of V with $\dim(W) = |S^1|$, where V is any vector space which contains W.*

Lemma 3.2. *Let S be any semigroup and $x \in S$. We write $S^1 = \{a_i : i \in I\}$ and define $\rho_x : S^1 \to S^1$ by $a_i\rho_x = a_i x$ for all $i \in I$. Let F be any field and V the external direct sum $\sum F_i$ with the copies of F indexed by S^1. Then*:

(1) *ρ_x can be extended by linearity to an element of $T(V)$,*

(2) *the mapping* $\rho\colon S \to T(V)$ *is given by* $x\rho = \rho_x$ *for all* $x \in S$, *is a monomorphism.*

Proof. See the proof as given in [16] (Theorem 3).

Lemma 3.3. *Let* $\sum F_i$ *be the external direct sum of the copies of a field* F *indexed by some set* I *with* $|I| \geq 2$. *We fix* $k \in I$ *and let* $J = I \setminus \{k\}$. *Let* G *be the external direct sum of* $\{0\} \cup \{F_j : j \in J\}$, *where* $0 \in F_k = F$, *and* $\sum F_j$ *is the external direct sum of the copies of a field* F *indexed by the set* J. *Then*:

(1) G *is a subspace of* $\sum F_i$,

(2) $\sum F_j$ *is isomorphic to* G.

Proof. (1) It is easy to verify that G is a subspace of $\sum F_i$.

(2) For each $(r_j) \in \sum F_j$, we construct an element (r_i') in G by

$$(r_i') = \begin{cases} 0, & \text{if } i = k, \\ r_j, & \text{if } i \in I \setminus \{k\} = J. \end{cases}$$

Define $\varphi\colon \sum F_j \to G$ by $(r_j)\varphi = (r_i')$ for all $(r_j) \in \sum F_j$. Then φ is bijective. Let $(r_j), (s_j) \in \in \sum F_j$ and $c \in F$. It is routine to show $[(r_j) + (s_j)]\varphi = (r_j)\varphi + (s_j)\varphi$ and $[c(r_j)]\varphi = c[(r_j)\varphi]$. Thus, φ is an isomorphism and so $\sum F_j \cong G$.

Lemma 3.3 is proved.

Theorem 3.2. *Any regular semigroup* S *can be embedded in* $F(V, W)$ *for some subspace* W *of a vector space* V, *where* $\dim(V) = |S^1|$ *and* $\dim(W) = |S|$.

Proof. Assume that S is a regular semigroup and let V be the external direct sum $\sum F_i$ with the copies of a field F indexed by S^1. We note that $V = \langle S^1 \rangle$ and $\dim(V) = |S^1|$ by Remark 3.1. There are two cases to consider.

Case 1: $1 \in S$. Then we have $S^1 = S$. Let $W = V$. It follows from Lemma 3.1 that S can be embedded in $T(V) = F(V, W)$ such that $\dim(W) = \dim(V) = |S^1| = |S|$.

Case 2: $1 \notin S$. This implies that $|S^1| \geq 2$ and $S = S^1 \setminus \{1\}$. Let G be the external direct sum of $\{0\} \cup \{F_j : j \in S\}$, where $0 \in F_1 = F = F_j$ for all $j \in S$. It follows from Lemma 3.3 that $\sum F_j \cong G \subseteq V$, where $\sum F_j$ is the external direct sum of $\{F_j : j \in S\}$ with the copies of the field F indexed by S. Here, we let $W = \sum F_j$. Thus, we have $W = \sum F_j = \langle S \rangle$, $\dim(W) = |S|$ and $W \subseteq V$ in the sense of embedding. Now, we write $S^1 = \{a_i : i \in I\}$. For each $x \in S$, define $\rho_x : S^1 \to S^1$ by $a_i\rho_x = a_i x$ for all $i \in I$. Then by Lemma 3.2 (1), we obtain $\rho_x \in T(V)$ and it is clear that $a_i\rho_x = a_i x \in S$ for all $i \in I$. Notice that there exists $t \in S$ such that $x = xtx$ since S is regular. We prove $\rho_x \in F(V, W)$. Let $v\rho_x \in V\rho_x$ for some $v \in V = \langle S^1 \rangle$. So, we can write $v = \sum d_i a_i$ and $v\rho_x = \sum d_i(a_i\rho_x) \in \langle S \rangle = W$. Whence, $V\rho_x \subseteq W$. Next, we prove $V\rho_x \subseteq W\rho_x$. If $v = \sum d_i a_i$ for some $a_i \in S$, then $v\rho_x = \left(\sum d_i a_i\right)\rho_x \in \langle S \rangle \rho_x = W\rho_x$. If $v = d \cdot 1$ for some scalar $d \in F$, then $v\rho_x = (d \cdot 1)\rho_x = d(1\rho_x) = dx = d(xtx) = d((xt)x) = d((xt)\rho_x) = (d(xt))\rho_x \in \langle S \rangle \rho_x = W\rho_x$. Hence, $V\rho_x \subseteq W\rho_x$ and so $\rho_x \in F(V, W)$. We define $\rho\colon S \to F(V, W)$ by $x\rho = \rho_x$ for all $x \in S$. Then by Lemma 3.2 (2), we have ρ is a monomorphism. Therefore, we conclude that S can be embedded in $F(V, W)$.

Theorem 3.2 is proved.

4. Maximal subsemigroups. In 2017, Sommanee and Sangkhanan determined the maximal regular subsemigroups of $F(V, W)$, when W is a finite dimensional subspace of a vector space V over a finite field F (see [14], Theorem 4.9).

In general, if S is a regular semigroup and T is a maximal regular subsemigroup of S, then T may not be a maximal subsemigroup of S (see [19, 20], Theorem 2). Here, we prove that the maximal subsemigroups and the maximal regular subsemigroups of $F(V, W)$ coincide.

We begin by recalling some notations and results from [14] that will be useful in this section.

Lemma 4.1 ([14], Lemma 2.3)**.** *Let W be a subspace of a vector space V and $\alpha, \beta \in F(V, W)$. Then*:

(1) $\alpha \mathcal{J} \beta$ *if and only if* $\dim(V\alpha) = \dim(V\beta)$,

(2) $\alpha \mathcal{H} \beta$ *if and only if* $V\alpha = V\beta$ *and* $\ker \alpha = \ker \beta$,

where $\ker \alpha = \{v \in V : v\alpha = 0\}$.

Lemma 4.2 ([14], Theorem 2.4)**.** *Let W be a subspace of a vector space V. Then the ideals of $F(V, W)$ are precisely the sets $Q_k = \{\alpha \in F(V, W) \colon \dim(V\alpha) \leq k\}$, where $0 \leq k \leq \dim(W)$.*

We note that Q_k is a regular subsemigroup of $F(V, W)$ (see [14], Lemma 2.5).

Let $n \geq 0$ be an integer and W an n-dimensional subspace of a vector space V over a finite field F.

For $0 \leq k \leq n = \dim(W)$, define $J(k) = \{\alpha \in F(V, W) \colon \dim(V\alpha) = k\}$. Then $J(k)$ is a $\mathcal{J}$-class of $F(V, W)$. Let Q_k be defined as in Lemma 4.2. We have $Q_k = J(0) \cup J(1) \cup \ldots \cup J(k)$ and $Q_n = F(V, W)$.

***Remark* 4.1.** The following facts are directly obtained from the definitions of $J(k)$ and Q_k:

(1) $Q_0 = J(0)$ contains exactly one element Θ_V, the zero map;

(2) for each $\alpha \in J(n)$, $V\alpha = W$ since $V\alpha \subseteq W$ and $\dim(V\alpha) = n = \dim(W)$ is finite.

We will use the notation $GL(U)$ as a set of all automorphisms of a vector space U over a field F. It is well-known that $GL(U)$ is a group under the composition of functions.

Lemma 4.3 ([14], Lemma 3.2)**.** *Let $\varepsilon \in F(V, W)$ be an idempotent. Then $H_\varepsilon \cong GL(V\varepsilon)$.*

From now on, we suppose that $n \geq 1$ and let $E\big(J(n)\big) = \{\varepsilon_p \colon p \in P\}$ be the set of all idempotents in $J(n)$. Then we have

$$J(n) = \bigcup_{p \in P} H_{\varepsilon_p}$$

is a disjoint union of groups all of which are isomorphic (see [14], Lemma 3.3). Moreover, $J(n)$ is a regular subsemigroup of $F(V, W)$ (see [14], Lemma 3.6).

Lemma 4.4 ([14], Lemma 4.1)**.** $J(n-1) \subseteq J(n)\alpha J(n)$ *for all* $\alpha \in J(n-1)$.

Lemma 4.5 ([14], Theorem 4.2)**.** *For $n \geq 2$, the set $Q_{n-2} \cup J(n)$ is a maximal regular subsemigroup of $F(V, W)$.*

For each $\varepsilon_p \in E\big(J(n)\big)$, $H_{\varepsilon_p} \cong GL(V\varepsilon_p) = GL(W)$ by Lemma 4.3 and Remark 4.1 (2). We let $\Phi_p \colon H_{\varepsilon_p} \to GL(W)$ be an isomorphism and U a fixed maximal subgroup of $GL(W)$. For each $p \in P$, we define

$$M_p = U\Phi_p^{-1}.$$

Then M_p is a maximal subgroup of H_{ε_p} for all $p \in P$ (for details, see [14, p. 409]).

Lemma 4.6 ([14], Lemma 4.3)**.** *Let M_p be defined as above and $M = \bigcup_{p \in P} M_p$. Then M is a maximal regular subsemigroup of $J(n)$.*

Lemma 4.7 ([14], Theorem 4.4)**.** *Let M be as in Lemma 4.6. Then $Q_{n-1} \cup M$ is a maximal regular subsemigroup of $F(V, W)$.*

Lemma 4.8 ([14], Lemma 4.6)**.** *T is a maximal regular subsemigroup of $J(n)$ if and only if there is a maximal subgroup U of $GL(W)$ such that $T = \bigcup_{p\in P} M_p$ with $M_p = U\Phi_p^{-1}$, where Φ_p is defined as previous Lemma* 4.6 $(p \in P)$.

Recall that if A is a subset of a semigroup S, then $\langle A\rangle$ denotes the subsemigroup of S generated by A.

Lemma 4.9 ([14], Lemma 4.7)**.** *For* $0 \le k \le n-1,\ Q_k = \langle J(k)\rangle$.

To prove the main results, we prepare the following two lemmas.

Lemma 4.10. *Every subsemigroup of $J(n)$ is a regular subsemigroup of $J(n)$.*

Proof. Assume that T is a subsemigroup of $J(n) = \bigcup_{p\in P} H_{\varepsilon_p}$. Let $R = \{r \in P : T\cap H_{\varepsilon_r} \neq \varnothing\}$ and $T_r = T \cap H_{\varepsilon_r}$ for all $r \in R$. It is clear that $T = \bigcup_{r\in R} T_r$. Since $T_r = T \cap H_{\varepsilon_r} \neq \varnothing$, we obtain T_r is a finite subsemigroup of the group H_{ε_r}. Thus, T_r is a subgroup of H_{ε_r} and so T_r is a regular subsemigroup of H_{ε_r} for all $r \in R$. Therefore, T is a regular subsemigroup of $J(n)$.

Lemma 4.10 is proved.

From Lemma 4.10, we easily verify the following lemma.

Corollary **4.1.** *The maximal subsemigroups and the maximal regular subsemigroups of $J(n)$ coincide.*

The following lemma is directly obtained from Lemma 4.8 and Corollary 4.1.

Lemma 4.11. *T is a maximal subsemigroup of $J(n)$ if and only if there is a maximal subgroup U of $GL(W)$ such that $T = \bigcup_{p\in P} M_p$ with $M_p = U\Phi_p^{-1}$ where Φ_p is defined as previous Lemma* 4.6 $(p \in P)$.

Lemma 4.12. *For $n \ge 2$, $Q_{n-2} \cup J(n)$ is a maximal subsemigroup of $F(V,W)$.*

Proof. Let $n \ge 2$. Then we have $Q_{n-2} \cup J(n)$ is a regular subsemigroup of $F(V,W)$ by Lemma 4.5. Thus, it is a subsemigroup of $F(V,W)$. To prove that $Q_{n-2} \cup J(n)$ is a maximal subsemigroup of $F(V,W)$, suppose that there is a subsemigroup S of $F(V,W)$ such that $Q_{n-2} \cup$ $\cup\, J(n) \subsetneq S \subseteq F(V,W)$. We prove that S is a regular subsemigroup of $F(V,W)$. Let α be any element in S. Then there exists $\alpha' \in F(V,W)$ such that $\alpha = \alpha\alpha'\alpha$, since $F(V,W)$ is regular. We note that if $\alpha \in Q_{n-2} \cup J(n)$, then α is a regular element in S, since $Q_{n-2} \cup J(n)$ is regular and $Q_{n-2} \cup J(n) \subseteq S$. Suppose that $\alpha \notin Q_{n-2} \cup J(n)$, that is, $\alpha \in J(n-1)$. We assume that $\alpha' \notin S$. Thus, $\alpha' \in J(n-1) \setminus S$ and we can write $\alpha' = \beta\alpha\gamma$ for some $\beta,\gamma \in J(n) \subseteq S$ by Lemma 4.4. This implies that $\alpha' \in S$, a contradiction. Whence, $\alpha' \in S$ and so S is a regular subsemigroup of $F(V,W)$. Since $Q_{n-2} \cup J(n)$ is a maximal regular subsemigroup of $F(V,W)$, we get $S = F(V,W)$. Therefore, $Q_{n-2} \cup J(n)$ is a maximal subsemigroup of $F(V,W)$.

Lemma 4.12 is proved.

Lemma 4.13. *Let M be as in Lemma* 4.6. *Then $Q_{n-1} \cup M$ is a maximal subsemigroup of $F(V,W)$.*

Proof. Since $Q_{n-1} \cup M$ is a regular subsemigroup of $F(V,W)$ by Lemma 4.7, it is a subsemigroup of $F(V,W)$. We prove that $Q_{n-1} \cup M$ is a maximal subsemigroup of $F(V,W)$. Let S be a subsemigroup of $F(V,W)$ such that $Q_{n-1} \cup M \subsetneq S \subseteq F(V,W)$. We see that $S \cap J(n) \neq \varnothing$. It follows that $S\cap J(n)$ is a subsemigroup of $J(n)$. Then by Lemma 4.10, we get that $S\cap J(n)$ is a regular subsemigroup of $J(n)$. Thus, $S = Q_{n-1} \cup (S \cap J(n))$ is a regular subsemigroup of $F(V,W)$. Since $Q_{n-1} \cup M$ is a maximal regular subsemigroup of $F(V,W)$, we obtain $S = F(V,W)$. Therefore, $Q_{n-1} \cup M$ is a maximal subsemigroup of $F(V,W)$.

Lemma 4.14. *Let S be any maximal subsemigroup of $F(V, W)$. Then the following statements hold*:

(1) $S \cap J(n) \neq \varnothing$,

(2) $S \cap J(n)$ *is a maximal subsemigroup of* $J(n)$.

Proof. (1) If $S \cap J(n) = \varnothing$, we get $S \subseteq Q_{n-1} \subsetneq Q_{n-1} \cup M \subsetneq F(V, W)$, where M is defined in Lemma 4.6. But $Q_{n-1} \cup M$ is a maximal subsemigroup of $F(V, W)$ by Lemma 4.13, which contradicts the maximality of S. Therefore, $S \cap J(n) \neq \varnothing$.

(2) It follows from (1) that $S \cap J(n)$ is a subsemigroup of $J(n)$. If $S \cap J(n)$ is not maximal, then there exists a maximal subsemigroup T of $J(n)$ such that $S \cap J(n) \subsetneq T \subsetneq J(n)$. It is easy to see that $Q_{n-1} \cup T$ is a subsemigroup of $F(V, W)$ with $S \subsetneq Q_{n-1} \cup T \subsetneq F(V, W)$, which contradicts the maximality of S. Therefore, $S \cap J(n)$ is a maximal subsemigroup of $J(n)$.

Lemma 4.14 is proved.

Theorem 4.1. *Let $n \geq 2$ and S a maximal subsemigroup of $F(V, W)$. Then S is either of the form*:

(1) $Q_{n-2} \cup J(n)$

or

(2) $Q_{n-1} \cup M$, *where M is defined in Lemma* 4.6.

Proof. By Lemmas 4.12 and 4.13, we have $Q_{n-2} \cup J(n)$ and $Q_{n-1} \cup M$ are maximal subsemigroups of $F(V, W)$. On the other hand, since $S \cap J(n) \neq \varnothing$ by Lemma 4.14 (1). So, we consider the following two cases.

Case 1: $S \cap J(n) = J(n)$. Hence, $J(n) \subseteq S$. We suppose that $S \nsubseteq Q_{n-2} \cup J(n)$. Then there exists $\alpha \in S$ and $\alpha \notin Q_{n-2} \cup J(n)$, that is, $\alpha \in J(n-1)$. It follows from Lemma 4.4 that $J(n-1) \subseteq J(n)\alpha J(n) \subseteq S\alpha S \subseteq S$, and so $Q_{n-1} = \langle J(n-1)\rangle \subseteq S$ by Lemma 4.9. Whence, $F(V, W) = Q_{n-1} \cup J(n) \subseteq S \subseteq F(V, W)$. Thus, $S = F(V, W)$, which contradicts the maximality of S. Therefore, $S \subseteq Q_{n-2} \cup J(n)$. But, $Q_{n-2} \cup J(n)$ is a maximal subsemigroup of $F(V, W)$ by Lemma 4.12. This implies that $S = Q_{n-2} \cup J(n)$.

Case 2: $S \cap J(n) \subsetneq J(n)$. By Lemma 4.14 (2), we have $S \cap J(n)$ is a maximal subsemigroup of $J(n)$. Then by Lemma 4.11, we get that $S \cap J(n) = \bigcup_{p \in P} M_p$, where $M_p = U\Phi_p^{-1}$ for all $p \in P$ with a fixed maximal subgroup U of $GL(W)$. We let $M = \bigcup_{p \in P} M_p$. Then $M = S \cap J(n)$. Since $S \subseteq Q_{n-1} \cup (S \cap J(n)) = Q_{n-1} \cup M$ and $Q_{n-1} \cup M$ is a maximal subsemigroup of $F(V, W)$ by Lemma 4.13, we obtain $S = Q_{n-1} \cup M$.

Theorem 4.1 is proved.

***Corollary* 4.2.** *For $n = 1$, each maximal subsemigroup of $F(V, W)$ must be one of the forms: $J(1)$ or $\{\Theta_V\} \cup M$, where M is defined in Lemma* 4.6.

Proof. Assume that $n = 1$. By Lemma 4.13, we obtain that $Q_0 \cup M = \{\Theta_V\} \cup M$ is a maximal subsemigroup of $F(V, W)$ where M is defined in Lemma 4.6. Furthermore, if $n = 1$, then $F(V, W) = J(0) \cup J(1) = \{\Theta_V\} \cup J(1)$, that is, $J(1) = F(V, W) \setminus \{\Theta_V\}$. And, since $J(1)$ is a subsemigroup of $F(V, W)$, it is clear that $J(1)$ is a maximal subsemigroup of $F(V, W)$.

Let S be any maximal subsemigroup of $F(V, W)$. Then we consider two cases.

Case 1: $\Theta_V \notin S$. Then $S \subseteq J(1)$. Since $S \subseteq J(1) \subsetneq F(V, W)$ and $J(1)$ is a subsemigroup of $F(V, W)$, whence $S = J(1)$.

Case 2: $\Theta_V \in S$. By Lemma 4.14 (1), we have $S \cap J(1) \neq \varnothing$. If $S \cap J(1) = J(1)$, we get that $S = F(V, W)$, a contradiction. Hence, $S \cap J(1) \subsetneq J(1)$. Then by the same argument as in the proof of Theorem 4.1 (Case 2), we obtain $S = Q_0 \cup M = \{\Theta_V\} \cup M$, where M is defined in Lemma 4.6.

Corollary 4.2 is proved.

Next, we consider the case when $V = W$ and V is a finite dimensional vector space with $\dim(V) = n$. Then we have $F(V,W) = T(V)$, and it is easy to verify that $J(n) = GL(V)$. So, we establish the following corollary.

***Corollary* 4.3.** *Let V be an n-dimensional vector space over a finite field F $(n \geq 2)$ and S a maximal subsemigroup of $T(V)$. Then S is either of the form*:

(1) $Q_{n-2} \cup GL(V)$

or

(2) $Q_{n-1} \cup M$, *where M is a maximal subgroup of $GL(V)$.*

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